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NONLINEAR EFFECT ON THE GROWTH RATE OF INTERNAL M=1 KINK INSTABILITY

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The growth rate of internal $m = 1$ kink instability in a tokamak plasma is modified by the nonlinear excitation of m = 2 mode and m = 0 mode which are stable in the linear stability theory. It is assumed that this nonlinear effect is due to the axisymmetric part of helical perturbation. The saturation amplitude of the instability is estimated to be about $\epsilon^2 a$, where a is the plasma radius and ϵ is a small parameter corresponding to the inverse aspect ratio.

m＝1　キンク不安定性の成長率に対する非線型効果

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m＝1キンク不安定性の成長率は，非線型効果によるm＝2，m＝0モードの励起によって修正を受ける。m＝2，m＝0モードは，線型安定性では，安定である。非線型効果は，ヘリカルな摂動の軸対称成分に原因があると考える。このキンク不安定性の振巾は，$\varepsilon^2 a$の程度になることがわかる。ここで，εはアスペクト比に対応し，aはプラズマ半径である。

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§1 Introduction

The MHD stability theory showed that the internal m =1 kink mode becomes unstable for $q(0) < 1$ in a tokamak plasma,[1),2)] where $q(0)$ denotes the safty factor at $r=0$. From MHD stability theory, it is preferable to concentrate current density in the central region of plasma column, because m $\geqslant$ 2 kink modes become stable. However excessive concentration of current density in the central region makes the internal m = 1 kink mode unstable.

Recently M.N.Rosenbluth et. al.[3)] investigated nonlinear properties of the internal m = 1 kink instability, after obtaining the nonlinear amplitude by use of a boundary layer theory. They showed that growing modes of this type produce negative voltage spikes and inwards shift in major radius. In this paper we study nonlinear effect on the growth rate of the internal m = 1 kink instability.

We take a perturbation theory by a small expansion parameter $\epsilon \sim ka \ll 1$, where k is a fundamental wave number along the longitudinal direction and a is the radius of plasma column. The perturbation is expanded by $\exp(im\theta - inkz)$ in the cylindrical geometry. Firstly the growth rate and the eigen function of m = 1, n = 1 instability are determined by linearized MHD equations for unstable configurations. In considering nonlinear effect, we take account of $j \times b$ term due to the growth of m = 1, n = 1 instability, where j is the perturbed current and b the perturbed magnetic field. As a poloidal magnetic field B_θ is smaller than a longitudinal magnetic field B_z in a tokamak equilibrium, perturbed magnetic fields b_r and b_θ are larger than b_z . We consider that the axially symmetric part of the helical perturbation is important in the nonlinear regime of stability in a tokamak plasma. From these considerations, we have assumed the $j \times b$ term produces a source term exciting m = 2, n = 1 mode or m = 0, n = 1 mode, after the internal m = 1, n = 1 instability grows to a finite amplitude. And we have assumed nonlinear effect producing harmonics such as m = 2, n = 2 mode is not important . Here we neglect other modes such as m = 3, n = 1 mode which are higher order in the perturbation expansion. As the m = 2, n = 1 mode and the m = 0, n = 1 mode are stable for $q(0) \lesssim 1$ in the linear stability theory,[1] excitation of these stable modes suppresses the growth of m = 1, n = 1 instability. The formulation by these procedures is descrived in § 2.

We choose the tokamak equilibrium of uniform current with the fixed boundary (i.e., a conducting wall is located at the surface of plasma column),[1] as an application. In this case the linear growth rate and the eigen function of m = 1, n = 1 instability are obtained by analitic calculation. Then we can solve the differential equations for the m = 2, n = 1 mode and the m = 0, n = 1 mode and estimate the modification of its growth rate due to the nonlinear effect. These calculations are carried out in § 3. Discussion is given in § 4.

§2 Perturbation expansion of MHD equations

The equations describing an incompressible ideal fluid are

$$\left.\begin{aligned} &\frac{d\mathbb{V}}{dt} = \mathbb{J}\times\mathbb{B} - \nabla p \\ &\frac{\partial \mathbb{B}}{\partial t} = \nabla\times(\mathbb{V}\times\mathbb{B}) \\ &\nabla\cdot\mathbb{V} = 0 \end{aligned}\right\} \qquad (2.1)$$

The static equilibrium is defined by $\mathbb{V}=0$ and $\mathbb{J}\times\mathbb{B}=\nabla p$. In the cylindrical geometry, the equilibrium magnetic field is given by $\mathbb{B} = (0, B_\theta(r), B_z)$. Here $B_z = const.$ is available for a tokamak plasma with $\beta_p \simeq 1$. We take an ordering $B_z \sim O(\epsilon^{-1})$. In order to satisfy Kruskal Shafranov limit, $B_\theta \sim O(1)$. Here we consider an equilibrium unstable against m = 1, n = 1 mode.

