JAERI-M
5793

FINITE ELEMENT METHOD FOR SOLVING NEUTRON TRANSPORT PROBLENS IN TWO-DIMENSIONAL CYLINDRICAL GEOMETRY

July 1974

Kunihiko HORIKAMI, Yasuaki NAKAHARA,
Toichiro FUJIMURA and Tadahiro OHNISHI*

日本原子力研究所
Japan Atomic Energy Research Institute

Finite Element Method for Solving Neutron Transport Problems in Two-Dimensional Cylindrical Geometry.

Kunihiko HORIKAMI, Yasuaki NAKAHARA,
Toichiro FUJIMURA and Tadahiro OHNISHI*

Division of Reactor Engineering, Tokai, JAERI,

(Received July 6, 1974)

Algorithms based on the finite element method have been developed for solving the two-dimensional multi-group neutron transport equation in (r,z) geometry. In the algorithms the finite element method is applied only to the spatial-variables in regular torus cells with rectangular cross sections. Angular variables are treated in the discrete ordinate approximation.

The formulations both in the continuous and discontinuous methods are performed by making use of the bilinear, cubic and biquadratic Lagrange polynomials, respectively, for the cases of four, eight and nine mesh points per rectangular subregion on the (r,z) plane.

The Galerkin scheme is adopted for eliminating the residuals of approximate equations in the continuous and discontinuous methods.

The algorithm using the neutron balance equation is also given for the continuous method.

*Atomic Energy Research Laboratory, Hitachi, Ltd.

有限要素法による2次元（r，z）中性子輸送方程式の解法

日本原子力研究所東海研究所原子炉工学部
堀上邦彦・中原康明
藤村統一郎・大西忠博*

（1974年7月6日受理）

2次元（r，z）体系での中性子輸送方程式を有限要素法を用いて解くアルゴリズムを開発した。有限要素法は空間変数に対してのみ適用し，角度変数に対してはSN法を用いた。(r, z)平面を幾つかの長方形に分割し，それぞれの長方形の上でラグランジ補間多項式を前もって作っておき，角度依存の中性子束をそれらの一次結合で表現する。一つの長方形上で定義されるラグランジ多項式の数は4, 8, 9の場合を考慮し，多項式の次数はr，zの双一次三次，双二次をそれぞれ対応させた。連続解を得るアルゴリズムと不連続解を得るアルゴリズムとを分けて説明し，両者いづれの場合においても，結合係数を決めるために適当な剰余を定義し，ガレルキン法を適用した。また連続解を得る解法の一つとして，中性子の保存則を表わす式を解くアルゴリズムも示した。

* 日立原子力研究所

Contents

目　　次

1. Introduction

The discrete ordinate S_n approximation for the angular variables has become a very important numerical method to solve the radiation transport equations and the S_n computer codes have been widely used in reactor and shielding calculations[(1)-(6)].

The finite element method, on the other hand, which originated in early times in the field of structure analysis and has been proved to be a powerful tool, is now attracting much attention of reactor physicists and mathematicians as a tool for solving multi-dimensional diffusion and transport equations[(6)-(10)]. The finite element method has two main advantages over the finite difference method; from the mathematical point of view a particular class of higher order approximation for the numerical solution can be applied and from the practical point of view any complex geometrical configulations can be simulated. In 1971 Ohnishi applied the finite element method to the neutron diffusion and transport equations in two-dimensional (x,y) geometries by dividing the reactor system into triangular subregions. In 1973 Reed et al. published a two-dimensional transport code TRIPLET based on the finite element method[(6)].

We are interested in combining these two familiar S_n and finite element methods to develop a computer program for solving multi-group neutron transport equations in two-dimensional (r,z) geometry.

The TRIPLET developed in Los Alamos Scientific Laboratory is just such a transport program as we are interested in, but it deals with only triangular meshes in a planar geometry. Triangular mesh causes rather complex algorithms, although it has higher flexibility

to geometrically complicated reactor system. On the other hand, the presence of the angular derivative in the transport equation for (r,z) geometry makes the algorithm somewhat more complicated than that for (x,y) geometry.

For these reasons we divide the whole system into a number of subregions whose shape in (r,z) plane is a regular torus with rectangular cross section. Lagrange's interpolating polynomials are used as expansion functions to represent angular fluxes in a form to be a linear combination of them in each subregion. Three types of Lagrange polynomials are used: bilinear, cubic and biquadratic polynomials, respectively, for the cases of four, eight and nine space mesh points per rectangular elementary subregion in (r,z) plane. The coefficients of the Lagrange expansion are the values of the angular fluxes at these mesh points.

In combining the discrete ordinate S_n and finite element method, several types of the formulation can be considered. We divide them into two categories depending on whether the fluxes obtained are continuous over the whole system or not. The method which gives a continuous solution is called as the continuous method. The discontinuous method, on the other hand, gives the discontinuous fluxes at the boundaries of the subregions. The latter is rather troublesome to formulate and requires the large amount of computer core storage, but it gives much more stable algorithm than the continuous one[6].

To eliminate the residual resulting from the interpolation of the exact solution with low order polynomials, we adopted a Galerkin type scheme for both the continuous and discontinuous methods.

A scheme using the balance equation is also formulated for the continuous method.

2. Fundamental Equations

The time independent two-dimensional transport equation in (r,z) geometry is written as follows:

$$\frac{\mu}{r}\cdot\frac{\partial(r\psi^g)}{\partial r}-\frac{1}{r}\frac{\partial(\xi\psi^g)}{\partial\omega}+\eta\frac{\partial\psi^g}{\partial z}+\sigma_t^g\psi^g=S^g,\quad (g=1\sim G), \tag{1}$$

where the energy domain is divided into G intervals of width ΔE_g, and ψ^g stands for the angular flux:

$$\psi^g=\psi^g(r,z,\mu,\varphi)=\int_{\Delta E_g}\psi(r,z,\mu,\varphi,E)\,dE\ .$$

For convenience of ready reference to and comparison with the TRIPLET and TWOTRAN[(1),(2)] which is a discrete ordinate S_n code for general two-dimensional geometries, we use the same notations and coordinate system as used in these two codes.

Integrating Eq.(1) over solid angle interval $\Delta\Omega_m$, we have

$$W_m\mu_m\frac{1}{r}\frac{\partial(r\psi_m^g)}{\partial r}+W_m\eta_m\frac{\partial\psi_m^g}{\partial z}+\frac{1}{r}(\alpha_{m+1/2}\psi_{m+1/2}^g-\alpha_{m-1/2}\psi_{m-1/2}^g)+\sigma_t^gW_m\psi_m^g=W_mS_m^g\ ,$$
$$(g=1\sim G,\ m=1\sim MT), \tag{2}$$

where the discrete ordinate S_n approximation is used for angular variables (solid angle space is partitioned into MT intervals). In Eq.(2) the quantities with the angular mesh index m(m=1~MT) are defined as follows;

$$W_m=\iint_{\Delta\Omega_m}d\mu\,d\varphi/2\pi\ ,$$

$$\psi_m^g=\psi_m^g(r,z)=\left[\iint_{\Delta\Omega_m}\psi^g(r,z,\mu,\varphi)\,d\mu\,d\varphi/2\pi\right]\Big/W_m\ .$$

A scheme using the balance equation is also formulated for the continuous method.

2. Fundamental Equations

The time independent two-dimensional transport equation in (r,z) geometry is written as follows:

$$\frac{\mu}{r}\cdot\frac{\partial(r\psi^g)}{\partial r}-\frac{1}{r}\frac{\partial(\xi\psi^g)}{\partial\omega}+\eta\frac{\partial\psi^g}{\partial z}+\sigma_t^g\psi^g=S^g,\quad (g=1\sim G), \tag{1}$$

where the energy domain is divided into G intervals of width ΔE_g, and ψ^g stands for the angular flux:

$$\psi^g=\psi^g(r,z,\mu,\varphi)=\int_{\Delta E_g}\psi(r,z,\mu,\varphi,E)\,dE\ .$$

For convenience of ready reference to and comparison with the TRIPLET and TWOTRAN[(1),(2)] which is a discrete ordinate S_n code for general two-dimensional geometries, we use the same notations and coordinate system as used in these two codes.

Integrating Eq.(1) over solid angle interval $\Delta\Omega_m$, we have

$$W_m\mu_m\frac{1}{r}\frac{\partial(r\psi_m^g)}{\partial r}+W_m\eta_m\frac{\partial\psi_m^g}{\partial z}+\frac{1}{r}(\alpha_{m+1/2}\psi_{m+1/2}^g-\alpha_{m-1/2}\psi_{m-1/2}^g)+\sigma_t^gW_m\psi_m^g=W_mS_m^g\ ,$$
$$(g=1\sim G,\ m=1\sim MT), \tag{2}$$

where the discrete ordinate S_n approximation is used for angular variables (solid angle space is partitioned into MT intervals). In Eq.(2) the quantities with the angular mesh index m(m=1~MT) are defined as follows;

$$W_m=\iint_{\Delta\Omega_m}d\mu\,d\varphi/2\pi\ ,$$

$$\psi_m^g=\psi_m^g(r,z)=\left[\iint_{\Delta\Omega_m}\psi^g(r,z,\mu,\varphi)\,d\mu\,d\varphi/2\pi\right]\Big/W_m\ .$$

The coefficients $\alpha_{m\pm 1/2}$ are assumed to satisfy the following conditions[1],[2]. Demanding that there is no net particle loss due to angular redistribution the initial α values vanish on each η level, and requiring $\psi_m^g(r,z)$=constant as the solution of Eq.(2) for an infinite system, we can write

$$\alpha_{m+1/2} - \alpha_{m-1/2} = -W_m \mu_m \ .$$

Now, we attempt to get an approximate solution $\tilde{\psi}_m^g(r,z)$ to Eq.(2) by representing the unknown discretized angular flux as a linear combination of Lagrange's interpolating polynomials defined in each rectangular subregion with NL mesh points;

$$\tilde{\psi}_m^g(r,z) = \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} L^{(\ell)}(r,z) \ , \tag{3}$$

where $\psi_m^{g(\ell)}$ and $L^{(\ell)}(r,z)$ stand for combining coefficients and Lagrange polynomials, respectively. The unknowns are now $\psi_m^{g(\ell)}$, which have the physical meaning that they are the approximate values of the angular flux at the point (r_ℓ, z_ℓ) in the subregion. The polynomial $L^{(\ell)}(r,z)$, explicit representations of which are listed in Appendix I, has a property that the value is unity at the point (r_ℓ, z_ℓ) and zero at all other (NL-1) points in the subregion.

Substituting Eq.(3) into Eq.(2), we get

$$W_m \mu_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r L^{(\ell)}(r,z) \right) \right] + W_m \eta_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[\frac{\partial L^{(\ell)}(r,z)}{\partial z} \right] +$$
$$+ \sum_{\ell=1}^{NL} \left(\alpha_{m+1/2} \psi_{m+1/2}^{g(\ell)} - \alpha_{m-1/2} \psi_{m-1/2}^{g(\ell)} \right) \left[\frac{1}{r} L^{(\ell)}(r,z) \right] + \sigma_t^g W_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[L^{(\ell)}(r,z) \right] = W_m S_m^g(r,z) \ ,$$
$$(g=1\sim G;\ m=1\sim MT) \ . \tag{4}$$

A spherical harmonic expansion is used for the source term on the right hand side of Eq.(4) (see the references[1],[2]):

$$S_m^g(r,z)=\sum_{g'=1}^{G}\sum_{n=0}^{ISCT}(2n+1)\,\sigma_{s,n}^{g'\to g}(r,z)\sum_{K=0}^{n}R_n^K(\mu_m,\varphi_m)\Phi_n^{K,g'}(r,z)+\chi_g\sum_{g'=1}^{G}\nu\sigma_f^{g'}(r,z)\Phi_0^{0,g'}(r,z)+$$
$$+\sum_{n=0}^{IQAN}(2n+1)\sum_{K=0}^{n}R_n^K(\mu_m,\varphi_m)Q_n^{K,g}(r,z)\ , \tag{5}$$

where associated Legendre polynomials R_n^K are defined by

$$R_n^K(\mu_m,\varphi_m)=\left[\frac{(2-\delta_{K0})(n-K)!}{(n+K)!}\right]^{\frac{1}{2}}P_n^K(\mu_m)\cos K\varphi_m\ , \tag{6}$$

$$\varphi_m=\begin{cases}\tan^{-1}[(1-\mu_m^2-\eta_m^2)^{\frac{1}{2}}/\eta_m]\ , & \text{for } \eta_m>0\ ,\\ \tan^{-1}[(1-\mu_m^2-\eta_m^2)^{\frac{1}{2}}/\eta_m]+\pi, & \text{for } \eta_m<0\ .\end{cases}$$

Furthermore, the functions $\Phi_n^{K,g'}(r,z)$ are defined by

$$\Phi_n^{K,g'}(r,z)=\int_{-1}^{1}d\mu\int_0^{\pi}d\varphi\,R_n^K(\mu,\varphi)\psi^{g'}(r,z,\mu,\varphi)/2\pi=$$
$$=\frac{1}{2\pi}\sum_{m'=1}^{MT}W_mR_n^K(\mu_{m'},\varphi_{m'})\left[\sum_{l=1}^{NL}\psi_m^{g'(l)}L^{(l)}(r,z)\right]\ . \tag{7}$$

In Eq.(4), we used NL Lagrange polynomials in each subregions whose shape in (r,z) plane is rectangular. There is a great choice of the degree of Lagrange polynomials (say N) and the number of mesh points on the rectangle with a point at each vertex. For N=2, we may choose eight points or nine points as shown in Fig.1. Bilinear, cubic and biquadratic polynomials correspond to these cases with four, eight and nine points, respectively.

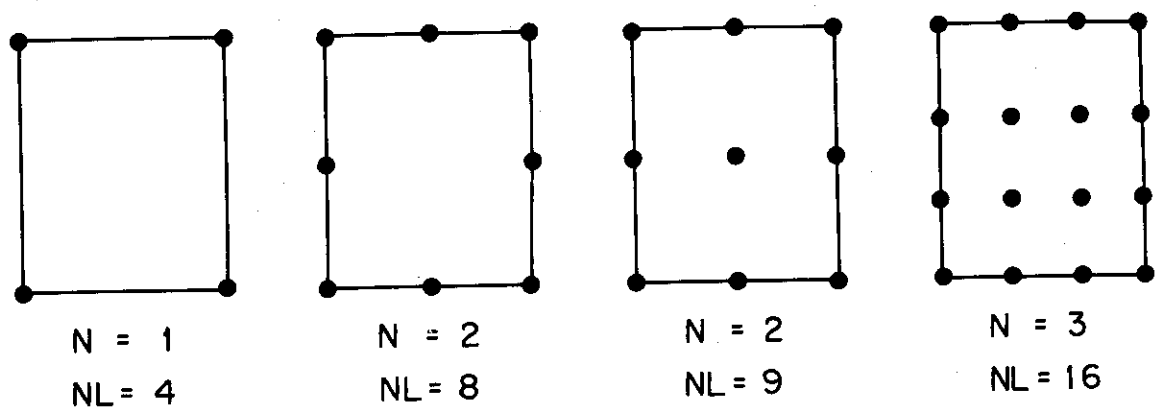

Fig. 1 Mesh point arrangement of a unit rectangle for a few low-order Lagrange polynomials.

The formulation becomes more complicated for nine points than for eight points, but the eight-points-algorithm may fail to simulate angular flux distributions with a steep peak near a center of the rectangle. Once the formulation has been performed, these two cases do not differ much in a labor needed for programming the algorithms. From these points of view, the nine-point-algorithm may be preferable to the eight points.

