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FORMALIZATION FOR OPTIMAL FEEDBACK CONTROL OF PLASMA CURRENT AND POSITION IN A TOKAMAK

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Formalization for Optimal Feedback Control of Plasma Current and Position in a Tokamak

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Simultaneous control of the plasma current and position in a tokamak is formalized to apply an optimal feedback rule. The optimal feedback rule is first explained. A physical model of the tokamak system is then modified to be combined with the feedback rule: A linearized state description is derived from circuit equations of plasma, air-core transformer coil and vertical field coil, with equilibrium equation. Possible extension of the method to a more complicated system is also described.

トカマクのプラズマ電流・位置の最適フィードバック制御のための定式化
（臨界プラズマ試験装置設計報告・XXXX）

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トカマク中のプラズマの電流値・位置をフィードバック制御する際に，最適フィードバック則を適用できるように，問題を定式化した。まず応用の観点から最適フィードバック則を概説した。つぎに，制御系への入力（コイル電圧）・出力（プラズマ電流，位置）間の関係が回路方程式および平衡の方程式で表わされることを示し，これらの方程式を線形化して最適フィードバック則が適用できる形を導いた。トカマクのポロイダルコイルとしては，空心変流器コイルと垂直磁場コイルのみを考慮に入れたが，より複雑なコイル系を持つトカマクへの応用のための拡張や，真空容器等の渦電流を考慮に入れるための拡張についてもふれた。

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1. Introduction

A few studies have been made on the control of the plasma position in a tokamak.[1)-3)] Most of them applied the classical control theory. Though optimal control model is formalized for the plasma position control of the ORMAK[3)], it does not suit to the real-time feedback control. However, recent control requirements come to point the application of the optimal feedback control theory. A reason is that the plasma current has also become a control object as well as the plasma position, and the two quantities, plasma current and position, are related each other. The system thus becomes multivariable system. The classical control theory cannot afford a design principle for such a multivariable problem. Recent developments of mini-computers are also helpful to the implementation of the optimal control law, because they have made it possible to carry out on-line real-time calculations of matrix algebra.

There exists, however, a border between plasma physics and control theory which hinders us from applicating the optimal control theory on our problem. This paper aims to translate physical requirements and constraints of the plasma current and position control into the control language. Once this crucial translation is accomplished, the optimal control theory will readily afford the details of the control system design.

In section 2 the rule of the optimal feedback control is briefly described. This section must be helpful in understanding an outline of the control process. Section 3 gives basic relations between the quantities to be controlled, plasma current and position, and related variables, coil current, voltage and plasma parameters. In section 4, the basic relations are modified and the rule in section 2 is applied. Section 5 contains discussion.

2. Optimal feedback rule

In this section, the optimal feedback control rule is outlined for plasma physisists. Those who are familiar to the optimal control rule can skip this section and have only to refer it just occasionary. All the procedures given here are available in the control literature[4]. The emphasis is on the application to our problem and no proofs are included.

The rule outlined does not actively take into consideration uncertainties of the control object, such as actuator errors, sensor errors, parameter errors and external disturbances. It is proved that the rule is optimal even in the existence of the uncertainties, in spite that it is constructed on the neglect of them.[4] The stochastic control theory can afford the design procedure taking the uncertainties into account more positively[4], but its description falls outside the scope of this paper.

We make further assumptions to simplify the treatment : 1) actuator dynamics are neglected ; 2) sensor dynamics are neglected. We denote the state vector by $x(t)=(x_1(t), x_2(t), ---, x_n(t))$, the control vector by $u(t)=(u_1(t), u_2(t), ---, u_m(t))$ and output vector by $y(t)=(y_1(t), y_2(t), ---, y_r(t))$. We start from the physical model which is described by a pair basic relations between the quantities to be controlled, plasma current and position, and related variables, coil current, voltage and plasma parameters. In section 4, the basic relations are modified and the rule in section 2 is applied. Section 5 contains discussion.