When we give the displacement ξ to the equilibrium, we obtain the linearized MHD equations,

$$\left.\begin{aligned} &\rho\omega^2\xi = \mathbb{J}\times\mathbb{B} + \mathbb{J}\times b + \nabla P_1 \\ &b = \nabla\times(\xi\times\mathbb{B}) \\ &\nabla\cdot\xi = 0 \end{aligned}\right\} \qquad (2.2)$$

where ω denotes a growth rate. To solve our problem, ξ is considered as

$$\xi = \xi_1(r)\exp[i(\theta - kz)] + \xi_2(r)\exp[i(2\theta - kz)] + \xi_0(r)\exp(-ikz) + \cdots \qquad (2.3)$$

By use of the small expansion parameter ϵ , we write the component of the displacement vector ξ ,

$$\xi_r = \epsilon\xi_{r1}^{(1)} + \epsilon^3\tilde{\xi}_{r1}^{(3)} + \epsilon^2\xi_{r2}^{(2)} + \epsilon^2\xi_{r0}^{(2)} + \cdots \qquad (2.4)$$

$$\xi_\theta = \epsilon\xi_{\theta1}^{(1)} + \epsilon^3\tilde{\xi}_{\theta1}^{(3)} + \epsilon^2\xi_{\theta2}^{(2)} + \epsilon^2\xi_{\theta0}^{(2)} + \cdots \qquad (2.5)$$

$$\xi_z = \epsilon^2\xi_{z1}^{(2)} + \epsilon^4\tilde{\xi}_{z1}^{(4)} + \epsilon^3\xi_{z2}^{(3)} + \epsilon^3\xi_{z0}^{(3)} + \cdots \qquad (2.6)$$

The z component of the displacement is smaller than other two components due to the condition $B_\theta \ll B_z$ of tokamak equilibrium. The m = 2, n = 1 mode and the m = 0, n = 1 mode in (2.4) ~ (2.6) are excited by the nonlinear term $j \times b$. The second term in (2.4) ~ (2.6) means the residual small term of m = 1, n = 1 mode appeared as a result of small parameter expansion and the necessity of this term will be discussed in § 3. Furthermore we expand the growth rate

$$\omega = \omega_{(0)} + \epsilon^2\omega_{(2)} + \cdots \qquad (2.7)$$

where $\omega_{(0)}$ denotes the linear growth rate of m = 1, n = 1 instability and $\omega_{(2)}$ is the modification of growth rate due to the nonlinear effect. In the following, we assume plasma density is uniform over the plasma column.

i) In the lowest order of ϵ , from the linearized equations (2.2) , we obtain

$$\rho\omega_{(0)}^2\xi_{r1}^{(1)} = \left(-ikb_{r1}^{(1)} - \frac{db_{z1}^{(2)}}{dr}\right)B_z - \left(\frac{1}{r}\frac{d(rb_{\theta1}^{(1)})}{dr} - \frac{i}{r}b_{r1}^{(1)}\right)B_\theta$$
$$- J_z b_{\theta1}^{(1)} + \frac{dP_1^{(1)}}{dr} \qquad (2.8)$$

$$\rho\omega_{(c)}^{2}\,\xi_{\theta 1}^{(1)} = -i\left(\frac{b_{z1}^{(2)}}{r} + k b_{\theta 1}^{(1)}\right)B_z + J_z\, b_{r1}^{(1)} + \frac{1}{r}P_1^{(1)} , \qquad (2.9)$$

$$\rho\omega_{(c)}^{2}\,\xi_{z1}^{(2)} = -i\left(\frac{b_{z1}^{(2)}}{r} + k b_{\theta 1}^{(1)}\right)B_\theta - ikP_1^{(1)} , \qquad (2.10)$$

$$b_{r1}^{(1)} = iF_1\,\xi_{r1}^{(1)} , \qquad (2.11)$$

$$b_{\theta 1}^{(1)} = iF_1\,\xi_{\theta 1}^{(1)} - r\,\xi_{r1}^{(1)}\left(\frac{B_\theta}{r}\right)' , \qquad (2.12)$$

$$b_{z1}^{(2)} = iF_1\,\xi_{z1}^{(2)} \qquad (2.13)$$

and

$$\xi_{\theta 1}^{(1)} = i\frac{d}{dr}\left(r\,\xi_{r1}^{(1)}\right) , \qquad (2.14)$$

where

$$F_1 = \frac{B_\theta}{r} - kB_z .$$

From eqs. (2.8) $\sim$ (2.14), we can find the differential equation for $\xi_{r1}^{(1)}$,

$$\frac{d}{dr}\left\{r\left(F_1^2+\rho\omega_{(c)}^2\right)\frac{d}{dr}\left(r\,\xi_{r1}^{(1)}\right)\right\} - \left\{\rho\omega_{(c)}^2 + F_1^2 - 2B_\theta\left(\frac{B_\theta}{r}\right)' + 2r\frac{d}{dr}\left(\frac{B_\theta F_1}{r}\right)\right\}\xi_{r1}^{(1)} = 0 . \qquad (2.15)$$