3. Solution Algorithms

We describe here the solution algorithms for solving Eq.(4), in which the source term $S_m^g(r,z)$ is assumed to be given. In the beginning, we define the inner products in (r,z) geometry by

$$\langle f, g \rangle = \langle f(r,z), g(r,z) \rangle \equiv \iint_{V_{ij}} r f(r,z) g(r,z) \, dr \, dz \ , \tag{8}$$

where Vij stands for the rectangular cell which is bounded by the i-th interval in r-direction and the j-th interval in z-direction, i.e.,

$$V_{ij} = \left\{ (r,z) \middle| \ r_{i-1} \leq r \leq r_i \ , \ z_{j-1} \leq z \leq z_j \right\} .$$

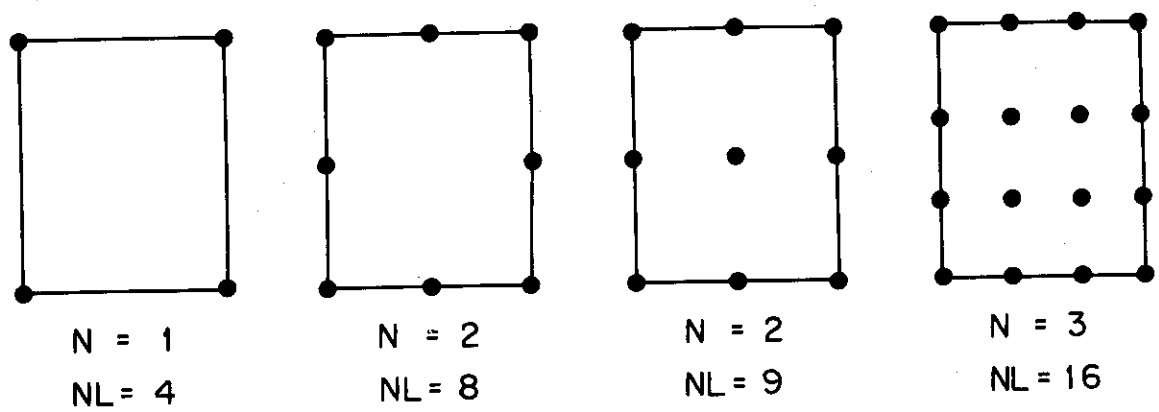

Fig. 1 Mesh point arrangement of a unit rectangle for a few low-order Lagrange polynomials.

The formulation becomes more complicated for nine points than for eight points, but the eight-points-algorithm may fail to simulate angular flux distributions with a steep peak near a center of the rectangle. Once the formulation has been performed, these two cases do not differ much in a labor needed for programming the algorithms. From these points of view, the nine-point-algorithm may be preferable to the eight points.

3. Solution Algorithms

We describe here the solution algorithms for solving Eq.(4), in which the source term $S_m^g(r,z)$ is assumed to be given. In the beginning, we define the inner products in (r,z) geometry by

$$\langle f, g \rangle = \langle f(r,z), g(r,z) \rangle \equiv \iint_{V_{ij}} r f(r,z) g(r,z) \, dr \, dz \ , \tag{8}$$

where Vij stands for the rectangular cell which is bounded by the i-th interval in r-direction and the j-th interval in z-direction, i.e.,

$$V_{ij} = \left\{ (r,z) \middle| \ r_{i-1} \leq r \leq r_i \ , \ z_{j-1} \leq z \leq z_j \right\} .$$

According to this definition, we prepare several kinds of the inner product of Lagrange polynomials which we will use later. The explicit expressions of the inner products defined below are given in Appendix III.

$$\langle 1, L^{(K)}(r,z)\rangle = \iint_{V_{ij}} r L^{(K)}(r,z)\,dr\,dz\ , \tag{9-1}$$

$$\langle 1, \frac{1}{r} L^{(K)}(r,z)\rangle = \iint_{V_{ij}} L^{(K)}(r,z)\,dr\,dz\ , \tag{9-2}$$

$$\langle L^{(K')}(r,z), L^{(K)}(r,z)\rangle = \iint_{V_{ij}} r L^{(K')}(r,z) L^{(K)}(r,z)\,dr\,dz\ , \tag{9-3}$$

$$\langle L^{(K')}(r,z), \frac{1}{r} L^{(K)}(r,z)\rangle = \iint_{V_{ij}} L^{(K')}(r,z) L^{(K)}(r,z)\,dr\,dz\ , \tag{9-4}$$

$$\langle L^{(K')}(r,z), \frac{\partial}{\partial r} L^{(K)}(r,z)\rangle = \iint_{V_{ij}} r L^{(K')}(r,z) \left(\frac{\partial}{\partial r} L^{(K)}(r,z)\right) dr\,dz\ , \tag{9-5}$$

$$\langle L^{(K')}(r,z), \frac{\partial}{\partial z} L^{(K)}(r,z)\rangle = \iint_{V_{ij}} r L^{(K')}(r,z) \left(\frac{\partial}{\partial z} L^{(K)}(r,z)\right) dr\,dz\ , \tag{9-6}$$

$$K', K = 1, 2, \cdots, NL\ .$$

Additional four special expressions which turn out the line integration are defined as follows:

$$\langle 1, \frac{1}{r} L^{(K)}(r_0,z)\rangle = \int_{z_{j-1}}^{z_j} L^{(K)}(r,z)\,dz\ , \tag{9-7}$$

$$\langle 1, L^{(K)}(r,z_0)\rangle = \int_{r_{i-1}}^{r_i} r L^{(K)}(r,z_0)\,dr\ , \tag{9-8}$$

$$\langle L^{(K')}(r_0,z), \frac{1}{r} L^{(K)}(r_0,z)\rangle = \int_{z_{j-1}}^{z_j} L^{(K')}(r_0,z) L^{(K)}(r_0,z)\,dz\ , \tag{9-9}$$

$$\langle L^{(K')}(r,z_0), L^{(K)}(r,z_0)\rangle = \int_{r_{i-1}}^{r_i} r L^{(K')}(r,z_0) L^{(K)}(r,z_0)\,dr\ , \tag{9-10}$$

where r_o and Z_o take some fixed values.

As illustrated in Appendix III, if the symmetric properties of these inner products are taken into consideration, the number of (k',k) combinations, for which the explicit expressions must be prepared, can be reduced considerably.

3.1 Continuous Method

We develop here the solution algorithm which gives the continuous solution to Eq.(4) over the whole system. Now we define a residual $R_m^g(r,z)$ as follows;

$$R_m^g(r,z) = W_m \mu_m \sum_{l=1}^{NL} \psi_m^{g(l)} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r L^{(l)}(r,z) \right) \right] + W_m \eta_m \sum_{l=1}^{NL} \psi_m^{g(l)} \left[\frac{\partial L^{(l)}(r,z)}{\partial z} \right] +$$
$$+ \sum_{l=1}^{NL} \left(\alpha_{m+1/2} \psi_{m+1/2}^{g(l)} - \alpha_{m-1/2} \psi_{m-1/2}^{g(l)} \right) \left[\frac{1}{r} L^{(l)}(r,z) \right] + \sigma_t^g W_m \sum_{l=1}^{NL} \psi_m^{g(l)} \left[L^{(l)}(r,z) \right] -$$
$$- W_m S_m^g(r,z) \ , \qquad (r,z) \in V_{ij} \ , \tag{10}$$

and we seek the solution which satisfies the relations

$$\langle W^{(k')}(r,z) , R_m^g(r,z) \rangle = 0 \qquad (k' = 1, 2, \cdots, K) \ , \tag{11}$$

where $W(r,z)$ are appropriate weight functions which are chosen to be linearly independent low order polynomials in r and Z, and K is so determind as to accord with the number of unknowns. We can develop different algorithms depending upon the selection of the practical form of the weight functions. Substitution of Eq.(10) into Eq.(11) gives

$$W_m \mu_m \sum_{l=1}^{NL} \psi_m^{g(l)} \left\langle W^{(K')}(r,z), \frac{1}{r}\frac{\partial}{\partial r}\left(r L^{(l)}(r,z)\right)\right\rangle + W_m \eta_m \sum_{l=1}^{NL} \psi_m^{g(l)} \left\langle W^{(K')}(r,z), \frac{\partial}{\partial z} L^{(l)}(r,z)\right\rangle +$$
$$+ \sum_{l=1}^{NL} \left(\alpha_{m+1/2}\psi_{m+1/2}^{g(l)} - \alpha_{m-1/2}\psi_{m-1/2}^{g(l)}\right)\left\langle W^{(K')}(r,z), \frac{1}{r} L^{(l)}(r,z)\right\rangle + \sigma_t^g W_m \sum_{l=1}^{NL} \psi_m^{g(l)} \left\langle W^{(K')}(r,z), L^{(l)}(r,z)\right\rangle -$$
$$- W_m \left\langle W^{(K')}(r,z), S_m^g(r,z)\right\rangle = 0 \ , \qquad (r,z) \in V_{ij} \ , \ (K' = 1,2,\cdots,K). \qquad (12)$$

Note that we assumed here $\sigma_t^g(r,z)$ are constant in each subregion. For the calculation of the source term $S_m^g(r,z)$ from Eq.(5), we assume that also $\sigma_{s,n}^{g'\to g}(r,z)$, $\nu\sigma_f^g(r,z)$ and $Q_n^{K,g}(r,z)$ are constant in each subregion. The inner product contained in the first term of Eq.(12) can be rewritten as the sum of two inner products:

$$\left\langle W^{(K')}(r,z), \frac{1}{r}\frac{\partial}{\partial r}\left(r L^{(l)}(r,z)\right)\right\rangle = \left\langle W^{(K')}(r,z), \frac{1}{r} L^{(l)}(r,z)\right\rangle + \left\langle W^{(K')}(r,z), \frac{\partial}{\partial r} L^{(l)}(r,z)\right\rangle. \qquad (13)$$

3.1.1 Continuous Method using Balance Equation

To solve Eq.(12), we can choose a constant function or zero-degree polynomial as the weight function. Putting K=1 and W(r,z)=1 over the whole system, we can write a single equation which is equivalent to the balance equation obtained by integrating Eq.(4) over each subregion. The inner products in the first and second term of Eq.(12) in this case are given by

$$\left\langle 1, \frac{1}{r} L^{(\ell)}(r,z) \right\rangle + \left\langle 1, \frac{\partial}{\partial r} L^{(\ell)}(r,z) \right\rangle \quad \text{and} \quad \left\langle 1, \frac{\partial}{\partial z} L^{(\ell)}(r,z) \right\rangle ,$$

respectively. These types of the inner product however, have not been prepared in Eq.(9), so we rewrite them with expressions used in Eqs.(9-7) and (9-8), i.e.,

$$\left\langle 1, \frac{1}{r}\frac{\partial}{\partial r}\left(r L^{(\ell)}(r,z)\right) \right\rangle = r_i \left\langle 1, \frac{1}{r} L^{(\ell)}(r_i,z) \right\rangle - r_{i-1} \left\langle 1, \frac{1}{r} L^{(\ell)}(r_{i-1},z) \right\rangle ,$$

$$\left\langle 1, \frac{\partial}{\partial z} L^{(\ell)}(r,z) \right\rangle = \left\langle 1, L^{(\ell)}(r,z_j) \right\rangle - \left\langle 1, L^{(\ell)}(r,z_{j-1}) \right\rangle .$$

Now, we can write the full expression of Eq.(12) explicitly;

$$\begin{aligned}
& W_m \mu_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[r_i \left\langle 1, \frac{1}{r} L^{(\ell)}(r_i,z) \right\rangle - r_{i-1} \left\langle 1, \frac{1}{r} L^{(\ell)}(r_{i-1},z) \right\rangle \right] + \\
& + W_m \eta_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[\left\langle 1, L^{(\ell)}(r,z_j) \right\rangle - \left\langle 1, L^{(\ell)}(r,z_{j-1}) \right\rangle \right] + \\
& + \sum_{\ell=1}^{NL} \left(\alpha_{m+\frac{1}{2}} \psi_{m+\frac{1}{2}}^{g(\ell)} - \alpha_{m-\frac{1}{2}} \psi_{m-\frac{1}{2}}^{g(\ell)} \right) \left\langle 1, \frac{1}{r} L^{(\ell)}(r,z) \right\rangle + \sigma_t^g W_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left\langle 1, L^{(\ell)}(r,z) \right\rangle - \\
& - W_m \left\langle 1, S_m^g(r,z) \right\rangle = 0 \quad . \qquad (14)
\end{aligned}$$

If we use here the "diamond difference" scheme for the angular variable;

$$\psi_{m+\frac{1}{2}}^{g(\ell)} = 2\psi_m^{g(\ell)} - \psi_{m-\frac{1}{2}}^{g(\ell)} \quad , \tag{15}$$

then the substitution of the relation into Eq.(14) gives

$$\begin{aligned} & W_m \mu_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[r_i \left\langle 1, \frac{1}{r} L^{(\ell)}(r_i, z) \right\rangle - r_{i-1} \left\langle 1, \frac{1}{r} L^{(\ell)}(r_{i-1}, z) \right\rangle \right] + \\ & + W_m \eta_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[\left\langle 1, L^{(\ell)}(r, z_j) \right\rangle - \left\langle 1, L^{(\ell)}(r, z_{j-1}) \right\rangle \right. + \\ & + 2\alpha_{m+\frac{1}{2}} \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left\langle 1, \frac{1}{r} L^{(\ell)}(r, z) \right\rangle + \sigma_t^g W_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left\langle 1, L^{(\ell)}(r, z) \right\rangle = \\ & = \left(\alpha_{m+\frac{1}{2}} + \alpha_{m-\frac{1}{2}} \right) \sum_{\ell=1}^{NL} \psi_{m-\frac{1}{2}}^{g(\ell)} \left\langle 1, \frac{1}{r} L^{(\ell)}(r, z) \right\rangle + W_m \left\langle 1, S_m^g(r, z) \right\rangle \quad , \end{aligned} \tag{16}$$

where $\psi_{m-\frac{1}{2}}^{g(\ell)}$'s are assumed to be known.

Eq.(16) can be solved only when N=1 (NL=4) because in this case a single unknown is remained to be determined for each sweep of the space-angle mesh (Fig.2).

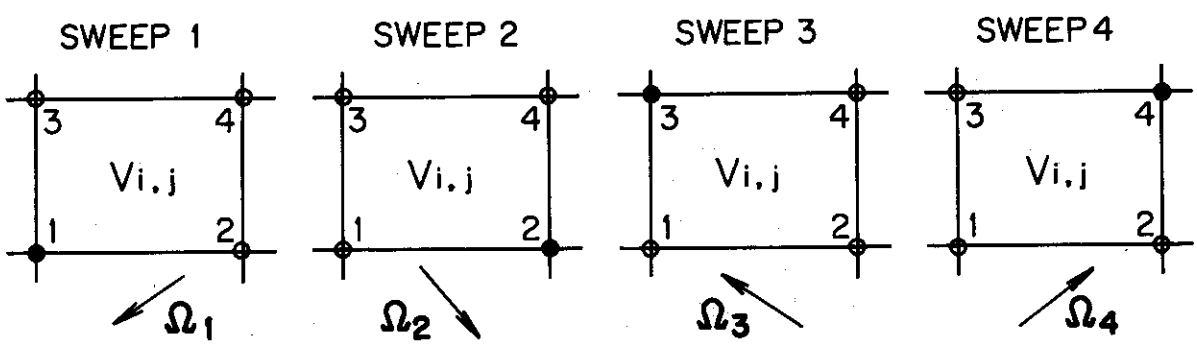

Fig. 2 Unknown to be determined for each sweep for N=1; a black point indicates the location with an unknown coefficient.