2. Optimal feedback rule

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$$f(\dot{x}(t),\ x(t),\ u(t))=0, \tag{1}$$

$$y(t)=g(x(t)). \tag{2}$$

where "·" denotes the operation d/dt.

We interpret the control problem as to find u(t) and the resultant x(t) which minimize a scalar-valued cost function appropriately selected. Several techniques are available to solve the problem by off-line computations. The representatives are the Pontryagin maximum principle[11] and the dynamic programming.[12]

We assume that we have somehow obtained the off-line solution and denote them by $u_0(t)$ and its resultant by $x_0(t)$ and $y_0(t)$. In general, we cannot expect the application of $u(t)=u_0(t)$ leads to the results $x(t)=x_0(t)$ and $y(t)=y_0(t)$ in the real system. The reasons are that : 1) $x_0(t)$ and $y_0(t)$ were computed using a mathematical model of the physical process, which was arrived at through some approximations ; 2) the values of the parameters used in the mathematical model are nominal ones and the true values may be different ; 3) the actual initial condition $x(t_0)$ may be different from the assumed one $x_0(t_0)$; 4) the uncertainties are not taken into consideration.

It then follows that errors in the model may by themselves contribute to deviate the true state x(t) from the nominal one $x_0(t)$ so that small initial deviations may get larger and larger as time goes on. So the actual input $u(t)$ must be different from the precomputed input $u_0(t)$. The control correction $\delta u(t)=u(t)-u_0(t)$ should be calculated based on the state perturbation vector $\delta x(t)=x(t)-x_0(t)$ and the output perturbation vector $\delta y(t)=y(t)-y_0(t)$. The control problem then becomes a problem to find the control correction $\delta u(t)$ in real-time operation.

To solve the problem we have to know the relationship between $\delta x(t)$, $\delta y(t)$ and $\delta u(t)$. We again start from the eqs. (1) and (2) which are assumed to give the relationship between true quantities; $x(t)$, $y(t)$ and $u(t)$, because we have no other ways. Let us remember nominal quantities are related by

$$f(\dot{x}_0(t),\ x_0(t),\ u_0(t))=0, \tag{3}$$

$$y_0(t)=g(x_0(t)). \tag{4}$$

Expanding $f(\dot{x}(t),\ x(t),\ u(t))$ and $g(x(t))$ about $\dot{x}_0(t)$, $x_0(t)$ and $u_0(t)$ in the Taylor series, we obtain

$$\begin{aligned} &f(\dot{x}(t),\ x(t),\ u(t)) \\ &\sim f(\dot{x}_0(t),\ x_0(t),\ u_0(t)) \\ &+\left[\frac{\partial f}{\partial \dot{x}}\right]_0 \delta\dot{x}(t)+\left[\frac{\partial f}{\partial x}\right]_0 \delta x(t)+\left[\frac{\partial f}{\partial u}\right]_0 \delta u(t), \end{aligned} \tag{5}$$

$$g(x(t)) \sim g(x_0(t))+\left[\frac{\partial g}{\partial x}\right]_0 \delta x(t), \tag{6}$$

where

$$\delta x(t)=x(t)-x_0(t) \ ,$$
$$\delta y(t)=y(t)-y_0(t) \ , \qquad (7)$$
$$\delta u(t)=u(t)-u_0(t) \ .$$

and $[\ \ ]_0$ denotes a differential coefficient at $(\dot{x}_0(t),\ x_0(t),\ u_0(t))$

Defining n×n matrices D(t) and E(t) by

$$D(t)=-[\frac{\partial f}{\partial \dot{x}}]_0 \ , \qquad (8)$$

$$E(t)=[\frac{\partial f}{\partial x}]_0 \ , \qquad (9)$$

and an n×m matrix H(t) by

$$H(t)=[\frac{\partial f}{\partial u}]_0 \ , \qquad (10)$$

we have

$$D(t)\delta\dot{x}(t)=E(t)\delta x(t)+H(t)\delta u(t). \qquad (11)$$

If D(t) is regular, we can effect $D^{-1}(t)$ on both sides of eq. (11) to obtain

$$\delta\dot{x}(t)=A(t)\delta x(t)+B(t)\delta u(t) \ , \qquad (12)$$

where $A(t)=D^{-1}(t)E(t)$ and $B(t)=D^{-1}(t)H(t)$ are n×n and n×m matrices, respectively. Treatment of the case where D(t) is not regular is discussed in ref. 13.