The other components of the displacement vector are found from the above equations, after the solution of eq. (2.15) satisfying boundary conditions is obtained. It is noted that eq. (2.15) is also obtained from the equation derived by Freidberg[5]

$$\frac{d}{dr}\left[a\frac{d}{dr}(r\xi_r)\right]-b\xi_r=0 \quad , \qquad (2\text{-}16)$$

where

$$a=\frac{r(\rho\omega^2+F^2)}{m^2+k^2r^2}$$

and

$$b=\rho\omega^2+F^2-2B_\theta\left(\frac{B_\theta}{r}\right)'+2mr\left[\frac{FB_\theta}{r(m^2+k^2r^2)}\right]'-\frac{4k^2F^2B_\theta^2}{(k^2r^2+m^2)(\rho\omega^2+F^2)} \quad ,$$

under the ordering $kr \ll 1$ and $B_\theta \ll B_z$. We consider that eq. (2.16) is recovered, when the residual terms $\tilde{\xi}^{(3)}_{r1}$, $\tilde{\xi}^{(3)}_{\theta 1}$ and $\tilde{\xi}^{(4)}_{z1}$ are included in eq. (2.15).

ii) In the next order of ϵ, we take account of $b^{(1)}_{r1}$ and $b^{(1)}_{\theta 1}$ of perturbed magnetic field due to the growth of m = 1, n = 1 instability but neglect $b^{(2)}_z$. Following the assumption that $\mathbf{j}\times\mathbf{b}$ term produces source terms of m = 2, n = 1 mode, we derive equations for this mode. We expand eq. (2.1) by (2.3), (2.4) and find

$$\rho\omega_{(0)}^2\xi^{(2)}_{r2}=-\left(ikb^{(2)}_{r2}+\frac{db^{(3)}_{z2}}{dr}\right)B_z-\left(\frac{1}{r}\frac{d}{dr}(rb^{(2)}_{\theta 2})-\frac{2i}{r}b^{(2)}_{r2}\right)B_\theta$$

$$-J_zb^{(2)}_{\theta 2}+\frac{dP^{(2)}_1}{dr}+T^{(2)}_{r2} \quad , \qquad (2.17)$$

$$\rho\omega_{(0)}^2\xi^{(2)}_{\theta 2}=-i\left(\frac{2}{r}b^{(3)}_{z2}+kb^{(2)}_{\theta 2}\right)B_z+J_zb^{(2)}_{r2}+\frac{2i}{r}P^{(2)}_1+T^{(2)}_{\theta 2} \quad , \qquad (2.18)$$

$$\rho\omega_{(0)}^2 \xi_{z2}^{(3)} = i\left(\frac{2}{r}b_{z2}^{(3)} + kb_{\theta 2}^{(2)}\right)B_\theta - ikP_1^{(2)} + T_{z2}^{(3)} \,, \qquad (2.19)$$

where
$$T_{r2}^{(2)} = -\left(\frac{1}{r}\frac{d(rb_{\theta 1}^{(1)})}{dr} - \frac{i}{r}b_{r1}^{(1)}\right)b_{\theta 1}^{(1)} \,,$$

$$T_{\theta 2}^{(2)} = \left(\frac{1}{r}\frac{d(rb_{\theta 1}^{(1)})}{dr} - \frac{i}{r}b_{r1}^{(1)}\right)b_{r1}^{(1)} \,,$$

$$T_{z2}^{(3)} = \left(\frac{i}{r}b_{z1}^{(2)} + ikb_{\theta 1}^{(2)}\right)b_{\theta 1}^{(1)} - \left(-ikb_{r1}^{(2)} - \frac{db_{z1}^{(2)}}{dr}\right)b_{r1}^{(1)} \,.$$

Here $T_{r2}^{(2)}$, $T_{\theta 2}^{(2)}$ and $T_{z2}^{(3)}$ come from the nonlinear term $\mathbf{j} \times \mathbf{b}$ and have the phase factor $\exp[i(2\theta - kz)]$.

We also find

$$b_{r2}^{(2)} = i\left(\frac{2B_\theta}{r} - kB_z\right)\xi_{r2}^{(2)} + S_{r2}^{(2)} \,, \qquad (2.20)$$

$$b_{\theta 2}^{(2)} = i\left(\frac{2B_\theta}{r} - kB_z\right)\xi_{\theta 2}^{(2)} - r\xi_{r2}^{(2)}\left(\frac{B_\theta}{r}\right)' + S_{\theta 2}^{(2)} \,, \qquad (2.21)$$

$$b_{z2}^{(3)} = i\left(\frac{2B_\theta}{r} - kB_z\right)\xi_{z2}^{(3)} + S_{z2}^{(3)} \,, \qquad (2.22)$$

where
$$S_{r2}^{(2)} = -\frac{i}{r}\left(\xi_{r1}^{(1)}b_{\theta 1}^{(1)} - b_{r1}^{(1)}\xi_{\theta 1}^{(1)}\right) \,,$$