In the case of the sweep k (k=1,2,3,4), the unknown coefficient is $\psi_m^{g(K)}$ which is calculated from

$$\psi_m^{g(K)} = X_{m,g}^{(K)} / Y_{m,g}^{(K)} \quad ,$$

$$X_{m,g}^{(K)} = -W_m \mu_m \sum_{\ell \neq K}^{NL} \psi_m^{g(\ell)} \left[r_i \left\langle 1, \frac{1}{r} L^{(\ell)}(r_i, z) \right\rangle - r_{i-1} \left\langle 1, \frac{1}{r} L^{(\ell)}(r_{i-1}, z) \right\rangle \right] -$$

$$- W_m \eta_m \sum_{\ell \neq K}^{NL} \psi_m^{g(\ell)} \left[\left\langle 1, L^{(\ell)}(r, z_j) \right\rangle - \left\langle 1, L^{(\ell)}(r, z_{j-1}) \right\rangle \right] -$$

$$-2\alpha_{m+1/2} \sum_{\ell \neq K}^{NL} \psi_m^{g(\ell)} \left\langle 1, \frac{1}{r} L^{(\ell)}(r, z) \right\rangle - \sigma_t^g W_m \sum_{\ell \neq K}^{NL} \psi_m^{g(\ell)} \left\langle 1, L^{(\ell)}(r, z) \right\rangle +$$

$$+ (\alpha_{m+1/2} + \alpha_{m-1/2}) \sum_{\ell \neq K}^{NL} \psi_{m-1/2}^{g(\ell)} \left\langle 1, \frac{1}{r} L^{(\ell)}(r, z) \right\rangle + W_m \left\langle 1, S_m^g(r, z) \right\rangle \quad ,$$

$$Y_{m,g}^{(K)} = W_m \mu_m \left[r_i \left\langle 1, \frac{1}{r} L^{(K)}(r_i, z) \right\rangle - r_{i-1} \left\langle 1, \frac{1}{r} L^{(K)}(r_{i-1}, z) \right\rangle \right] +$$

$$+ W_m \eta_m \left[\left\langle 1, L^{(K)}(r, z_j) \right\rangle - \left\langle 1, L^{(K)}(r, z_{j-1}) \right\rangle \right] + 2\alpha_{m+1/2} \left\langle 1, \frac{1}{r} L^{(K)}(r, z) \right\rangle +$$

$$+ \sigma_t^g W_m \left\langle 1, L^{(K)}(r, z) \right\rangle \quad , \qquad (r, z) \in V_{ij} \ , \ (g = 1 \sim G \ , \ m = 1 \sim MT) \ . \qquad (17)$$

Using $\psi_m^{g(K)}$ given above, we can obtain $\psi_{m+1/2}^{g(K)}$ from Eq.(15). The term $\langle 1, S_m^g(r,z) \rangle$ which is contained in $X_{m,g}^{(K)}$ is calculated from Eqs.(5), (6) and (7). Refering to Eq.(7), we can write

$$\left\langle 1, \Phi_n^{K,g'}(r, z) \right\rangle = \frac{1}{2\pi} \sum_{m'=1}^{MT} W_{m'} R_n^K(\mu_{m'}, \varphi_{m'}) \sum_{\ell=1}^{NL} \psi_m^{g'(\ell)} \left\langle 1, L^{(\ell)}(r, z) \right\rangle$$

The continuous method described here may be the simplest one. Some other algorithms of the continuous method will be discussed in the following section.

3.1.2 Continuous Method with Galerkin-type Scheme

We return to Eq.(12) again. If we do not use the diamond difference equation (15), the number of unknowns contained in Eq.(12) is two for NL=4 (N=1), six for NL=8 (N=2) and eight for NL=9 (N=2). Consider the direction of the sweep 1, for example. The unknown coefficients to be determined are $\psi_m^{g(1)}$ and $\psi_{m+1/2}^{g(1)}$ for N=1, $\psi_m^{g(\ell')}$ and $\psi_{m+1/2}^{g(\ell')}$ (ℓ'=1,2,4) for NL=8, and $\psi_m^{g(\ell')}$ and $\psi_{m+1/2}^{g(\ell')}$ (ℓ'=1,2,4,5) for NL=9 (see Fig.3). The other coefficients can be assumed to be known from the previous step of the sweep over the space-angle mesh.

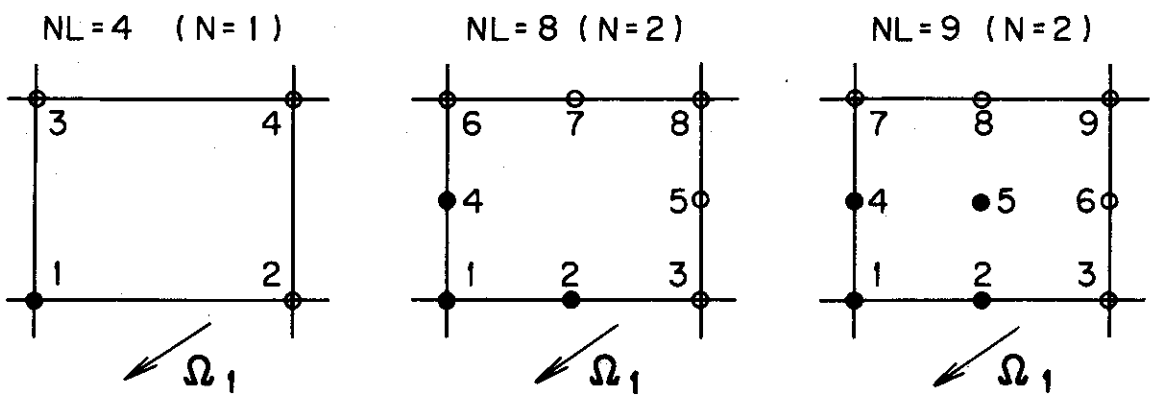

Fig. 3 Unknowns to be determined for the direction of the sweep 1; black points indicate the location with unknown coefficients.

To determine these unknowns, the appropriate number of the weight functions is selected probably to be low-order polynomials in r and z. In this case Eq.(12) are represented as a linear system of algebraic equations whose order is the same as the number of the unknowns. Next we show a little different approach to the determination of the unknowns. Let us consider the case of NL=9.

The number of the unknowns is eight as mentioned above. If we use Lagrange polynomials themselves as the weight functions (there are nine Lagrange polynomials in this case), then Eq.(12) for the sweep 1 can be expressed as follows;

$$\sum_{l=1,2,4,5} \psi_m^{g(l)} \Big[W_m \mu_m \Big\langle L^{(k')}(r,z), \frac{1}{r}\frac{\partial}{\partial r}\big(r L^{(l)}(r,z)\big) \Big\rangle + W_m \eta_m \Big\langle L^{(k')}(r,z), \frac{\partial}{\partial z} L^{(l)}(r,z) \Big\rangle + \sigma_t^g W_m \Big\langle L^{(k')}(r,z), L^{(l)}(r,z) \Big\rangle \Big] + \alpha_{m+1/2} \sum_{l=1,2,4,5} \psi_{m+1/2}^{g(l)} \Big\langle L^{(k')}(r,z), \frac{1}{r} L^{(l)}(r,z) \Big\rangle =$$

$$= W_m \Big\langle L^{(k')}(r,z), S_m^g(r,z) \Big\rangle + \alpha_{m-1/2} \sum_{l=1}^{9} \psi_{m-1/2}^{g(l)} \Big\langle L^{(k')}(r,z), \frac{1}{r} L^{(l)}(r,z) \Big\rangle -$$

$$- \sum_{l=3,6,7,8,9} \psi_m^{g(l)} \Big[W_m \mu_m \Big\langle L^{(k')}(r,z), \frac{1}{r}\frac{\partial}{\partial r}\big(r L^{(l)}(r,z)\big) \Big\rangle + W_m \eta_m \Big\langle L^{(k')}(r,z), \frac{\partial}{\partial z} L^{(l)}(r,z) \Big\rangle + \sigma_t^g W_m \Big\langle L^{(k')}(r,z), L^{(l)}(r,z) \Big\rangle \Big] - \alpha_{m+1/2} \sum_{l=3,6,7,8,9} \psi_{m+1/2}^{g(l)} \Big\langle L^{(k')}(r,z), \frac{1}{r} L^{(l)}(r,z) \Big\rangle, \quad (k' = 1 \sim 9). \tag{18}$$

In Eq.(18), the number of equations is larger than the number of unknowns by one. To take this redundancy away, the least squares method may be applied. By making use of the matrix representation, Eq.(18) can be rewritten as

$$A \vec{\Psi} = \vec{b} \ , \tag{19-1}$$

where A is a rectangular matrix of order 9x8, $\vec{\Psi}$ and $\vec{b}$ are the column vectors of order 8 and 9, respectively. If we write

$$\vec{\Psi} = \Big(\psi_m^{g(1)}, \psi_m^{g(2)}, \psi_m^{g(4)}, \psi_m^{g(5)}, \psi_{m+1/2}^{g(1)}, \psi_{m+1/2}^{g(2)}, \psi_{m+1/2}^{g(4)}, \psi_{m+1/2}^{g(5)} \Big)^T \ , \tag{19-2}$$

then the elements a_{ij} of A and b_i of $\vec{b}$ are easily defined by making comparison between Eqs.(18) and (19). The vector $\vec{\Psi}$ can now be obtained in a sense of the least squares method;

$$B \vec{\Psi} = \vec{C} \ , \tag{20}$$

where $B=A^T A$ is a square matrix of order 8 and $\vec{\mathfrak{C}}=A\vec{\mathfrak{b}}$ is a column vector of order 8, where A^T means transpose matrix of A.

3.2 Discontinuous Method with Galerkin-type Scheme

In this section, we describe another method to obtain the solution to Eq.(4). This method can be compared with the discontinuous method formulated in TRIPLET. We first redefine the residual $\mathbf{R}_m^q(r,z)$ in a form different from Eq.(10) on the rectangular V_{ij}. To take account of the possible discontinuity of the flux along the boundaries of V_{ij}, we assume $\mathfrak{g}(r_i,z)$, $\mathfrak{f}(r,z_j)$, $\mathfrak{e}(r_{i-1},z)$ $\mathfrak{h}(r,z_{j-1})$ to be the flux at the left boundary of $V_{i+1,j}$, bottom boundary of $V_{i,j+1}$, right boundary of $V_{i-1,j}$ and top boundary of $V_{i,j-1}$, respectively (see Fig.4).

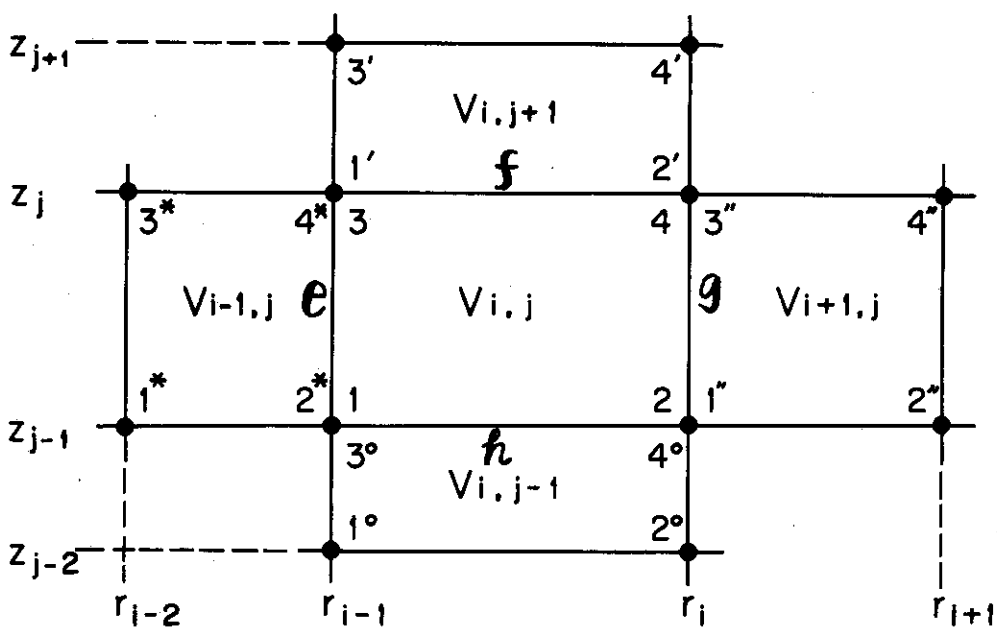

Fig. 4 Allocation of the fluxes on the boundaries and the numbering of mesh points (N=1).

Note that $g(r_i, z)$ is the flux calculated on the rectangular $V_{i+1,j}$, $f(r, z_j)$ is the flux calculated on the rectangular $V_{i,j+1}$ and so forth. For each direction of the sweep, only two of them are assumed to be known. Now we define the residual by

$$
\begin{aligned}
R_m^g(r,z) = {} & W_m \mu_m \Big[\sum_{l=1}^{NL} \psi_m^{g(l)} L^{(l)}(r,z) - g_m^g(r,z) \Big] \delta(r-r_i) \big[\delta_{d1} + \delta_{d3} \big] + \\
& + W_m \eta_m \Big[\sum_{l=1}^{NL} \psi_m^{g(l)} L^{(l)}(r,z) - f_m^g(r,z) \Big] \delta(z-z_j) \big[\delta_{d1} + \delta_{d2} \big] + \\
& + W_m \mu_m \Big[\sum_{l=1}^{NL} \psi_m^{g(l)} L^{(l)}(r,z) - e_m^g(r,z) \Big] \delta(r-r_{i-1}) \big[\delta_{d2} + \delta_{d4} \big] + \\
& + W_m \eta_m \Big[\sum_{l=1}^{NL} \psi_m^{g(l)} L^{(l)}(r,z) - h_m^g(r,z) \Big] \delta(z-z_{j-1}) \big[\delta_{d3} + \delta_{d4} \big] + \\
& + W_m \mu_m \sum_{l=1}^{NL} \psi_m^{g(l)} \Big[\frac{1}{r} \frac{\partial}{\partial r} \big(r L^{(l)}(r,z) \big) \Big] + W_m \eta_m \sum_{l=1}^{NL} \psi_m^{g(l)} \Big[\frac{\partial}{\partial z} L^{(l)}(r,z) \Big] + \\
& + \sum_{l=1}^{NL} \Big[2\alpha_{m+\frac{1}{2}} \psi_m^{g(l)} - (\alpha_{m+\frac{1}{2}} + \alpha_{m-\frac{1}{2}}) \psi_{m-\frac{1}{2}}^{g(l)} \Big] \Big[\frac{1}{r} L^{(l)}(r,z) \Big] + \\
& + \sigma_t^g W_m \sum_{l=1}^{NL} \psi_m^{g(l)} L^{(l)}(r,z) - W_m S_m^g(r,z), \quad (g = 1 \sim G,\ m = 1 \sim MT),
\end{aligned}
\tag{21}
$$

where $\delta(r-r_i)$ and $\delta(z-z_j)$ are the Dirac delta functions and δ_{dp} (p=1,2,3,4) are the Kronecker deltas in which d takes the value 1,2,3 or 4 corresponding to the direction of the sweeps, i.e.,

$$
d = \begin{cases} 1 & \text{for sweep } 1, \\ 2 & \text{" \quad " } 2, \\ 3 & \text{" \quad " } 3, \\ 4 & \text{" \quad " } 4. \end{cases}
$$

Some remarks on the discontinuities introduced above are pertinent here. According to the numerical experiments by Reed et al[6], the discontinuous finite element method has been shown to yield an accurate solution of the discrete ordinate equation and to be much more stable than the continuous method. Mathematical foundations of these experimental facts have not been given yet.