Similarly, defining an r×n matrix C(t) by

$$C(t)=\left[\frac{\partial g}{\partial x}\right]_0 \quad , \tag{13}$$

we have

$$\delta y(t)=C(t)\,\delta x(t) \quad . \tag{14}$$

We assume rank C=n in the following treatment.
The pair of equations (12) and (14) is a standard state description of a linear time-varying system.

In estimating the performance of the control, we employ three quantities : final state, integrated squared error and integrated squared transient responce. These correspond to the first term, the first and the second terms in the integral of the following equation, respectively:

$$J=\delta x'(t_1)F\delta x(t_1) + \int_{t_0}^{t_1} [\delta x'(t)Q(t)\delta x(t)+\delta u'(t)R(t)\delta u(t)]dt, \tag{15}$$

where $F=F' \geq 0$ (an n×n matrix), $Q(t)=Q'(t) \geq 0$ for $t\in[t_0, t_1]$ (an n×n matrix) and $R(t)=R'(t) \geq 0$ for $t\in[t_0, t_1]$ (an m×n matrix), and "'" denotes transposing operation.

We have now arrived at the following mathematical optimization problem : Given the system described by eq.(12) and (14) and given a fixed time interval $t\in[t_0, t_1]$, find the control correction $\delta u(t)$ for $t \in [t_0, t_1]$ such that the quadratic cost function eq.(15) is minimized.

It is known that the optimal control correction vector is related to the state perturbation vector by the linear time-varying feedback relationship:

$$\begin{aligned}\delta u &= -G(t)\delta x(t)\\ &= -G(t)C^{-1}(t)\delta y(t) \quad , \end{aligned} \tag{16}$$

where G(t) is an m×n time varying control gain matrix. The value of G(t) is given by

$$G(t)=R^{-1}(t)B'(t)K(t) \quad , \tag{17}$$

where the n×n matrix K(t) is the solution of the Riccati matrix differential equation:

$$\begin{aligned}\frac{d}{dt}K(t) = &-K(t)A(t)-A'(t)K(t)-Q(t)\\ &+K(t)B(t)R^{-1}(t)B'(t)K(t) \quad , \end{aligned} \tag{18}$$

which subject to the boundary condition at $t=t_1$:

$$K(t_1)=F \quad . \tag{19}$$

If the system is time-invariant or stationary, A, B, C becomes constant matrices. Equation (11) becomes

$$D\delta\dot{x}(t)=E\delta x(t)+H\delta u(t) \quad . \tag{20}$$

The standard state description is given by:

$$\delta\dot{x}(t)=A\delta x(t)+B\delta u(t) \quad , \tag{21}$$

$$\delta y(t)=C\delta x(t) \quad . \tag{22}$$

The cost function is

$$J=\int_{t_0}^{\infty}(\delta x'(t)Q\delta x(t)+\delta u'(t)R\delta u(t))dt \ , \tag{23}$$

where Q and R are constant matrices. The optimal control correction vector is

$$\begin{aligned}\delta u(t)&= -G\delta x(t)\\ &= -GC^{-1}\delta y(t) \quad ,\end{aligned} \tag{24}$$

where

$$G=R^{-1}BK \quad , \tag{25}$$

and K is the solution of algebraic matrix Riccati equation:

$$-KA-A'K-Q+KBR^{-1}B'K=0 \ . \tag{26}$$

A time-invariant system is controllable and observable when eqs.(27) and (28) are applicable, respectively:

$$\text{rank } [B,\ AB,\cdots,\ A^{n-1}B]=n \quad , \tag{27}$$

$$\text{rank}\ [C', A'C', \cdots, A'^{(n-1)}C'] = n \quad . \tag{28}$$

It is verified that there exists the unique stable solution of eq.(26) under the conditions given by eq.(27) and (28).[4)] A few computer algorithm are available to solve eq.(26) by off-line computers.[7)] Controllability, observability and stability of a time-varying system are discussed in ref. 4).