$$S_{\theta 2}^{(2)} = -\frac{d}{dr}\left(\xi_{r1}^{(1)}b_{\theta 1}^{(1)} - \xi_{\theta 1}^{(1)}b_{r1}^{(1)}\right) \,,$$

$$S_{z2}^{(3)} = \frac{1}{r}\frac{d}{dr}r\left(\xi_{z1}^{(2)}b_{r1}^{(1)} - \xi_{r1}^{(1)}b_{z1}^{(2)}\right) - \frac{i}{r}\left(\xi_{\theta 1}^{(1)}b_{z1}^{(2)} - \xi_{z1}^{(2)}b_{\theta 1}^{(1)}\right) \,.$$

Here $S_{r2}^{(2)}$, $S_{\theta 2}^{(2)}$ and $S_{z2}^{(3)}$ come from the nonlinear term $\boldsymbol{\xi} \times \mathbf{b}$ and has the phase factor $\exp[i(2\theta - kz)]$.

From incompressibility

$$\xi_{\theta 2}^{(2)} = \frac{i}{2}\frac{d}{dr}(r\xi_{r2}^{(2)}) \qquad (2.23)$$

is found. The differential equation for $\xi_{r2}^{(2)}$ is derived from eqs. (2.17) ~ (2.23),

$$\frac{d}{dr}\left\{\frac{r(F_2^2+\rho\omega_{(0)}^2)}{4}\frac{d}{dr}(r\xi_{r2}^{(2)})\right\} - \left\{\rho\omega_{(0)}^2 + F_2^2 - 2B_\theta\left(\frac{B_\theta}{r}\right)' + 2r\frac{d}{dr}\left(\frac{B_\theta F_2}{4r}\right)\right\}\xi_{r2}^{(2)}$$

$$= \frac{F_2}{2}\frac{d}{dr}(rS_{\theta 2}^{(2)}) - \frac{i}{2}\frac{d}{dr}(rT_{\theta 2}^{(2)}) + \frac{1}{r}\frac{d}{dr}(rB_\theta)S_{\theta 2}^{(2)} - iF_2S_{r2}^{(2)} - T_{r2}^{(2)}, \qquad (2.24)$$

where

$$F_2 = \frac{2B_\theta}{r} - kB_z.$$

The right hand side of eq. (2.24) comes from the nonlinear term and gives the source term for $\xi_{r2}^{(2)}$. The θ component of m = 2, n = 1 mode is obtained from (2.23).

iii) We note that assumption of incompressibility is invalid for the m = 0, n = 1 mode, because

$$\nabla\cdot\xi \simeq \frac{1}{r}\frac{d}{dr}(r\xi_{r0}^{(2)}) = 0 \qquad (2.25)$$

gives a singular solution at $r=0$. Thus we consider incompressibility for this mode. From the second equation of (2.1), we find

$$b_{r0}^{(2)} = -ik\xi_{r0}^{(2)}B_z, \qquad (2.26)$$

$$b_{\theta 0}^{(2)} = -ik\xi_{\theta 0}^{(2)}B_z - \frac{d\xi_{r0}^{(2)}}{dr}B_\theta - \frac{dB_\theta}{dr}\xi_{r0}^{(2)} + S_{\theta 0}^{(2)}, \qquad (2.27)$$

where

$$S_{\theta 0}^{\prime(2)} = -\frac{d}{dr}\left(\xi_{r1}^{(1)} b_{\theta 1}^{(1)} - \xi_{\theta 1}^{(1)} b_{r1}^{(1)}\right)$$

For the m = 0, n = 1 mode, $b_{r0}^{(2)}$ must be equal to zero to satisfy

$$\nabla \cdot \mathbf{b} \simeq \frac{1}{r}\frac{d}{dr}(r b_{r0}^{(2)}) = 0 \qquad (2.28)$$

from this and (2.26), $\xi_{r0}^{(2)} = 0$. The equation of MHD motion gives for $\xi_{\theta 0}^{(2)}$,

$$\rho \omega_{(0)}^2 \xi_{\theta 0}^{(2)} = -ik b_{\theta 0}^{(2)} B_z + \overline{T}_{\theta 0}^{(2)} \qquad (2.29)$$

where

$$\overline{T}_{\theta 0}^{(2)} = \left(\frac{1}{r}\frac{d(r b_{\theta 1}^{(1)})}{dr} - \frac{i}{r} b_{r1}^{(1)}\right) b_{r1}^{(1)}$$

Here $\overline{T}_{\theta 0}^{(2)}$ comes from the $\mathbf{j} \times \mathbf{b}$ term and has the phase factor $\exp(-ikz)$. Eqs. (2.27) and (2.29) give

$$\xi_{\theta 0}^{(2)} = \frac{\overline{T}_{\theta 0}^{(2)}}{\rho \omega_{(0)}^2 + k^2 B_z^2} \qquad (2.30)$$