For proceeding the formulation, it should be noted that the functions $L^{(\ell)}(r_i, z)$, $L^{(\ell)}(r, z_j)$, $L^{(\ell)}(r_{i-1}, z)$ and $L^{(\ell)}(r, z_{j-1})$ are identically zero unless the ℓ-th point is located on the line $r=r_i$, $z=z_j$, $r=r_{i-1}$ and $z=z_{j-1}$, respectively. More explicitly (refer to Fig.4),

$$\begin{cases} L^{(1)}(r_i, z) = L^{(3)}(r_i, z) = L^{(2')}(r_i, z) = L^{(4')}(r_i, z) = 0 \ , \\ L^{(1)}(r, z_j) = L^{(2)}(r, z_j) = L^{(3')}(r, z_j) = L^{(4')}(r, z_j) = 0 \ , \\ L^{(2)}(r_{i-1}, z) = L^{(4)}(r_{i-1}, z) = L^{(1')}(r_{i-1}, z) = L^{(3')}(r_{i-1}, z) = 0 \ , \\ L^{(3)}(r, z_{j-1}) = L^{(4)}(r, z_{j-1}) = L^{(1')}(r, z_{j-1}) = L^{(2')}(r, z_{j-1}) = 0 \ . \end{cases}$$

Taking account of this situation, the first term of Eq.(21), for example, can be written as follows;

$$W_m \mu_m \Big[\sum_{\ell=2,4} \psi_m^{g(\ell)} L^{(\ell)}(r, z) - \sum_{\ell'=1',3'} \psi_m^{g(\ell')} L^{(\ell')}(r, z) \Big] \delta(r - r_i) \big[\delta_{d1} + \delta_{d3} \big] \ .$$

For the residual defined by Eq.(21), we insist that the solution satisfies Eq.(11) in which Lagrange polynomials take the place of the weight functions $W^{(K')}(r, z)$ (Galerkin method):

$$\Big\langle L^{(\ell')}(r, z), R_m^g(r, z) \Big\rangle = 0 \quad , \qquad (\ell' = 1, 2, \cdots, NL) \ . \tag{22}$$

From Eqs.(13), (21) and (22), the equation to be solved is now given by

$$T^{N}_{m,g} + W_m \mu_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left[\left\langle \overset{(\ell')}{L}(r,z), \frac{1}{r} \overset{(\ell)}{L}(r,z) \right\rangle + \left\langle \overset{(\ell')}{L}(r,z), \frac{\partial}{\partial r} \overset{(\ell)}{L}(r,z) \right\rangle \right] +$$

$$+ W_m \eta_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left\langle \overset{(\ell')}{L}(r,z), \frac{\partial}{\partial z} \overset{(\ell)}{L}(r,z) \right\rangle + \sum_{\ell=1}^{NL} \left[2\alpha_{m+1/2} \psi_m^{g(\ell)} - (\alpha_{m+1/2} + \alpha_{m-1/2}) \psi_{m-1/2}^{g(\ell)} \right] \times$$

$$\times \left\langle \overset{(\ell')}{L}(r,z), \frac{1}{r} \overset{(\ell)}{L}(r,z) \right\rangle + \sigma_t^g W_m \sum_{\ell=1}^{NL} \psi_m^{g(\ell)} \left\langle \overset{(\ell')}{L}(r,z), \overset{(\ell)}{L}(r,z) \right\rangle - W_m \left\langle \overset{(\ell')}{L}(r,z), S_m^g(r,z) \right\rangle = 0,$$

$$(\ell' = 1, 2, \cdots, NL), \qquad (23)$$

where $T^{N}_{m,g}$ is the term resulting from the discontinuity along the boundary.

For N=1,

$$T^{1}_{m,g} = W_m \mu_m r_i \left[\sum_{K=2,4} \psi_m^{g(K)} \left\langle \overset{(\ell')}{L}(r_i,z), \frac{1}{r} \overset{(K)}{L}(r_i,z) \right\rangle - \sum_{K''=1'',3''} \psi_m^{g(K'')} \left\langle \overset{(\ell')}{L}(r_i,z), \frac{1}{r} \overset{(K'')}{L}(r_i,z) \right\rangle \right] (\delta_{d1} + \delta_{d3}) +$$

$$+ W_m \eta_m \left[\sum_{K=3,4} \psi_m^{g(K)} \left\langle \overset{(\ell')}{L}(r,z_j), \overset{(K)}{L}(r,z_j) \right\rangle - \sum_{K'=1',2'} \psi_m^{g(K')} \left\langle \overset{(\ell')}{L}(r,z_j), \overset{(K')}{L}(r,z_j) \right\rangle \right] (\delta_{d1} + \delta_{d2}) +$$

$$+ W_m \mu_m r_{i-1} \left[\sum_{K=1,3} \psi_m^{g(K)} \left\langle \overset{(\ell')}{L}(r_{i-1},z), \frac{1}{r} \overset{(K)}{L}(r_{i-1},z) \right\rangle - \sum_{K^*=2^*,4^*} \psi_m^{g(K^*)} \left\langle \overset{(\ell')}{L}(r_{i-1},z), \frac{1}{r} \overset{(K^*)}{L}(r_{i-1},z) \right\rangle \right] (\delta_{d2} + \delta_{d4}) +$$

$$+ W_m \eta_m \left[\sum_{K=1,2} \psi_m^{g(K)} \left\langle \overset{(\ell')}{L}(r,z_{j-1}), \overset{(K)}{L}(r,z_{j-1}) \right\rangle - \sum_{K^\circ=3^\circ,4^\circ} \psi_m^{g(K^\circ)} \left\langle \overset{(\ell')}{L}(r,z_{j-1}), \overset{(K^\circ)}{L}(r,z_{j-1}) \right\rangle \right] (\delta_{d3} + \delta_{d4}). \qquad (24)$$

Since, as readily seen, we have following relations;

$$\overset{(2)}{L}(r_i,z) = \overset{(1^*)}{L}(r_i,z), \quad \overset{(4)}{L}(r_i,z) = \overset{(3^*)}{L}(r_i,z),$$

$$\overset{(3)}{L}(r,z_j) = \overset{(1')}{L}(r,z_j), \quad \overset{(4)}{L}(r,z_j) = \overset{(2')}{L}(r,z_j),$$

$$\overset{(1)}{L}(r_{i-1},z) = \overset{(2^*)}{L}(r_{i-1},z), \quad \overset{(3)}{L}(r_{i-1},z) = \overset{(4^*)}{L}(r_{i-1},z),$$

$$\overset{(1)}{L}(r,z_{j-1}) = \overset{(3^\circ)}{L}(r,z_{j-1}), \quad \overset{(2)}{L}(r,z_{j-1}) = \overset{(4^\circ)}{L}(r,z_{j-1}),$$

all the inner products contained in Eq.(24) have already been defined by Eq.(9).

For N=2, we consider two cases depending upon the number of points located on the rectangle (see Fig.3).

In order to unify the expression $T^{2}_{m,g}$ for these two cases, we renumber the points in the eight points scheme in such a fashion as shown in Fig.5 instead of the numbering in Fig.3.

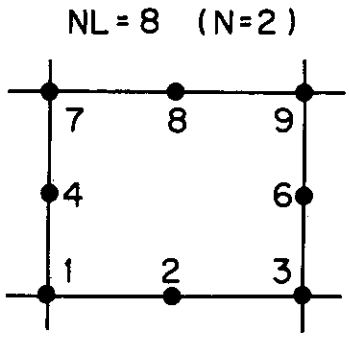

Fig. 5 Renumbering of the eight mesh points for N=2.

It should be noted that $T^{N}_{m,g}$ is not affected by the Lagrange polynomials originated at the interior points of the rectangle, because the value of these polynomials vanishes along the boundary. Now, we can write the expression for $T^{2}_{m,g}$ explicitly as follows;

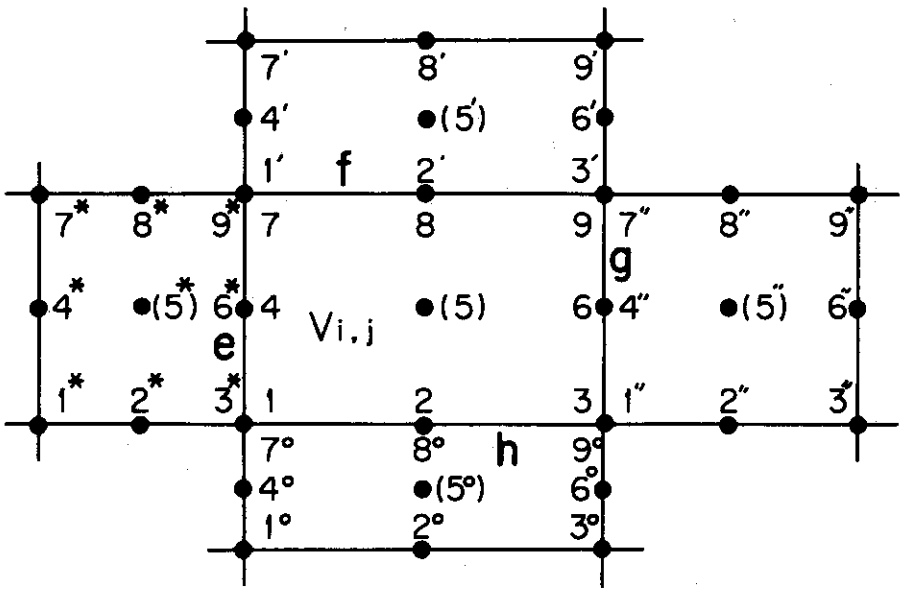

Fig. 6 Allocation of the fluxes on the boundaries and the numbering of mesh points (N=2).

$$
\begin{aligned}
T^{2}_{m,g} = {} & W_m \mu_m r_i \Big[\sum_{K=3,6,9} \psi_m^{g(K)} \Big\langle L^{(\ell')}(r_i, Z), \frac{1}{r} L^{(K)}(r_i, Z) \Big\rangle - \sum_{K^*=1^*,4^*,7^*} \psi_m^{g(K^*)} \Big\langle L^{(\ell')}(r_i, Z), \frac{1}{r} L^{(K^*)}(r_i, Z) \Big\rangle \Big] (\delta_{d1} + \delta_{d3}) + \\
& + W_m \eta_m \Big[\sum_{K=7,8,9} \psi_m^{g(K)} \Big\langle L^{(\ell')}(r, Z_j), L^{(K)}(r, Z_j) \Big\rangle - \sum_{K'=1',2',3'} \psi_m^{g(K')} \Big\langle L^{(\ell')}(r, Z_j), L^{(K')}(r, Z_j) \Big\rangle \Big] (\delta_{d1} + \delta_{d2}) + \\
& + W_m \mu_m r_{i-1} \Big[\sum_{K=1,4,7} \psi_m^{g(K)} \Big\langle L^{(\ell')}(r_{i-1}, Z), \frac{1}{r} L^{(K)}(r_{i-1}, Z) \Big\rangle - \sum_{K^*=3^*,6^*,9^*} \psi_m^{g(K^*)} \Big\langle L^{(\ell')}(r_{i-1}, Z), \frac{1}{r} L^{(K^*)}(r_{i-1}, Z) \Big\rangle \Big] (\delta_{d2} + \delta_{d4}) + \\
& + W_m \eta_m \Big[\sum_{K=1,2,3} \psi_m^{g(K)} \Big\langle L^{(\ell')}(r, Z_{j-1}), L^{(K)}(r, Z_{j-1}) \Big\rangle - \sum_{K^\circ=7^\circ,8^\circ,9^\circ} \psi_m^{g(K^\circ)} \Big\langle L^{(\ell')}(r, Z_{j-1}), L^{(K^\circ)}(r, Z_{j-1}) \Big\rangle (\delta_{d3} + \delta_{d4}) .
\end{aligned}
\tag{25}
$$

Thus for any N, the definition of $T^{N}_{m,g}$ is quite straightforward. Generally, Eq.(23) together with $T^{N}_{m,g}$ leads to NL linear algebraic equations for the coefficients $\psi_m^{g(\ell)}$. We write this system of equations as

$$A \vec{\Psi} = \vec{b} \tag{26}$$

where A is a square matrix of order NL, and $\vec{\Psi}$ and $\vec{b}$ are column vectors of order NL;

$$
\begin{aligned}
A &= (a_{IJ}^{m,g}) \quad , \\
\vec{\Psi} &= (\psi_m^{g(1)}, \psi_m^{g(2)}, \cdots, \psi_m^{g(NL)})^T , \\
\vec{b} &= (b_1^{m,g}, b_2^{m,g}, \cdots, b_{NL}^{m,g})^T .
\end{aligned}
$$

It must be emphasized here that the elements $a_{IJ}^{m,g}$ and $b_i^{m,g}$ take different values depending on the direction of the space-angle sweep because of the existence of Kronecker deltas δ_{dk} (k=1~4) in Eqs.(24) and (25).

We write here the expression of all elements explicitly only for N=1. The expressions for N=2 are presented in Appendix II. The types of the inner products used in the discontinuous method are those defined by Eqs. (9-1), (9-3), (9-4), (9-5), (9-6), (9-9) and (9-10). The explicit expressions for these inner products are given in Appendix III for N=1 and N=2.

By the use of the definitions;

$$\begin{cases} \tilde{a}_{IJ}^{m,g} = W_m \mu_m \left[\left\langle \overset{(I)}{L}(r,z), \frac{1}{r}\overset{(J)}{L}(r,z) \right\rangle + \left\langle \overset{(I)}{L}(r,z), \frac{\partial}{\partial r}\overset{(J)}{L}(r,z) \right\rangle \right] + W_m \eta_m \left\langle \overset{(I)}{L}(r,z), \frac{\partial}{\partial z}\overset{(J)}{L}(r,z) \right\rangle + \\ \qquad + 2\alpha_{m+1/2} \left\langle \overset{(I)}{L}(r,z), \frac{1}{r}\overset{(J)}{L}(r,z) \right\rangle + \sigma_t^g W_m \left\langle \overset{(I)}{L}(r,z), \overset{(J)}{L}(r,z) \right\rangle , \\ \tilde{b}_I^{m,g} = (\alpha_{m+1/2} + \alpha_{m-1/2}) \sum_{k=1}^{NL} \psi_{m-1/2}^{g(k)} \left\langle \overset{(I)}{L}(r,z), \frac{1}{r}\overset{(J)}{L}(r,z) \right\rangle + W_m \left\langle \overset{(I)}{L}(r,z), S_m^g(r,z) \right\rangle , \end{cases} \quad (27)$$

all elements of A and $\vec{b}$ for N=1 can now be written down for each direction of the sweep as follows.