Let us summarize the calculations necessary in the design stage : 1) We start from a physical model given in eqs.(1) and (2) ; 2) we determine the functions $u_0(t)$ and $x_0(t)$ by selecting the cost function and solving the resultant optimal control problem ; 3) we obtain the standard state description of the functions $\delta x(t)$, $\delta y(t)$, $\delta u(t)$ (eqs.(12) and (14) or (21) and (22)) by computing the matrices $A(t)$, $B(t)$ and $C(t)$; 4) We select the weighting matrices $Q(t)$, $R(t)$ and F to define the cost function ; 5) we solve the Riccati equation (18) or (26) to obtain $G(t)$ or $-G(t)C^{-1}(t)$ defined by eq.(17) or (25).

Necessary real-time calculations are as follows : 1) Measure the true output $y(t)$ and obtain $\delta y(t)$; 2) Compute $-G(t)C^{-1}(t)\delta y(t)$; only a matrix-vector multiplication is required in real time, because $-G(t)C^{-1}(t)$ has been precomputed in the design stage. 3) Compute the true control output $u(t)=u_0(t)+\delta u(t)$.

It is clear in the preceding description that, once the standard state description is made, the design can proceed straightforward. The following sections aim derivation of the standard state description for the control of the plasma current and position, starting from the construction of a physical model.

3. Physical model

The first step necessary in the formalization of our control problem is to obtain eqs.(1) and (2). We follow the procedure given in ref. 5 in this step, introducing some simplifications just to make understanding of the outline of the formalization intuitive.

Following the model in ref. 5, we regard a tokamak as a transformer with the secondary circuit of a plasma to describe the plasma behavior by equivalent electrical circuit equations. We take into account only the horizontal plasma displacement Δ and neglect the vertical displacement.

We take into account the following tokamak components, which are identified by the subscripts given in the respective parentheses : plasma (p), ohmic heating coil with air core (f), vertical field coil (v). These components are assumed to be located on the toroidal surface whose major and minor radii are denoted by R and r with their subscripts, respectively. Effects caused by the toroidal coil and the vacuum chamber are neglected.

We now write circuit equations for the tokamak components: For the plasma channel,

$$\frac{d}{dt}(L_p(t)I_p(t))+M_{fp}\frac{dI_f(t)}{dt}+\frac{d}{dt}(M_{pv}(t)I_v(t))$$

$$+\eta_p(t)I_p(t)=0, \qquad (29)$$

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$$+\eta_p(t)I_p(t)=0, \qquad (29)$$

for the ohmic heating coil with air core,

$$M_{fp}\frac{dI_p(t)}{dt}+L_f\frac{dI_f(t)}{dt}+\eta_f I_f(t)-V_f(t)=0\ , \tag{30}$$

and for the vertical field coil,

$$\frac{d}{dt}(M_{pv}(t)I_p(t))+L_v\frac{dI_v(t)}{dt}+\eta_v I_v(t)-V_v(t)=0\ , \tag{31}$$