The z component of m = 0, n = 1 mode is negligible within this ordering.

iv) Considering the nonlinear effect, we have shown that the m = 2, n = 1 mode and m = o, n = 1 mode are excited. These modes, however, are stable in the linear regime of stability. And they suppress the growth rate of m = 1,n = 1 instability. We can estimate the modification of growth rate, $\omega_{(2)}$, by use of (2.3). We write the first equation of (2.2) in the form,

$$\rho \omega^2 \xi = F(\xi) \qquad (2.31)$$

Multiplying eq. (2.31) by ξ^* and integrating over the plasma column, we obtain

$$\rho\omega^2\int|\xi|^2 d\mathbf{r} = \int\xi^*\cdot F(\xi)\, d\mathbf{r} \,. \tag{2.32}$$

The right hand side is related to the well-known potential energy in the MHD stability theory, $W(\xi,\xi)$. We substitute (2.4) ~ (2.6) into eq. (2.32) , perform integration over θ and z components and find

$$(\rho\omega_{(0)}^2 + 2\epsilon^2\rho\omega_{(0)}\omega_{(2)})\int|\xi_1|^2 r\,dr + \epsilon^2\rho\omega_{(0)}^2\int(|\xi_2|^2+|\xi_0|^2)\, r\,dr$$

$$= -W(\xi_1,\xi_1) - \epsilon^2 W(\xi_2,\xi_2) - \epsilon^2 W(\xi_0,\xi_0) \,. \tag{2.33}$$

Minimization of the growth rate $\omega_{(0)}$ by varying ξ_1 in the lowest order terms of (2.33) determines the growth rate and the gigen function, which are equivalent to those obtained in i). In the order of ϵ^2 , we find the modification of the growth rate $\omega_{(2)}$,

$$\omega_{(2)} = -\left\{W(\xi_2,\xi_2) + W(\xi_0,\xi_0) + \rho\omega_{(0)}^2\int(|\xi_2|^2+|\xi_0|^2)\, r\,dr\right\}\Big/ 2\rho\omega_0\int|\xi_1|^2 r\,dr. \tag{2.34}$$

It is easy to know $W(\xi_2,\xi_2)$ and $W(\xi_0,\xi_0)$ are positive definite for ξ_2 and ξ_0 excited by the nonlinear effect, because the m = 2, n = 1 mode and m = 0, n = 1 mode are stable in the linear stability theory. therefore $\omega_{(2)}$ is negative quantity.

§3 Application to equilibrium of uniform current

In this section we apply the method shown in § 2 to the tokamak equilibrium of uniform current profile with the fixed boundary. The internal m = 1 kink mode becomes unstable in this equilibrium configuration.[1] The equilibrium quantities are given by

$$\left.\begin{aligned} B_\theta &= \frac{r}{a} B_a , \\ B_z &= \text{const.}, \\ p &= \beta_p \left(1-\frac{r^2}{a^2}\right) . \end{aligned}\right\} \qquad (3.1)$$

For the fixed boundary, $\xi_r(a)=0$ and $\xi_r(0)$ is not singular .

We first take up the differential equation (2.15) . For (3.1), there is no solution satisfying the boundary conditions. This is due to that we neglect $\tilde{\xi}_{r1}^{(1)}$, $\tilde{\xi}_{\theta 1}^{(1)}$ and $\tilde{\xi}_{z1}^{(2)}$ in (2.15) . As we point out before, eq. (2.16) includes these higher order terms. Here eq. (2.16) takes the place of eq. (2.15) . The solution of eq. (2.16) for the eqiilibrium (3.1) is

$$\xi_{r1}^{(2)} = \xi_c \frac{a}{r} J_1\left(z\frac{r}{a}\right) ; \qquad (3.2)$$

where z is the zero point of the Bessel finction and has the relation

$$z = \frac{2|1-q(a)|}{\omega_{(0)}^2 \frac{\rho a^2}{B_a^2} + (1-q(a))^2} ka \qquad (3.3)$$

and

$$q(a) = kaB_z/B_a .$$

Here $\omega_{(0)} > 0$ corresponds to instability. We take the first zero point z = 3.83 hereafter. Using (3.3) , we can write the source term of the differential equation (2.24) or (2.30) by the Bessel functions. However we cannot obtain its solution by analytic calculations. Then we take an approximation to the Bessel function,

$$J_1(z\frac{r}{a}) \simeq x\,(1 - x^2/2 + x^4/12 - x^6/144) \qquad (3.4)$$

and we write the solution for m = 1, n = 1 mode

$$\xi_{r1}^{(1)} = \xi_0 a \frac{z}{2a}(1 - x^2/2 + x^4/12 - x^6/144) \qquad (3.5)$$

and

$$\xi_{\theta 1}^{(1)} = i\xi_0 a \frac{z}{2a}(1 - 3x^2/2 + 5x^4/12 - 7x^6/144), \qquad (3.6)$$