(a) Sweep 1 (Ω_1)

$$a_{I1}^{m,g} = \tilde{a}_{I1}^{m,g} ,$$

$$a_{I2}^{m,g} = \tilde{a}_{I2}^{m,g} + W_m \mu_m r_i \sum_{K=2,4} \left\langle \overset{(K)}{L}(r_i,z), \frac{1}{r}\overset{(2)}{L}(r_i,z) \right\rangle \delta_{IK} ,$$

$$a_{I3}^{m,g} = \tilde{a}_{I3}^{m,g} + W_m \eta_m \sum_{K=3,4} \left\langle \overset{(K)}{L}(r,z_j), \overset{(3)}{L}(r,z_j) \right\rangle \delta_{IK} ,$$

$$a_{I4}^{m,g} = \tilde{a}_{I4}^{m,g} + W_m \mu_m r_i \sum_{K=2,4} \left\langle \overset{(K)}{L}(r_i,z), \frac{1}{r}\overset{(4)}{L}(r_i,z) \right\rangle \delta_{IK} + W_m \eta_m \sum_{K=3,4} \left\langle \overset{(K)}{L}(r,z_j), \overset{(4)}{L}(r,z_j) \right\rangle \delta_{IK} ,$$

$$b_I^{m,g} = \tilde{b}_I^{m,g} + W_m \mu_m r_i \sum_{K=2,4} \sum_{K''=1'',3''} \left\langle \overset{(K)}{L}(r_i,z), \frac{1}{r}\overset{(K'')}{L}(r_i,z) \right\rangle \psi_m^{g(K'')} \cdot \delta_{IK} +$$

$$+ W_m \eta_m \sum_{K=3,4} \sum_{K'=1',2'} \left\langle \overset{(K)}{L}(r,z_j), \overset{(K')}{L}(r,z_j) \right\rangle \psi_m^{g(K')} \cdot \delta_{IK} . \quad (28\text{-}1)$$

(b) Sweep 2 (Ω_2)

$$a_{I1}^{m,g} = \widetilde{a}_{I1}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,3} \left\langle L^{(K)}(r_{i-1},z), \frac{1}{r} L^{(1)}(r_{i-1},z) \right\rangle \delta_{IK} ,$$

$$a_{I2}^{m,g} = \widetilde{a}_{I2}^{m,g} ,$$

$$a_{I3}^{m,g} = \widetilde{a}_{I3}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,3} \left\langle L^{(K)}(r_{i-1},z), \frac{1}{r} L^{(3)}(r_{i-1},z) \right\rangle \delta_{IK} + W_m \eta_m \sum_{K=3,4} \left\langle L^{(K)}(r,z_j), L^{(3)}(r,z_j) \right\rangle \delta_{IK} ,$$

$$a_{I4}^{m,g} = \widetilde{a}_{I4}^{m,g} + W_m \eta_m \sum_{K=3,4} \left\langle L^{(K)}(r,z_j), L^{(4)}(r,z_j) \right\rangle \delta_{IK} ,$$

$$b_I^{m,g} = \widetilde{b}_I^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,3} \sum_{K^*=2^*,4^*} \left\langle L^{(K)}(r_{i-1},z), \frac{1}{r} L^{(K^*)}(r_{i-1},z) \right\rangle \psi_m^{g(K^*)} \cdot \delta_{IK} +$$

$$+ W_m \eta_m \sum_{K=3,4} \sum_{K'=1',2'} \left\langle L^{(K)}(r,z_j), L^{(K')}(r,z_j) \right\rangle \psi_m^{g(K')} \cdot \delta_{IK} . \qquad (28\text{-}2)$$

(c) Sweep 3 (Ω_3)

$$a_{I1}^{m,g} = \widetilde{a}_{I1}^{m,g} + W_m \eta_m \sum_{K=1,2} \left\langle L^{(K)}(r,z_{j-1}), L^{(1)}(r,z_{j-1}) \right\rangle \delta_{IK} ,$$

$$a_{I2}^{m,g} = \widetilde{a}_{I2}^{m,g} + W_m \mu_m r_i \sum_{K=2,4} \left\langle L^{(K)}(r_i,z), \frac{1}{r} L^{(2)}(r_i,z) \right\rangle \delta_{IK} + W_m \eta_m \sum_{K=1,2} \left\langle L^{(K)}(r,z_{j-1}), L^{(2)}(r,z_{j-1}) \right\rangle \delta_{IK} ,$$

$$a_{I3}^{m,g} = \widetilde{a}_{I3}^{m,g} ,$$

$$a_{I4}^{m,g} = \widetilde{a}_{I4}^{m,g} + W_m \mu_m r_i \sum_{K=2,4} \left\langle L^{(K)}(r_i,z), \frac{1}{r} L^{(4)}(r_i,z) \right\rangle \delta_{IK}$$

$$b_I^{m,g} = \widetilde{b}_I^{m,g} + W_m \mu_m r_i \sum_{K=2,4} \sum_{K''=1'',3''} \left\langle L^{(K)}(r_i,z), \frac{1}{r} L^{(K'')}(r_i,z) \right\rangle \psi_m^{g(K'')} \cdot \delta_{IK} +$$

$$+ W_m \eta_m \sum_{K=1,2} \sum_{K^\circ=3^\circ,4^\circ} \left\langle L^{(K)}(r,z_{j-1}), L^{(K^\circ)}(r,z_{j-1}) \right\rangle \psi_m^{g(K^\circ)} \cdot \delta_{IK} . \qquad (28\text{-}3)$$

(d) Sweep 4 (Ω_4)

$$a_{I1}^{m,g} = \widetilde{a}_{I1}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,3} \left\langle L^{(K)}(r_{i-1},z), \frac{1}{r} L^{(1)}(r_{i-1},z) \right\rangle \delta_{IK} + W_m \eta_m \sum_{K=1,2} \left\langle L^{(K)}(r,z_{j-1}), L^{(1)}(r,z_{j-1}) \right\rangle \delta_{IK} ,$$

$$a_{I2}^{m,g} = \widetilde{a}_{I2}^{m,g} + W_m \eta_m \sum_{K=1,2} \left\langle L^{(K)}(r,z_{j-1}), L^{(2)}(r,z_{j-1}) \right\rangle \delta_{IK} ,$$

$$a_{I3}^{m,g} = \widetilde{a}_{I3}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,3} \left\langle L^{(K)}(r_{i-1},z), \frac{1}{r} L^{(3)}(r_{i-1},z) \right\rangle \delta_{IK} ,$$

$$a_{I4}^{m,g} = \widetilde{a}_{I4}^{m,g} ,$$

$$b_I^{m,g} = \widetilde{b}_I^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,3} \sum_{K^*=2^*,4^*} \left\langle L^{(K)}(r_{i-1},z), \frac{1}{r} L^{(K^*)}(r_{i-1},z) \right\rangle \psi_m^{g(K^*)} \delta_{IK}$$

$$+ W_m \eta_m \sum_{K=1,2} \sum_{K^\circ=3^\circ,4^\circ} \left\langle L^{(K)}(r,z_{j-1}), L^{(K^\circ)}(r,z_{j-1}) \right\rangle \psi_m^{g(K^\circ)} \cdot \delta_{IK} . \qquad (28\text{-}4)$$

From Eqs.(5), (6) and (7), the second term in the definition of $\widetilde{b}_I^{m,g}$, Eq.(27), is calculated by

$$\left\langle L^{(I)}(r,z), S_m^g(r,z) \right\rangle = \sum_{g'=1}^{G} \sum_{n=0}^{ISCT} (2n+1) \sigma_{sn}^{g'\to g} \sum_{K=0}^{n} R_n^K(\mu_m,\varphi_m) \left\langle L^{(I)}(r,z), \Phi_n^{K,g'}(r,z) \right\rangle +$$

$$+ \chi_g \sum_{g'=1}^{G} \nu\sigma_f^{g'} \left\langle L^{(I)}(r,z), \Phi_0^{0,g'}(r,z) \right\rangle + \sum_{n=0}^{IQAN} (2n+1) \sum_{K=0}^{n} R_n^K(\mu_m,\varphi_m) Q_n^{K,g} \left\langle 1, L^{(I)}(r,z) \right\rangle , \qquad (29)$$

where

$$\left\langle L^{(I)}(r,z), \Phi_n^{K,g'}(r,z) \right\rangle = \frac{1}{2\pi} \sum_{m'=1}^{MT} W_{m'} R_n^K(\mu_{m'},\varphi_{m'}) \left[\sum_{\ell=1}^{NL} \psi_{m'}^{g'(\ell)} \left\langle L^{(I)}(r,z), L^{(\ell)}(r,z) \right\rangle \right] . \qquad (30)$$

4. Discussions

We have used a regular arrangement of the rectangular space subregions in our algorithm (regular in a sense that vertices of adjacent rectangles must coincide). More generally, it is also possible to use an irregular arrangement of the rectangular ones. However, the specification of irregular arrangement is much more complicated than that of regular one.

Another advantage of the regular rectangles over irregular ones is related to the discrete ordinate approximation applied to angular variables. For solving the discrete ordinate equation it is necessary to sweep the mesh in the direction of the neutron flight. There must be therefore a definite order of the rectangles, which is simply defined on the regular mesh but cannot be easily defined on the irregular one.

The triangularization of the system is generally used in two-dimensional (x,y) geometry. One of the advantages of triangular subregions is that one can simulate any complicated geometries. But the sweep through the triangular meshes is not simple even if the mesh is regular, because the orientations of the triangles to a direction of the sweep must be distinguished. On the rectanglar mesh, the orientations of the rectangles are irrelevant to the sweep directions. The fact makes the sweep on the rectangular mesh much simpler.

In two-dimensional cylindrical geometry, we rarely have to solve neutron transport problems with a complicated geometrical arrangement. Therefore, from the geometrical and practical points of view it is sufficient to use the rectangularization. Our algorithms can be said to have taken advantage of a higher order approximation.

We have not tried to make discussions on the merits and demerits coming from the choice of the weight functions used in our formulation for eliminating the residual. It is known that the Galerkin scheme is more general and straightforward. However, the lower order shemes may be convenient and accurate enough for practical purposes. It will be one of our further research items to analyse this problem.

We are now developing a new computer code based on our algorithms described in this article. Our efforts will be continued to develop our algorithm so that we shall be able to solve space dependent kinetics problems in two-dimensional cylindrical geometry.

5. Acknowledgements

We are much indebted to Dr. T. Asaoka for providing the opportunity and arrangement of the present work. Acknowledgements are also due to him for many valuable discussions. We are also thankful to Mr. K. Kobayashi for helpful discussions.

In two-dimensional cylindrical geometry, we rarely have to solve neutron transport problems with a complicated geometrical arrangement. Therefore, from the geometrical and practical points of view it is sufficient to use the rectangularization. Our algorithms can be said to have taken advantage of a higher order approximation.

We have not tried to make discussions on the merits and demerits coming from the choice of the weight functions used in our formulation for eliminating the residual. It is known that the Galerkin scheme is more general and straightforward. However, the lower order shemes may be convenient and accurate enough for practical purposes. It will be one of our further research items to analyse this problem.

We are now developing a new computer code based on our algorithms described in this article. Our efforts will be continued to develop our algorithm so that we shall be able to solve space dependent kinetics problems in two-dimensional cylindrical geometry.

5. Acknowledgements

We are much indebted to Dr. T. Asaoka for providing the opportunity and arrangement of the present work. Acknowledgements are also due to him for many valuable discussions. We are also thankful to Mr. K. Kobayashi for helpful discussions.

References

(1) Lathrop k.D., Brinkly F.W. : "TWOTRAN-II: An Interfaced, Exportable Version of the TWOTRAN Code for Two-Dimensional Transport", LA-4848-MS (July 1973)

(2) Lathrop K.D., Brinkly F.W. : "Theory and Use of the General Geometry TWOTRAN Program", LA-4432 (April 1970)

(3) Lathrop K.D. : "DTF-IV: A Fortran Program for Solving the Multigroup Transport Equation with Anisotropic Scattering", LA-3373 (November 1965)

(4) Saltesy R.G., Disney R.K. : "Revised WANL ANISN Program User's Mannual", WANL-TMI-1967 (April 1969)

(5) Mynatt F.R. : "A User's Manual for DOT", K-1694 (1967)

(6) Reed Wm.H., Hill T.R., Brinkly F.W., Lathrop K.D.: "TRIPLET: A Two-Dimensional, Multi-group, Triangular Mesh, Planar Geometry, Explicit Transport Code", LA-5428-MS (October 1973)

(7) Ohnishi T. : "Application of Finite Element Solution Technique to Neutron Diffusion and Transport Equations", Proceedings of Conference on New Developments in Reactor Mathematics and Applications. Idaho Falls, Idaho (March 1971)

(8) Chang Mu Kang, Hansen K.F. : "Finite Element Method for Space Time Reactor Analysis", MIT-3903-5, MITNE-135 (November 1971)

(9) Nakamura S., Ohnishi T. : "The Iterative Solutions of the Finite Element Method", Proceeding of a Seminar on Numerical Reactor Calculations, IAEA, Vienna, pp. 639-656 (1972)

(10) Kaper H.G., Leaf G.K., Lindeman A.J. : "Applications of Finite Element Methods in Reactor Mathematics. Numerical Solution of the Neutron Diffusion Equation", ANL-7925 (1972)

Appendix I Lagrange Polynomials

The explicit expressions for a few low-order Lagrange polynomials are listed below. As the order of the Lagrange polynomials become higher, it is not so easy to obtain the expressions of them intuitively. The systematical procedure to obtain the expressions is therefore also given in the following. In the beginning we give explicit expressions of the Lagrange polynomials for N=1 and N=2 (see Fig. A-1).

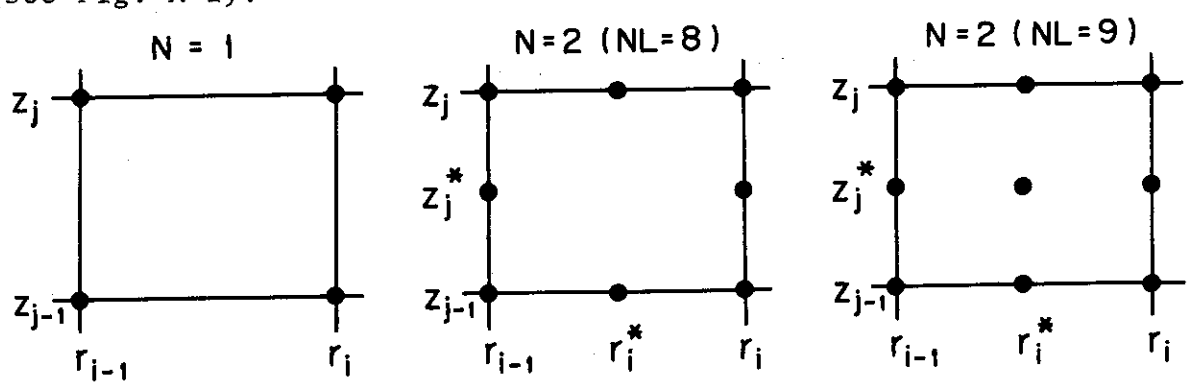

Fig. A-1 Arrangement of the mesh points on a unit rectangle.

(i) N=1 (NL=4)

$$L^{(1)}(r,z) = \frac{(r-r_i)(z-z_j)}{(r_{i-1}-r_i)(z_{j-1}-z_j)} \quad , \quad L^{(2)}(r,z) = \frac{(r-r_{i-1})(z-z_j)}{(r_i-r_{i-1})(z_{j-1}-z_j)} \quad ,$$

$$L^{(3)}(r,z) = \frac{(r-r_i)(z-z_{j-1})}{(r_{i-1}-r_i)(z_j-z_{j-1})} \quad , \quad L^{(4)}(r,z) = \frac{(r-r_{i-1})(z-z_{j-1})}{(r_i-r_{i-1})(z_j-z_{j-1})} \quad .$$

(ii) N=2 (NL=8)

We define $r_i^* = (r_{i-1} + r_i)/2$ and $z_j^* = (z_{j-1} + z_j)/2$. Note that the numbering of the points is in such a way as shown in Fig. A-1, instead of one in Fig. 3.

$$L^{(1)}(r,z) = \frac{(r-r_i)(z-z_j)}{(r_{i-1}-r_i)(z_{j-1}-z_j)}\left[\frac{r-r_i^*}{r_{i-1}-r_i^*} - \frac{z-z_{j-1}}{z_j^*-z_{j-1}}\right] , \quad L^{(2)}(r,z) = \frac{(r-r_{i-1})(r-r_i)(z-z_j)}{(r_i^*-r_{i-1})(r_i^*-r_i)(z_{j-1}-z_j)} ,$$

$$L^{(3)}(r,z) = \frac{(r-r_{i-1})(z-z_j)}{(r_i-r_{i-1})(z_{j-1}-z_j)}\left[\frac{r-r_i^*}{r_i-r_i^*} - \frac{z-z_{j-1}}{z_j^*-z_{j-1}}\right] , \quad L^{(4)}(r,z) = \frac{(r-r_i)(z-z_{j-1})(z-z_j)}{(r_{i-1}-r_i)(z_j^*-z_{j-1})(z_j^*-z_j)} ,$$

$$L^{(6)}(r,z) = \frac{(r-r_{i-1})(z-z_{j-1})(z-z_j)}{(r_i-r_{i-1})(z_j^*-z_{j-1})(z_j^*-z_j)} , \quad L^{(7)}(r,z) = \frac{(r-r_i)(z-z_{j-1})}{(r_{i-1}-r_i)(z_j-z_{j-1})}\left[\frac{r-r_i^*}{r_{i-1}-r_i^*} - \frac{z-z_j}{z_j^*-z_j}\right] ,$$

$$L^{(8)}(r,z) = \frac{(r-r_{i-1})(r-r_i)(z-z_{j-1})}{(r_i^*-r_{i-1})(r_i^*-r_i)(z_j-z_{j-1})} , \quad L^{(9)}(r,z) = \frac{(r-r_{i-1})(z-z_{j-1})}{(r_i-r_{i-1})(z_j-z_{j-1})}\left[\frac{r-r_i^*}{r_i-r_i^*} - \frac{z-z_j}{z_j^*-z_j}\right] .$$

(iii) N=2 (NL=9)

$$L^{(1)}(r,z) = \frac{(r-r_i^*)(r-r_i)(z-z_j^*)(z-z_j)}{(r_{i-1}-r_i^*)(r_{i-1}-r_i)(z_{j-1}-z_j^*)(z_{j-1}-z_j)} , \quad L^{(2)}(r,z) = \frac{(r-r_{i-1})(r-r_i)(z-z_j^*)(z-z_j)}{(r_i^*-r_{i-1})(r_i^*-r_i)(z_{j-1}-z_j^*)(z_{j-1}-z_j)} ,$$

$$L^{(3)}(r,z) = \frac{(r-r_{i-1})(r-r_i^*)(z-z_j^*)(z-z_j)}{(r_i-r_{i-1})(r_i-r_i^*)(z_{j-1}-z_j^*)(z_{j-1}-z_j)} , \quad L^{(4)}(r,z) = \frac{(r-r_i^*)(r-r_i)(z-z_{j-1})(z-z_j)}{(r_{i-1}-r_i^*)(r_{i-1}-r_i)(z_j^*-z_{j-1})(z_j^*-z_j)} ,$$

$$L^{(5)}(r,z) = \frac{(r-r_{i-1})(r-r_i)(z-z_{j-1})(z-z_j)}{(r_i^*-r_{i-1})(r_i^*-r_i)(z_j^*-z_{j-1})(z_j^*-z_j)} , \quad L^{(6)}(r,z) = \frac{(r-r_{i-1})(r-r_i^*)(z-z_{j-1})(z-z_j)}{(r_i-r_{i-1})(r_i-r_i^*)(z_j^*-z_{j-1})(z_j^*-z_j)} ,$$

$$L^{(7)}(r,z) = \frac{(r-r_i^*)(r-r_i)(z-z_{j-1})(z-z_j^*)}{(r_{i-1}-r_i^*)(r_{i-1}-r_i)(z_j-z_{j-1})(z_j-z_j^*)} , \quad L^{(8)}(r,z) = \frac{(r-r_{i-1})(r-r_i)(z-z_{j-1})(z-z_j^*)}{(r_i^*-r_{i-1})(r_i^*-r_i)(z_j-z_{j-1})(z_j-z_j^*)} ,$$

$$L^{(9)}(r,z) = \frac{(r-r_{i-1})(r-r_i^*)(z-z_{j-1})(z-z_j^*)}{(r_i-r_{i-1})(r_i-r_i^*)(z_j-z_{j-1})(z_j-z_j^*)} .$$

In order to explain the systematical procedure for the construction of the Lagrange polynomials of any order, it will be sufficient to show the procedure for a low-order case. We take N=2 (NL=8) for the example.

Let represent $L^{(K)}(r,z)$, the Lagrange polynomial generated at the k-th point, by

$$L^{(K)}(r,z) = a_K + b_K r + c_K z + d_K rz + e_K r^2 + f_K z^2 + g_K r^2 z + h_K r z^2 .$$

The coefficients $a_K, b_K, \cdots, h_K$ can readily be determined by the Cramer's method.

The following abbreviations are introduced;

$$a_k = \alpha_k / D \ , \ b_k = \beta_k / D \ , \ c_k = \gamma_k / D \ , \ d_k = \delta_k / D \ ,$$

$$e_k = \varepsilon_k / D \ , \ f_k = \eta_k / D \ , \ g_k = \zeta_k / D \ , \ h_k = \theta_k / D \ ,$$

where

$$D = \begin{vmatrix} 1 & r_1 & Z_1 & r_1 Z_1 & r_1^2 & Z_1^2 & r_1^2 Z_1 & r_1 Z_1^2 \\ 1 & r_2 & Z_2 & r_2 Z_2 & r_2^2 & Z_2^2 & r_2^2 Z_2 & r_2 Z_2^2 \\ 1 & r_3 & Z_3 & | & | & | & | & | \\ 1 & r_4 & Z_4 & | & | & | & | & | \\ 1 & r_6 & Z_6 & | & | & | & | & | \\ 1 & r_7 & Z_7 & | & | & | & | & | \\ 1 & r_8 & Z_8 & | & | & | & | & | \\ 1 & r_9 & Z_9 & r_9 Z_9 & r_9^2 & Z_9^2 & r_9^2 Z_9 & r_9 Z_9^2 \end{vmatrix}$$

$$= \alpha_k + \beta_k r_k + \gamma_k Z_k + \delta_k r_k Z_k + \varepsilon_k r_k^2 + \eta_k Z_k^2 + \zeta_k r_k^2 Z_k + \theta_k r_k Z_k^2 \ .$$

r_i and Z_i (i=1,2,3,4,6,7,8,9) stand for the coordinates of the i-th point.

Then the coefficients $\alpha_k, \beta_k, \cdots, \theta_k$ are calculated in cyclic scheme as follows, (k,l,m,n,q,r,s,t) being the cyclic permutation of (1,2,3,4,6,7,8,9).

$$\alpha_K = (-1)^{K-1} \begin{vmatrix} r_\ell & Z_\ell & r_\ell Z_\ell & r_\ell^2 & Z_\ell^2 & r_\ell^2 Z_\ell & r_\ell Z_\ell^2 \\ r_m & Z_m & r_m Z_m & r_m^2 & Z_m^2 & r_m^2 Z_m & r_m Z_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_t & Z_t & r_t Z_t & r_t^2 & Z_t^2 & r_t^2 Z_t & r_t Z_t^2 \end{vmatrix} ,$$

$$\beta_K = (-1)^{K} \begin{vmatrix} 1 & Z_\ell & r_\ell Z_\ell & r_\ell^2 & Z_\ell^2 & r_\ell^2 Z_\ell & r_\ell Z_\ell^2 \\ 1 & Z_m & r_m Z_m & r_m^2 & Z_m^2 & r_m^2 Z_m & r_m Z_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & Z_t & r_t Z_t & r_t^2 & Z_t^2 & r_t^2 Z_t & r_t Z_t^2 \end{vmatrix} ,$$

$$\gamma_K = (-1)^{K-1} \begin{vmatrix} 1 & r_\ell & r_\ell Z_\ell & r_\ell^2 & Z_\ell^2 & r_\ell^2 Z_\ell & r_\ell Z_\ell^2 \\ 1 & r_m & r_m Z_m & r_m^2 & Z_m^2 & r_m^2 Z_m & r_m Z_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r_t & r_t Z_t & r_t^2 & Z_t^2 & r_t^2 Z_t & r_t Z_t^2 \end{vmatrix} ,$$

$$\delta_K = (-1)^{K} \begin{vmatrix} 1 & r_\ell & Z_\ell & r_\ell^2 & Z_\ell^2 & r_\ell^2 Z_\ell & r_\ell Z_\ell^2 \\ 1 & r_m & Z_m & r_m^2 & Z_m^2 & r_m^2 Z_m & r_m Z_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r_t & Z_t & r_t^2 & Z_t^2 & r_t^2 Z_t & r_t Z_t^2 \end{vmatrix} ,$$

$$\varepsilon_K = (-1)^{K-1} \begin{vmatrix} 1 & r_\ell & z_\ell & r_\ell z_\ell & z_\ell^2 & r_\ell^2 z_\ell & r_\ell z_\ell^2 \\ 1 & r_m & z_m & r_m z_m & z_m^2 & r_m^2 z_m & r_m z_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r_t & z_t & r_t z_t & z_t^2 & r_t^2 z_t & r_t z_t^2 \end{vmatrix} ,$$

$$\eta_K = (-1)^{K} \begin{vmatrix} 1 & r_\ell & z_\ell & r_\ell z_\ell & r_\ell^2 & r_\ell^2 z_\ell & r_\ell z_\ell^2 \\ 1 & r_m & z_m & r_m z_m & r_m^2 & r_m^2 z_m & r_m z_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r_t & z_t & r_t z_t & r_t^2 & r_t^2 z_t & r_t z_t^2 \end{vmatrix} ,$$

$$\zeta_K = (-1)^{K-1} \begin{vmatrix} 1 & r_\ell & z_\ell & r_\ell z_\ell & r_\ell^2 & z_\ell^2 & r_\ell z_\ell^2 \\ 1 & r_m & z_m & r_m z_m & r_m^2 & z_m^2 & r_m z_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r_t & z_t & r_t z_t & r_t^2 & z_t^2 & r_t z_t^2 \end{vmatrix} ,$$

$$\theta_K = (-1)^{K} \begin{vmatrix} 1 & r_\ell & z_\ell & r_\ell z_\ell & r_\ell^2 & z_\ell^2 & r_\ell^2 z_\ell \\ 1 & r_m & z_m & r_m z_m & r_m^2 & z_m^2 & r_m^2 z_m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r_t & z_t & r_t z_t & r_t^2 & z_t^2 & r_t^2 z_t \end{vmatrix} .$$

Appendix II Matrix Elements used in Discontinuous Method

In the algorithm of the discontinuous method, we must solve the linear algebraic equations given by Eq.(26). It has been shown already that the elements of the matrix A and the vector depend on the direction of the space-angle sweep. The coefficients for each direction are listed below for the case of N=2.

Note that the index number 5 should be suppressed when NL=8.

(a) Sweep 1 (Ω_1)

$$a_{IK}^{m,g} = \widetilde{a}_{IK}^{m,g} \quad , \qquad \text{for} \quad K = 1,2,4,5 \quad ,$$

$$a_{I3}^{m,g} = \widetilde{a}_{I3}^{m,g} + W_m \mu_m r_i \sum_{K=3,6,9} \left\langle L^{(K)}(r_i,z), \frac{1}{r} L^{(3)}(r_i,z) \right\rangle \delta_{IK} \quad ,$$

$$a_{I6}^{m,g} = \widetilde{a}_{I6}^{m,g} + W_m \mu_m r_i \sum_{K=3,6,9} \left\langle L^{(K)}(r_i,z), \frac{1}{r} L^{(6)}(r_i,z) \right\rangle \delta_{IK} \quad ,$$

$$a_{I7}^{m,g} = \widetilde{a}_{I7}^{m,g} + W_m \eta_m \sum_{K=7,8,9} \left\langle L^{(K)}(r,z_j), L^{(7)}(r,z_j) \right\rangle \delta_{IK} \quad ,$$

$$a_{I8}^{m,g} = \widetilde{a}_{I8}^{m,g} + W_m \eta_m \sum_{K=7,8,9} \left\langle L^{(K)}(r,z_j), L^{(8)}(r,z_j) \right\rangle \delta_{IK} \quad ,$$

$$a_{I9}^{m,g} = \widetilde{a}_{I9}^{m,g} + \left[W_m \mu_m r_i \sum_{K=3,6,9} \left\langle L^{(K)}(r_i,z), \frac{1}{r} L^{(9)}(r_i,z) \right\rangle + W_m \eta_m \sum_{K=7,8,9} \left\langle L^{(K)}(r,z_j), L^{(9)}(r,z_j) \right\rangle \right] \delta_{IK} \quad ,$$

$$b_I^{m,g} = \widetilde{b}_I^{m,g} + \left[W_m \mu_m r_i \sum_{K=3,6,9} \sum_{K''=1',4',7'} \left\langle L^{(K)}(r_i,z), \frac{1}{r} L^{(K'')}(r_i,z) \right\rangle \psi_m^{g(K'')} + \right.$$

$$\left. + W_m \eta_m \sum_{K=7,8,9} \sum_{K'=1',2',3'} \left\langle L^{(K)}(r,z_j), L^{(K')}(r,z_j) \right\rangle \psi_m^{g(K')} \right] \delta_{IK} \quad .$$

(b) Sweep 2 (Ω_2)

$$a_{IK}^{m,g} = \widetilde{a}_{IK}^{m,g}, \qquad \text{for } K = 2,3,5,6,$$

$$a_{I1}^{m,g} = \widetilde{a}_{I1}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,4,7} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(1)}(r_{i-1}, z) \right\rangle \delta_{IK},$$

$$a_{I4}^{m,g} = \widetilde{a}_{I4}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,4,7} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(4)}(r_{i-1}, z) \right\rangle \delta_{IK},$$

$$a_{I7}^{m,g} = \widetilde{a}_{I7}^{m,g} + \left[W_m \mu_m r_{i-1} \sum_{K=1,4,7} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(7)}(r_{i-1}, z) \right\rangle + W_m \eta_m \sum_{K=7,8,9} \left\langle L^{(K)}(r, z_j), L^{(7)}(r, z_j) \right\rangle \right] \delta_{IK},$$

$$a_{I8}^{m,g} = \widetilde{a}_{I8}^{m,g} + W_m \eta_m \sum_{K=7,8,9} \left\langle L^{(K)}(r, z_j), L^{(8)}(r, z_j) \right\rangle \delta_{IK},$$

$$a_{I9}^{m,g} = \widetilde{a}_{I9}^{m,g} + W_m \eta_m \sum_{K=7,8,9} \left\langle L^{(K)}(r, z_j), L^{(9)}(r, z_j) \right\rangle \delta_{IK},$$

$$b_I^{m,g} = \widetilde{b}_I^{m,g} + \left[W_m \mu_m r_{i-1} \sum_{K=1,4,7} \sum_{K^*=3^*,6^*,9^*} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(K^*)}(r_{i-1}, z) \right\rangle \psi_m^{g(K^*)} + \right.$$
$$\left. + W_m \eta_m \sum_{K=7,8,9} \sum_{K'=1',2',3'} \left\langle L^{(K)}(r, z_j), L^{(K')}(r, z_j) \right\rangle \psi_m^{g(K')} \right] \delta_{IK}.$$