where

$$x = zr/2a.$$

To justify this approximation, we resort the property in the eigen-value problem. It is that an eigen value is obtained with sufficient accuracy by an eigen function with less accuracy.[6] From (3.4) we find $z \simeq 3.8$ for $J_1 = 0$. Fig. 1 shows the displacements given by (3.5) and (3.6). It is noted that $\xi_{r1}^{(1)}$ does not satisfy $\xi_{r1}^{(1)}(a) = 0$ exactly. For a uniform current model, we rewite the differential equation for m = 2, n = 1 mode,

$$\frac{d}{dr}\left[r\frac{d}{dr}(r\xi_{r2}^{(2)})\right] - 4\xi_{r2}^{(2)} = \frac{4}{\rho\omega_{(\omega)}^2 + F_2^2}\left\{-\frac{i}{2}\frac{d}{dr}(rT_{\theta 2}^{(2)}) - T_{r2}^{(3)}\right\}, \qquad (3.7)$$

because of $S_{r2}^{(2)} = S_{\theta 2}^{(2)} = 0$. From (3.5), (3.6) (2.11) and (2.12), we can derive

$$T_{r2}^{(2)} = \xi_0^2 a^2 F_1^2 \frac{z^3}{8a^3}(4x - 8x^3 + 5x^5 - 55x^7/36 + 19x^9/12 - 7x^{11}/432), \qquad (3.8)$$

and

$$T_{\theta 2}^{(1)} = i\xi_0^2 a^2 F_1^2 \frac{z^3}{8a^3}(4x - 4x^3 + 5x^5/3 - 3x^7/8 + x^9/24 - x^{11}/432). \quad (3.9)$$

From (3.8) and (3.9), the right hand side of eq. (3.7) is given by

$$\frac{4F_1^2}{\rho\omega_{(0)}^2 + F_2^2}\xi_0^2 a^2 \frac{z^3}{8a^3}\left(\frac{7}{36}x^7 - x^9/36 + x^{11}/432\right). \quad (3.10)$$

Then the general solution of eq. (3.7) is $\xi_{r2}^{(2)} \propto x$ and the particular solution of eq. (3.7) is

$$\frac{F_1^2}{\rho\omega_{(0)}^2 + F_2^2}\frac{\xi_0^2}{a}\frac{z^3}{864}\left(\frac{7x^7}{5} - x^9/8 - x^{11}/140\right). \quad (3.11)$$

The solution satisfying the boundary condition $\xi_{r2}^{(2)} = 0$ can be written,

$$\xi_{r2}^{(2)} = \frac{F_1^2}{\rho\omega_{(0)}^2 + F_2^2}\frac{\xi_0^2}{a}\frac{z^3}{864}\left\{7x(x^6 - x_a^6)/5 - x(x^8 - x_a^8)/8 + x(x^{10} - x_a^{10})/140\right\} \quad (3.12)$$

and

$$\xi_{\theta 2}^{(2)} = \frac{F_1^2}{\rho\omega_{(0)}^2 + F_2^2}\frac{\xi_0^2}{a}\frac{z^3}{864}\left\{7x(x^6 - x_a^6)/5 - x(5x^8 - x_a^8)/8 + x(6x^{10} - x_a^{10})/140\right\} \quad (3.13)$$

where

$x_a = z/2$.

From (3.9) and (2.30), we can write the m = 0, n = 1 mode,

$$\xi_{\theta 0}^{(2)} = \frac{iF_1^2}{\rho\omega_{(0)}^2 + k^2 B_z^2}\frac{\xi_0^2}{a}\frac{z^3}{8}(4x - 4x^3 + 5x^5/3 - 3x^7/8 + x^9/24 - x^{11}/432). \quad (3.14)$$

Fig. 2 shows the displacements given by (3.12), (3.13), (3.14).

Now we have obtained the m = 2, n = 1 mode and the m = 0, n = 1 mode doe to the nonlinear effect. We can estimate the modification of growth rate $\omega_{(2)}$ given by (2.34). In tne linear MHD stability theory, the potential energy can be written [7]

$$W(\xi,\xi)=\int_0^a r\,dr\left\{\frac{f}{r}\left(\frac{d\xi_r}{dr}\right)^2+\frac{g}{r}\xi_r^2+\gamma p(\nabla\cdot\xi)^2+\frac{m^2+k^2r^2}{r^2}\zeta^2\right\}, \quad (3.15)$$

where

$$f=\frac{r(mB_\theta-krB_z)^2}{m^2+k^2r^2},$$

$$g=\frac{2k^2r^2}{m^2+k^2r^2}\frac{dp}{dr}+\frac{1}{r}(mB_\theta-krB_z)^2\frac{m^2+k^2r^2-1}{m^2+k^2r^2}$$

$$+\frac{2k^2r}{(m^2+k^2r^2)^2}\left(k^2r^2B_z^2-m^2B_\theta^2\right),$$

and

$$\zeta=i\xi_\theta B_z-i\xi_zB_\theta+\frac{r}{m^2+k^2r^2}\left[(krB_\theta+mB_z)\frac{d\xi_r}{dr}+(mB_z-krB_\theta)\frac{\xi_r}{r}\right].$$