(c) Sweep 3 (Ω_3)

$$a_{IK}^{m,g} = \widetilde{a}_{IK}^{m,g}, \qquad \text{for } K = 4,5,7,8,$$

$$a_{I1}^{m,g} = \widetilde{a}_{I1}^{m,g} + W_m \eta_m \sum_{K=1,2,3} \left\langle L^{(K)}(r, z_{j-1}), L^{(1)}(r, z_{j-1}) \right\rangle \delta_{IK},$$

$$a_{I2}^{m,g} = \widetilde{a}_{I2}^{m,g} + W_m \eta_m \sum_{K=1,2,3} \left\langle L^{(K)}(r, z_{j-1}), L^{(2)}(r, z_{j-1}) \right\rangle \delta_{IK},$$

$$a_{I3}^{m,g} = \widetilde{a}_{I3}^{m,g} + \left[W_m \mu_m r_i \sum_{K=3,6,9} \left\langle L^{(K)}(r_i, z), \frac{1}{r} L^{(3)}(r_i, z) \right\rangle + W_m \eta_m \sum_{K=1,2,3} \left\langle L^{(K)}(r, z_{j-1}), L^{(3)}(r, z_{j-1}) \right\rangle \right] \delta_{IK},$$

$$a_{I6}^{m,g} = \widetilde{a}_{I6}^{m,g} + W_m \mu_m r_i \sum_{K=3,6,9} \left\langle L^{(K)}(r_i, z), \frac{1}{r} L^{(6)}(r_i, z) \right\rangle \delta_{IK} ,$$

$$a_{I9}^{m,g} = \widetilde{a}_{I9}^{m,g} + W_m \mu_m r_i \sum_{K=3,6,9} \left\langle L^{(K)}(r_i, z), \frac{1}{r} L^{(9)}(r_i, z) \right\rangle \delta_{IK} ,$$

$$b_I^{m,g} = \widetilde{b}_I^{m,g} + \Big[W_m \mu_m r_i \sum_{K=3,6,9} \sum_{K''=1'',4'',7''} \left\langle L^{(K)}(r_i, z), \frac{1}{r} L^{(K'')}(r_i, z) \right\rangle \psi_m^{g(K'')} +$$

$$+ W_m \eta_m \sum_{K=1,2,3} \sum_{K^\circ=7^\circ,8^\circ,9^\circ} \left\langle L^{(K)}(r, z_{j-1}), L^{(K^\circ)}(r, z_{j-1}) \right\rangle \psi_m^{g(K^\circ)} \Big] \delta_{IK} .$$

(d) Sweep 4 (Ω_4)

$$a_{IK}^{m,g} = \widetilde{a}_{IK}^{m,g} , \qquad \text{for } K = 5, 6, 8, 9 ,$$

$$a_{I1}^{m,g} = \widetilde{a}_{I1}^{m,g} + \Big[W_m \mu_m r_{i-1} \sum_{K=1,4,7} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(1)}(r_{i-1}, z) \right\rangle + W_m \eta_m \sum_{K=1,2,3} \left\langle L^{(K)}(r, z_{j-1}), L^{(1)}(r, z_{j-1}) \right\rangle \Big] \delta_{IK} ,$$

$$a_{I2}^{m,g} = \widetilde{a}_{I2}^{m,g} + W_m \eta_m \sum_{K=1,2,3} \left\langle L^{(K)}(r, z_{j-1}), L^{(2)}(r, z_{j-1}) \right\rangle \delta_{IK} ,$$

$$a_{I3}^{m,g} = \widetilde{a}_{I3}^{m,g} + W_m \eta_m \sum_{K=1,2,3} \left\langle L^{(K)}(r, z_{j-1}), L^{(3)}(r, z_{j-1}) \right\rangle \delta_{IK} ,$$

$$a_{I4}^{m,g} = \widetilde{a}_{I4}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,4,7} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(4)}(r_{i-1}, z) \right\rangle \delta_{IK} ,$$

$$a_{I7}^{m,g} = \widetilde{a}_{I7}^{m,g} + W_m \mu_m r_{i-1} \sum_{K=1,4,7} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(7)}(r_{i-1}, z) \right\rangle \delta_{IK} ,$$

$$b_I^{m,g} = \widetilde{b}_I^{m,g} + \Big[W_m \mu_m r_{i-1} \sum_{K=1,4,7} \sum_{K^*=3^*,6^*,9^*} \left\langle L^{(K)}(r_{i-1}, z), \frac{1}{r} L^{(K^*)}(r_{i-1}, z) \right\rangle \psi_m^{g(K^*)} +$$

$$+ W_m \eta_m \sum_{K=1,2,3} \sum_{K^\circ=7^\circ,8^\circ,9^\circ} \left\langle L^{(K)}(r, z_{j-1}), L^{(K^\circ)}(r, z_{j-1}) \right\rangle \psi_m^{g(K^\circ)} \Big] \delta_{IK} .$$

Appendix III Explicit Expressions for Inner Products

Several kinds of the inner products related to Lagrange polynomials have been defined by Eq.(9). Confining to the inner products used in discontinuous method, we present tables of the explicit expressions for these inner products obtained on the rectangular cell V_{ij} [Eqs. (9-1), (9-3), (9-4), (9-5), (9-6), (9-9) and (9-10)]. For each inner product there are two kinds of tables. One is a table which shows the correspondence between a integer (we call it the pointer) and the expression, and the other indicates the correspondence between the pairs (k',k) and the pointers. If the integer (pointer) is negative in the second table, it means that the minus sign must be attached to the expression in the first table corresponding to the absolute value of the integer.

In the tables for $\left\langle L^{(k')}(r_0,z), \frac{1}{r}L^{(k)}(r_0,z)\right\rangle$ and $\left\langle L^{(k')}(r,z_0), L^{(k)}(r,z_0)\right\rangle$, only non-zero values are indicated in the second tables. Thus the values which are disappeared in those tables are all zero. We use the following abbreviations;

$$\Delta r_i = r_i - r_{i-1},$$
$$\Delta Z_j = Z_j - Z_{j-1},$$
$$S_{ij} = \Delta r_i \times \Delta Z_j .$$

(A) N = 1 (NL = 4)

$$\langle 1 , L^{(k)}(r,z) \rangle$$

POINTER	VALUE
1	$\frac{1}{12}(r_{i-1}+2r_i)\Delta S_{ij}$
2	$\frac{1}{12}(2r_{i-1}+r_i)\Delta S_{ij}$

k	1	2	3	4
POINTER	2	1	2	1

$$\langle L^{(k')}(r,z) , L^{(k)}(r,z) \rangle$$

POINTER	VALUE
1	$\frac{1}{36}(r_{i-1}+r_i)\Delta S_{ij}$
2	$\frac{1}{36}(r_{i-1}+3r_i)\Delta S_{ij}$
3	$\frac{1}{36}(3r_{i-1}+r_i)\Delta S_{ij}$
4	$\frac{1}{72}(r_{i-1}+r_i)\Delta S_{ij}$
5	$\frac{1}{72}(r_{i-1}+3r_i)\Delta S_{ij}$
6	$\frac{1}{72}(3r_{i-1}+r_i)\Delta S_{ij}$

k' \ k	1	2	3	4
1	3	1	6	4
2		2	4	5
3			3	1
4				2

(SYMMETRIC)

$$\langle L^{(k')}(r,z) , \frac{1}{r} L^{(k)}(r,z) \rangle$$

POINTER	VALUE
1	$\frac{1}{9}\Delta S_{ij}$
2	$\frac{1}{18}\Delta S_{ij}$
3	$\frac{1}{36}\Delta S_{ij}$

k' \ k	1	2	3	4
1	1	2	2	3
2		1	3	2
3			1	2
4				1

(SYMMETRIC)

$$\left\langle L^{(k')}(r,z),\ \frac{\partial}{\partial r}L^{(k)}(r,z)\right\rangle$$

POINTER	VALUE
1	$\frac{1}{18}(r_{i-1}+2r_i)\Delta Z_j$
2	$\frac{1}{18}(2r_{i-1}+r_i)\Delta Z_i$
3	$\frac{1}{36}(r_{i-1}+2r_i)\Delta Z_j$
4	$\frac{1}{36}(2r_{i-1}+r_i)\Delta Z_j$

k' \ k	1	2	3	4
1	-2	2	-4	4
2	-1	1	-3	3
3	-4	4	-2	2
4	-3	3	-1	1

$$\left\langle L^{(k')}(r,z),\ \frac{\partial}{\partial z}L^{(k)}(r,z)\right\rangle$$

POINTER	VALUE
1	$\frac{1}{24}(r_{i-1}+r_i)\Delta r_i$
2	$\frac{1}{24}(r_{i-1}+3r_i)\Delta r_i$
3	$\frac{1}{24}(3r_{i-1}+r_i)\Delta r_i$

k' \ k	1	2	3	4
1	-3	-1	3	1
2	-1	-2	1	2
3	-3	-1	3	1
4	-1	-2	1	2

$$\left\langle L^{(k')}(r_0,z),\ \frac{1}{r}L^{(k)}(r_0,z)\right\rangle$$

POINTER	VALUE
1	$\frac{1}{3}\Delta Z_j$
2	$\frac{1}{6}\Delta Z_j$

k' \ k	1	3
1	1	2
3	2	1

$(r_0 = r_{i-1})$

k' \ k	2	4
2	1	2
4	2	1

$(r_0 = r_i)$

$$\left\langle L^{(k')}(r,z_0),\ L^{(k)}(r,z_0)\right\rangle$$

POINTER	VALUE
1	$\frac{1}{12}(r_{i-1}+r_i)\Delta r_i$
2	$\frac{1}{12}(r_{i-1}+3r_i)\Delta r_i$
3	$\frac{1}{12}(3r_{i-1}+r_i)\Delta r_i$

k' \ k	1	2
1	3	1
2	1	2

$(z_0 = z_{j-1})$

k' \ k	3	4
3	3	1
4	1	2

$(z_0 = z_j)$

(B) N = 2 (NL = 8)

$$\langle 1 , L^{(k)}(r,z) \rangle$$

POINTER	VALUE
1	$\frac{1}{6}(r_{i-1}+r_i)\Delta S_{ij}$
2	$\frac{1}{9}(r_{i-1}+2r_i)\Delta S_{ij}$
3	$\frac{1}{9}(2r_{i-1}+r_i)\Delta S_{ij}$
4	$-\frac{1}{36}(r_{i-1}+2r_i)\Delta S_{ij}$
5	$-\frac{1}{36}(2r_{i-1}+r_i)\Delta S_{ij}$

k	1	2	3	4	6	7	8	9
POINTER	4	1	5	3	2	4	1	5

$$\langle L^{(k')}(r,z), L^{(k)}(r,z) \rangle$$

k' \ k	1	2	3	4	6	7	8	9
1	10	-9	12	8	13	15	5	11
2		1	-10	7	6	5	2	5
3			9	14	8	11	5	16
4				4	2	8	7	14
6					3	13	6	8
7						10	-9	12
8							1	-10
9								9

(SYMMETRIC)

$\Longleftrightarrow$

POINTER	VALUE
1	$\frac{4}{45}(r_{i-1}+r_i)\Delta S_{ij}$
2	$\frac{2}{45}(r_{i-1}+r_i)\Delta S_{ij}$
3	$\frac{2}{45}(r_{i-1}+3r_i)\Delta S_{ij}$
4	$\frac{2}{45}(3r_{i-1}+r_i)\Delta S_{ij}$
5	$-\frac{1}{45}(r_{i-1}+r_i)\Delta S_{ij}$
6	$\frac{1}{45}(2r_{i-1}+3r_i)\Delta S_{ij}$
7	$\frac{1}{45}(3r_{i-1}+2r_i)\Delta S_{ij}$
8	$-\frac{1}{60}(r_{i-1}+r_i)\Delta S_{ij}$
9	$\frac{1}{90}(r_{i-1}+2r_i)\Delta S_{ij}$
10	$\frac{1}{90}(2r_{i-1}+r_i)\Delta S_{ij}$
11	$\frac{1}{120}(r_{i-1}+r_i)\Delta S_{ij}$
12	$\frac{1}{180}(r_{i-1}+r_i)\Delta S_{ij}$
13	$-\frac{1}{180}(3r_{i-1}+5r_i)\Delta S_{ij}$
14	$-\frac{1}{180}(5r_{i-1}+3r_i)\Delta S_{ij}$
15	$\frac{1}{360}(r_{i-1}+3r_i)\Delta S_{ij}$
16	$\frac{1}{360}(3r_{i-1}+r_i)\Delta S_{ij}$

$$\langle L^{(k')}(r,z), \frac{1}{r} L^{(k)}(r,z) \rangle$$

POINTER	VALUE
1	$\frac{1}{9}\Delta S_{ij}$
2	$\frac{1}{30}\Delta S_{ij}$
3	$-\frac{2}{45}\Delta S_{ij}$
4	$\frac{4}{45}\Delta S_{ij}$
5	$\frac{8}{45}\Delta S_{ij}$
6	$\frac{1}{60}\Delta S_{ij}$
7	$\frac{1}{90}\Delta S_{ij}$

k' \ k	1	2	3	4	6	7	8	9
1	2	-2	7	-2	3	7	3	6
2		5	-2	1	1	3	4	3
3			2	3	-2	6	3	7
4				5	4	-2	1	3
6					5	3	1	-2
7						2	-2	7
8							5	-2
9								2

(SYMMETRIC)

$$\langle L^{(k')}(r,z), \frac{\partial}{\partial r} L^{(k)}(r,z) \rangle$$

POINTER	VALUE				
1	$\frac{2}{9}(r_{i-1})\Delta Z_j$	9	$\frac{4}{45}(r_{i-1}+2r_i)\Delta Z_j$	18	$-\frac{1}{90}(9r_{i-1}+r_i)\Delta Z_j$
2	$-\frac{2}{9}(r_i)\Delta Z_j$	10	$\frac{4}{45}(2r_{i-1}+r_i)\Delta Z_j$	19	$\frac{1}{90}(r_{i-1}+12r_i)\Delta Z_j$
3	$\frac{1}{9}(r_{i-1}+r_i)\Delta Z_j$	11	$\frac{1}{60}(r_{i-1})\Delta Z_j$	20	$-\frac{1}{90}(12r_{i-1}+r_i)\Delta Z_j$
4	$\frac{1}{45}\Delta S_{ij}$	12	$-\frac{1}{60}(r_i)\Delta Z_j$	21	$\frac{1}{90}(3r_{i-1}+4r_i)\Delta Z_j$
5	$-\frac{1}{45}(2r_{i-1}+3r_i)\Delta Z_j$	13	$\frac{1}{60}(r_{i-1}+3r_i)\Delta Z_j$	22	$\frac{1}{90}(4r_{i-1}+3r_i)\Delta Z_j$
6	$\frac{1}{45}(3r_{i-1}+2r_i)\Delta Z_j$	14	$-\frac{1}{60}(3r_{i-1}+r_i)\Delta Z_j$	23	$\frac{1}{180}(r_{i-1}-4r_i)\Delta Z_j$
7	$-\frac{2}{45}\Delta S_{ij}$	15	$-\frac{1}{90}(r_{i-1}-8r_i)\Delta Z_j$	24	$\frac{1}{180}(4r_{i-1}-r_i)\Delta Z_j$
8	$\frac{4}{45}\Delta S_{ij}$	16	$-\frac{1}{90}(8r_{i-1}-r_i)\Delta Z_j$	25	$-\frac{1}{180}(3r_{i-1}+5r_i)\Delta Z_j$
		17	$\frac{1}{90}(r_{i-1}+9r_i)\Delta Z_j$	26	$\frac{1}{180}(5r_{i-1}+3r_i)\Delta Z_j$

k' \ k	1	2	3	4	6	7	8	9
1	14	6	25	21	-21	24	4	12
2	18	8	17	-3	3	4	7	4
3	26	5	13	22	-22	11	4	23
4	20	1	16	-10	10	20	1	16
6	15	2	19	-9	9	15	2	19
7	24	4	12	21	-21	14	6	25
8	4	7	4	-3	3	18	8	17
9	11	4	23	22	-22	26	5	13

$$\left\langle L^{(k')}(r,z),\ \frac{\partial}{\partial z} L^{(k)}(r,z) \right\rangle$$

POINTER	VALUE				
1	0	7	$\frac{1}{90}(9r_{i-1}+r_i)\Delta r_i$	14	$\frac{1}{120}(7r_{i-1}+r_i)\Delta r_i$
2	$\frac{2}{15}(r_{i-1}+r_i)\Delta r_i$	8	$\frac{1}{90}(3r_{i-1}+4r_i)\Delta r_i$	15	$\frac{1}{360}(r_{i-1}-7r_i)\Delta r_i$
3	$\frac{2}{45}(2r_{i-1}+3r_i)\Delta r_i$	9	$\frac{1}{90}(4r_{i-1}+3r_i)\Delta r_i$	16	$\frac{1}{360}(7r_{i-1}-r_i)\Delta r_i$
4	$\frac{2}{45}(3r_{i-1}+2r_i)\Delta r_i$	10	$\frac{1}{90}(4r_{i-1}+9r_i)\Delta r_i$	17	$\frac{1}{360}(r_{i-1}+15r_i)\Delta r_i$
5	$\frac{1}{90}(\Delta r_i)^2$	11	$\frac{1}{90}(9r_{i-1}+4r_i)\Delta r_i$	18	$\frac{1}{360}(15r_{i-1}+r_i)\Delta r_i$
6	$\frac{1}{90}(r_{i-1}+9r_i)\Delta r_i$	12	$\frac{1}{120}(r_{i-1}+r_i)\Delta r_i$		
		13	$\frac{1}{120}(r_{i-1}+7r_i)\Delta r_i$		

⇓

k' \ k	1	2	3	4	6	7	8	9
1	-14	8	-15	7	-5	-18	-8	-12
2	-11	-2	-10	4	3	-8	2	-9
3	16	9	-13	5	6	-12	-9	-17
4	-7	-4	-5	1	1	7	4	5
6	5	-3	-6	1	1	-5	3	6
7	18	8	12	-7	5	14	-8	15
8	8	-2	9	-4	-3	11	2	10
9	12	9	17	-5	-6	-16	-9	13

$$\left\langle L^{(k')}(r_0, z),\ \frac{1}{r} L^{(k)}(r_0, z) \right\rangle$$

POINTER	VALUE
1	$\frac{8}{15}\Delta Z_j$
2	$\frac{2}{15}\Delta Z_j$
3	$\frac{1}{15}\Delta Z_j$
4	$-\frac{1}{30}\Delta Z_j$

k' \ k	1	4	7
1	2	3	4
4		1	3
7			2

$(r_0 = r_{i-1})$

k' \ k	3	6	9
3	2	3	4
6		1	3
9			2

$(r_0 = r_i)$

$$\langle L^{(k')}(r,z_0), L^{(k)}(r,z_0)\rangle$$

POINTER	VALUE
1	$\frac{4}{15}(r_{i-1}+r_i)\Delta r_i$
2	$\frac{1}{15}(r_{i-1})\Delta r_i$
3	$\frac{1}{15}(r_i)\Delta r_i$
4	$-\frac{1}{60}(r_{i-1}+r_i)\Delta r_i$
5	$\frac{1}{60}(r_{i-1}+7r_i)\Delta r_i$
6	$\frac{1}{60}(7r_{i-1}+r_i)\Delta r_i$

k' \ k	1	2	3
1	6	2	4
2		1	3
3			5

$(z_0 = z_{j-1})$

k' \ k	7	8	9
7	6	2	4
8		1	3
9			5

$(z_0 = z_j)$

(C) N = 2 (NL = 9)

$$\langle 1, L^{(k)}(r,z)\rangle$$

POINTER	VALUE
1	$\frac{2}{9}(r_{i-1}+r_i)\Delta S_{ij}$
2	$\frac{1}{9}(r_{i-1})\Delta S_{ij}$
3	$\frac{1}{9}(r_i)\Delta S_{ij}$
4	$\frac{1}{18}(r_{i-1}+r_i)\Delta S_{ij}$
5	$\frac{1}{36}(r_{i-1})\Delta S_{ij}$
6	$\frac{1}{36}(r_i)\Delta S_{ij}$

k	1	2	3	4	5	6	7	8	9
POINTER	5	4	6	2	1	3	5	4	6

POINTER	VALUE
1	$\frac{32}{225}(r_{i-1}+r_i)\Delta S_{ij}$
2	$\frac{8}{225}(r_{i-1}+r_i)\Delta S_{ij}$
3	$\frac{8}{225}(r_{i-1})\Delta S_{ij}$
4	$\frac{8}{225}(r_i)\Delta S_{ij}$
5	$\frac{4}{225}(r_{i-1}+r_i)\Delta S_{ij}$
6	$-\frac{2}{225}(r_{i-1}+r_i)\Delta S_{ij}$
7	$\frac{2}{225}(r_{i-1})\Delta S_{ij}$
8	$\frac{2}{225}(r_i)\Delta S_{ij}$
9	$\frac{2}{225}(r_{i-1}+7r_i)\Delta S_{ij}$
10	$\frac{2}{225}(7r_{i-1}+r_i)\Delta S_{ij}$
11	$\frac{1}{225}(r_{i-1})\Delta S_{ij}$
12	$\frac{1}{225}(r_i)\Delta S_{ij}$
13	$-\frac{1}{450}(r_{i-1}+r_i)\Delta S_{ij}$
14	$-\frac{1}{450}(r_{i-1})\Delta S_{ij}$
15	$-\frac{1}{450}(r_i)\Delta S_{ij}$
16	$\frac{1}{450}(r_{i-1}+7r_i)\Delta S_{ij}$
17	$\frac{1}{450}(7r_{i-1}+r_i)\Delta S_{ij}$
18	$-\frac{1}{900}(r_{i-1}+r_i)\Delta S_{ij}$
19	$\frac{1}{900}(r_{i-1}+7r_i)\Delta S_{ij}$
20	$\frac{1}{900}(7r_{i-1}+r_i)\Delta S_{ij}$
21	$\frac{1}{1800}(r_{i-1}+r_i)\Delta S_{ij}$
22	$-\frac{1}{1800}(r_{i-1}+7r_i)\Delta S_{ij}$
23	$-\frac{1}{1800}(7r_{i-1}+r_i)\Delta S_{ij}$

$$\langle L^{(k')}(r,z), L^{(k)}(r,z) \rangle$$

k' \ k	1	2	3	4	5	6	7	8	9
1	17	7	13	20	11	18	23	14	21
2		2	8	11	5	12	14	6	15
3			16	18	12	19	21	15	22
4				10	3	6	20	11	18
5					1	4	11	5	12
6						9	18	12	19
7							17	7	13
8								2	8
9									16

(SYMMETRIC)

$$\left\langle L^{(k')}(r,z), \frac{1}{r} L^{(k)}(r,z) \right\rangle$$

POINTER	VALUE
1	$\frac{64}{225}\Delta S_{ij}$
2	$\frac{16}{225}\Delta S_{ij}$
3	$\frac{8}{225}\Delta S_{ij}$
4	$\frac{4}{225}\Delta S_{ij}$
5	$\frac{2}{225}\Delta S_{ij}$
6	$\frac{1}{225}\Delta S_{ij}$
7	$-\frac{1}{450}\Delta S_{ij}$
8	$\frac{1}{900}\Delta S_{ij}$

k' \ k	1	2	3	4	5	6	7	8	9
1	4	5	-6	5	6	7	-6	7	8
2		2	5	6	3	6	7	-4	7
3			4	7	6	5	8	7	-6
4				2	3	-4	5	6	7
5					1	3	6	3	6
6						2	7	6	5
7							4	5	-6
8								2	5
9									4

(SYMMETRIC)

$$\left\langle L^{(k')}(r,z),\ \frac{\partial}{\partial r}L^{(k)}(r,z)\right\rangle$$

POINTER	VALUE				
1	$-\frac{32}{225}\Delta S_{ij}$	13	$-\frac{4}{225}(13r_{i-1}+2r_i)\Delta Z_j$	25	$\frac{1}{225}(2r_{i-1}+13r_i)\Delta Z_j$
2	$-\frac{16}{225}(r_{i-1}+4r_i)\Delta Z_j$	14	$\frac{2}{225}\Delta S_{ij}$	26	$-\frac{1}{225}(13r_{i-1}+2r_i)\Delta Z_j$
3	$\frac{16}{225}(4r_{i-1}+r_i)\Delta Z_j$	15	$-\frac{2}{225}(r_{i-1}+4r_i)\Delta Z_j$	27	$\frac{1}{450}(2r_{i-1}+3r_i)\Delta Z_j$
4	$-\frac{8}{225}\Delta S_{ij}$	16	$\frac{2}{225}(4r_{i-1}+r_i)\Delta Z_j$	28	$-\frac{1}{450}(3r_{i-1}+2r_i)\Delta Z_j$
5	$\frac{8}{225}(3r_{i-1}+7r_i)\Delta Z_j$	17	$\frac{2}{225}(3r_{i-1}+7r_i)\Delta Z_j$	29	$-\frac{1}{450}(3r_{i-1}+7r_i)\Delta Z_j$
6	$-\frac{8}{225}(7r_{i-1}+3r_i)\Delta Z_j$	18	$-\frac{2}{225}(7r_{i-1}+3r_i)\Delta Z_j$	30	$\frac{1}{450}(7r_{i-1}+3r_i)\Delta Z_j$
7	$-\frac{4}{225}\Delta S_{ij}$	19	$\frac{1}{225}(r_{i-1}+4r_i)\Delta Z_j$	31	$\frac{1}{450}(2r_{i-1}+13r_i)\Delta Z_j$
8	$-\frac{4}{225}(r_{i-1}+4r_i)\Delta Z_j$	20	$-\frac{1}{225}(4r_{i-1}+r_i)\Delta Z_j$	32	$-\frac{1}{450}(13r_{i-1}+2r_i)\Delta Z_j$
9	$\frac{4}{225}(4r_{i-1}+r_i)\Delta Z_j$	21	$\frac{1}{225}(2r_{i-1}+3r_i)\Delta Z_j$	33	$-\frac{1}{900}(2r_{i-1}+3r_i)\Delta Z_j$
10	$\frac{4}{225}(2r_{i-1}+3r_i)\Delta Z_j$	22	$-\frac{1}{225}(3r_{i-1}+2r_i)\Delta Z_j$	34	$\frac{1}{900}(3r_{i-1}+2r_i)\Delta Z_j$
11	$-\frac{4}{225}(3r_{i-1}+2r_i)\Delta Z_j$	23	$\frac{1}{225}(3r_{i-1}+7r_i)\Delta Z_j$	35	$-\frac{1}{900}(2r_{i-1}+13r_i)\Delta Z_j$
12	$\frac{4}{225}(2r_{i-1}+13r_i)\Delta Z_j$	24	$-\frac{1}{225}(7r_{i-1}+3r_i)\Delta Z_j$	36	$\frac{1}{900}(13r_{i-1}+2r_i)\Delta Z_j$

k' \ k	1	2	3	4	5	6	7	8	9
1	26	9	22	32	16	28	36	20	34
2	18	4	17	24	7	23	30	14	29
3	21	8	25	27	15	31	33	19	35
4	32	16	28	13	3	11	32	16	28
5	24	7	23	6	1	5	24	7	23
6	27	15	31	10	2	12	27	15	31
7	36	20	34	32	16	28	26	9	22
8	30	14	29	24	7	23	18	4	17
9	33	19	35	27	15	31	21	8	25

$$\left\langle L^{(k')}(r,z), \frac{\partial}{\partial z} L^{(k)}(r,z) \right\rangle$$

POINTER	VALUE		
1	0	10	$\frac{1}{90}(r_{i-1})\Delta r_i$
2	$\frac{2}{15}(r_{i-1}+r_i)\Delta r_i$	11	$\frac{1}{90}(r_i)\Delta r_i$
3	$\frac{1}{30}(r_{i-1})\Delta r_i$	12	$\frac{1}{90}(r_{i-1}+7r_i)\Delta r_i$
4	$\frac{1}{30}(r_i)\Delta r_i$	13	$\frac{1}{90}(7r_{i-1}+r_i)\Delta r_i$
5	$\frac{8}{45}(r_{i-1}+r_i)\Delta r_i$	14	$\frac{1}{120}(r_{i-1}+r_i)\Delta r_i$
6	$\frac{2}{45}(r_{i-1}+r_i)\Delta r_i$	15	$\frac{1}{120}(r_{i-1}+7r_i)\Delta r_i$
7	$\frac{2}{45}(r_{i-1})\Delta r_i$	16	$\frac{1}{120}(7r_{i-1}+r_i)\Delta r_i$
8	$\frac{2}{45}(r_i)\Delta r_i$	17	$\frac{1}{360}(r_{i-1}+r_i)\Delta r_i$
9	$\frac{1}{90}(r_{i-1}+r_i)\Delta r_i$	18	$\frac{1}{360}(r_{i-1}+7r_i)\Delta r_i$
		19	$\frac{1}{360}(7r_{i-1}+r_i)\Delta r_i$

k' \ k	1	2	3	4	5	6	7	8	9
1	-16	-3	14	13	7	-9	-19	-10	17
2	-3	-2	-4	7	5	8	-10	-6	-11
3	14	-4	-15	-9	8	12	17	-11	-18
4	-13	-7	9	1	1	1	13	7	-9
5	-7	-5	-8	1	1	1	7	5	8
6	9	-8	-12	1	1	1	-9	8	12
7	19	10	-17	-13	-7	9	16	3	-14
8	10	6	11	-7	-5	-8	3	2	4
9	-17	11	18	9	-8	-12	-14	4	15

$$\langle L^{(k')}(r_0, z), \frac{1}{r} L^{(k)}(r_0, z) \rangle$$

POINTER	VALUE
1	$\frac{8}{15}\Delta Z_j$
2	$\frac{2}{15}\Delta Z_j$
3	$\frac{1}{15}\Delta Z_j$
4	$-\frac{1}{30}\Delta Z_j$

k' \ k	1	4	7
1	2	3	4
4		1	3
7			2

$(r_0 = r_{i-1})$

k' \ k	3	6	9
3	2	3	4
6		1	3
9			2

$(r_0 = r_i)$

$$\langle L^{(k)}(r, z_0), L^{(k)}(r, z_0) \rangle$$

POINTER	VALUE
1	$\frac{4}{15}(r_{i-1}+r_i)\Delta r_i$
2	$\frac{1}{15}(r_{i-1})\Delta r_i$
3	$\frac{1}{15}(r_i)\Delta r_i$
4	$-\frac{1}{60}(r_{i-1}+r_i)\Delta r_i$
5	$\frac{1}{60}(r_{i-1}+7r_i)\Delta r_i$
6	$\frac{1}{60}(7r_{i-1}+r_i)\Delta r_i$

k' \ k	1	2	3
1	6	2	4
2		1	3
3			5

$(Z_0 = Z_{j-1})$

k' \ k	7	8	9
7	6	2	4
8		1	3
9			5

$(Z_0 = Z_j)$