In (2.34), then, $W(\xi_2,\xi_2)$ and $W(\xi_0,\xi_0)$ are written in the following form,

$$W(\xi_2,\xi_2)=\int_0^a r\,dr\left\{\frac{r^2F_2^2}{4}\left(\frac{d\xi_{r2}^{(2)}}{dr}\right)^2+\frac{3}{4}F_2^2(\xi_{r2}^{(2)})^2+\frac{4B_z^2}{r^2}(\zeta_2)^2\right\}, \quad (3.16)$$

$$W(\xi_0,\xi_0)=\int_0^a r\,dr\,k^2B_z^2(\zeta_0)^2 \quad (3.17)$$

,

where $\zeta_2=i\xi_{\theta 2}^{(2)}+\frac{r}{2}\left(\frac{d\xi_{r2}^{(2)}}{dr}+\frac{\xi_{r2}^{(2)}}{r}\right)$

and $\zeta_0=i\xi_{\theta 0}^{(2)}$.

In (3.16), the largest contribution to the integral comes from the third term, because of $B_z^2 \sim O(\epsilon^{-2})$. The remaining terms has the order ϵ^0 and we can make an approximation,

$$W(\xi_2, \xi_2) + W(\xi_0, \xi_0) \simeq \int_0^a \frac{4B_z^2}{r}(\xi_2)^2 dr. \tag{3.18}$$

From (3.3), we can make an estimation for the growth rate of m =1, n = 1 instability,[1)]

$$\omega_{(0)}^2 \sim (ka)^2 \frac{B_a^2}{\rho a^2} \sim \epsilon^2 \frac{B_a^2}{\rho a^2} \sim \epsilon^4 \frac{B_z^2}{\rho a^2}. \tag{3.19}$$

From (2.34), (3.18) and (3.19),

$$\omega_{(2)} \simeq -\int_0^a \frac{4B_z^2}{r}(\xi_2)^2 dr \Big/ 2\rho\omega_{(0)} \int_0^a |\xi_1|^2 r dr \tag{3.20}$$

is found. From this the modification of growth rate can be esiimated

$$\omega_{(2)} \simeq \frac{B_z^2}{\rho a^2} \frac{F_1^4}{\omega_{(0)}(\rho\omega_{(0)}^2 + F_2^2)^2} \left(\frac{\xi_0}{a}\right)^2 \cdot G, \tag{3.21}$$

where G denotes the numerical factor by the spatial integration. In this exsample, we have $G \simeq 0.8$ for $z \simeq 3.8$. As

$F_1 \sim F_2$ and $\rho\omega_{(0)}^2 \ll F_2^2$,

$$\omega_{(2)} \sim -\frac{1}{\tau^2 \omega_{(0)}} \left(\frac{\xi_0}{a}\right)^2$$

is roughly valid, where $\tau = a \big/ \sqrt{B_z^2/\rho}$. From $\omega_{(0)} + \omega_{(2)} \sim 0$, the saturation amplitude of m = 1, n = 1 internal kink instability can be estimated,

$$\frac{\xi_0}{a} \sim \tau \omega_{(0)} \sim \epsilon^2$$

This saturation amplitude is very small but it is consistert with that obtained by Rosenbluth et.al., $\xi_0/a \sim \epsilon^2 \beta_p$, when $\beta_p \sim 1$.

§ 4 Discussion

It has been shown that the growth rate of m = 1, n = 1 internal kink instability is modified due to the nonlinear effect which we pick out. As the saturation amplitude is small, $\xi_0 \sim \epsilon^2 a$, the m = 1, n = 1 internal kink instability may not disrupt the plasma column.

In a tokamak plasma, we consider that the nonlinear effect coming from the axisymmetric part of the helical perturbation may be important in other MHD modes.

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Figure captions

Fig. 1 The m = 1, n = 1 internal kink mode in the plasma of uniform current with the fixed boundary: *a* shows

$$\xi_{r1}^{(1)} \propto 1 - x^2/2 + x^4/12 - x^6/144 ,$$

and *b* shows $\xi_{\theta 1}^{(1)} \propto 1 - 3x^2/2 + 5x^4/12 - 7x^6/144 .$

Fig. 2 The m = 2, n = 1 mode and the m = 0, n = 1 mode :

a shows $\xi_{r1}^{(2)} \propto 7x(x^6 - x_a^6)/5 - x(x^8 - x_a^8)/8 + x(x^{10} - x_a^{10})/140 ,$

b shows $\xi_{\theta 1}^{(2)} \propto 7x(4x^6 - x_a^6)/5 - x(5x^8 - x_a^8)/8 + x(6x^{10} - x_a^{10})/140 ,$

c shows $\xi_{\theta 0}^{(2)} \propto 108(4x - 4x^3 + 5x^5/3 - 3x^7/8 - x^9/24 - x^{11}/432) .$

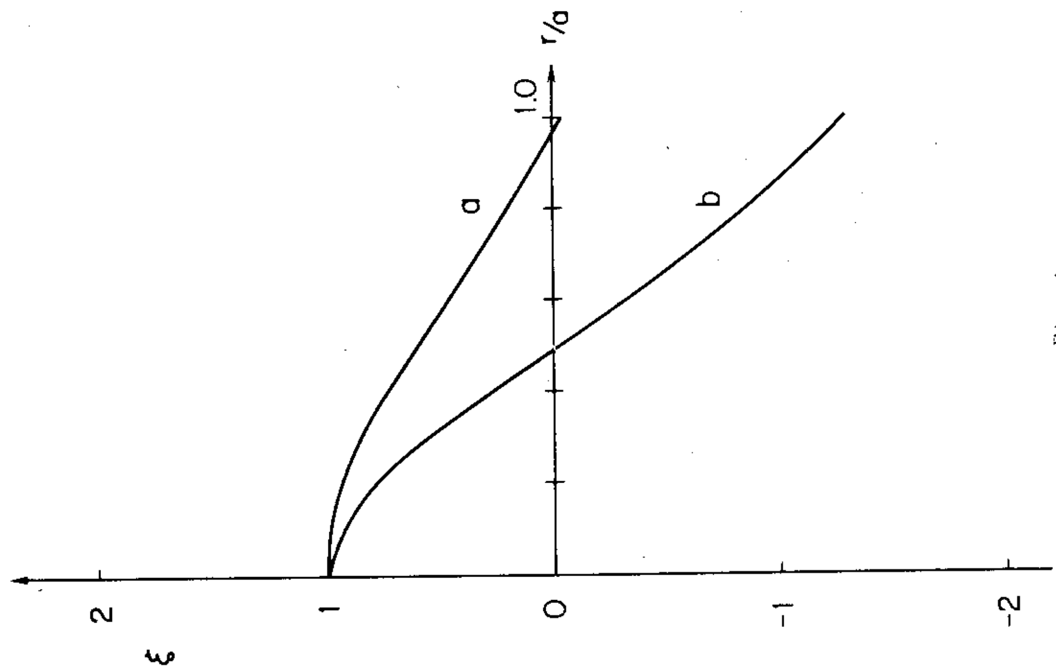

Fig. 1

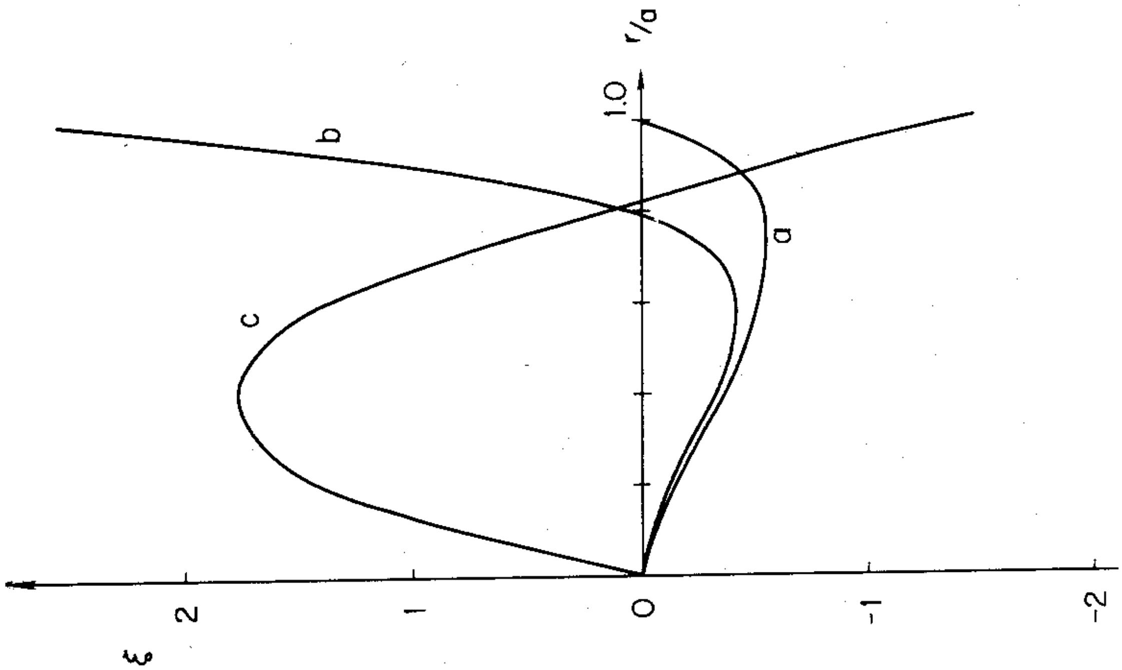

Fig. 2