In the above expressions, I is current, V is applied voltage on the coils, L and M are self and mutual inductances, and η is registance. Inductances are expressed as follows in the first order approximation of the expansion in the inverse aspect ratio $\varepsilon=r/R$:[5)]

$$L_p(t)=\mu_o R_p(t)(\ell n\frac{8R_p(t)}{r_p(t)}+\frac{\ell_i(t)}{2}-2)\ , \tag{32}$$

$$L_v=\frac{\mu_o\pi^2 R_v N_v^2}{4}\ , \tag{33}$$

$$L_f=\mu_o R_f N_f^2\ (\ell n\frac{8R_f}{r_f}-2)\ , \tag{34}$$

$$M_{pv}(t)=\frac{\mu_o\pi r_v N_v}{4}\{\ell n\frac{8R_v}{r_v}-1+\frac{r_p^2(t)}{r_v^2}(\Lambda_1(t)+\frac{1}{2})+\frac{2R_v}{r_v^2}\Delta(t)\}\ , \tag{35}$$

$$M_{pf}=\mu_o R_f N_f(\ell n\frac{8R_f}{r_f}-2)\ , \tag{36}$$

where $\Delta(t)=R_p(t)-R_v$ is the horizontal plasma displacement, N is turn number of a coil and

$$\Lambda_1(t)=\beta_p(t)+\frac{\ell_i(t)}{2}-1 \quad , \tag{37}$$

where β_p is the poloidal beta and ℓ_i is the internal inductance.

The equation which governs the horizontal plasma motion is given by Mukhovatov and Shafranov:[6)]

$$\frac{\mu_o I_p(t)}{2}\left[\ell n\frac{8R_p(t)}{r_p(t)}+\Lambda_1(t)-\frac{1}{2}\right)-2\pi R_p(t)B_z(t)=0, \tag{38}$$

The vertical field B_z is related to I_v by the following equation:

$$B_z(t)=\nu_v I_v(t) \quad , \tag{39}$$

where

$$\nu_v=\frac{\mu_o N_v}{4r_v} \quad .$$

Current in the plasma and coils are readily detected by the Rogowskii coil and current detectors. Magnetic probes are often used to detect the plasma position. After integration, the magnetic probe signal becomes[6)]

$$H_\omega(t)=\frac{\mu_o I_p(t)}{2\pi r_m}$$
$$+(\cos\ \omega)\left\{\frac{1}{2\pi R_v}\left(C_2(t)-\frac{C_1(t)}{r_m^2}\right)-\frac{\mu_o I_p(t)}{4\pi R_v}\ell n\frac{8R_v}{r_m}\right\}, \tag{40}$$

where

$$C_1(t)=-\frac{\mu_o I_p(t)}{2} r_p^2(t)\left(\frac{2R_v\Delta(t)}{r_p^2(t)}+\Lambda_1(t)+\frac{1}{2}\right) ,$$

$$C_2(t)=\frac{\mu_o I_p(t)}{2}\left(\ell n\frac{8R_v}{r_p(t)}+\Lambda_1(t)-\frac{1}{2}\right) ,$$

where ω is the poloidal angle of the probe position, and r_m is the minor radiusof the probe position. The difference signal of probes at $\omega=0$ and $\omega=\pi$ becomes

$$\begin{aligned}U(t)&=H_o(t)-H_\pi(t)\\&=-\frac{\mu_o I_p(t)}{2\pi R_v}\ell n\frac{8R_v}{r_m}+\frac{1}{\pi R_v}\left(C_2(t)-\frac{C_1(t)}{r_m^2}\right)\\&=-\frac{\mu_o I_p(t)}{2\pi R_v}\ell n\frac{8R_v}{r_m}\\&+\frac{1}{\pi R_v}\left\{\frac{\mu_o I_p(t)}{2}\left(\ell n\frac{8R_v}{r_p(t)}+\Lambda_1(t)-\frac{1}{2}\right)\right.\\&\left.-\frac{\mu_o I_p(t)}{2}\,\frac{r_p^2(t)}{r_m^2}\left(\frac{2R_v\Delta(t)}{r_p^2(t)}+\Lambda_1(t)+\frac{1}{2}\right)\right\} ,\end{aligned} \qquad (41)$$

Equations (29) to (31), and (38) give the state equation and eq.(41) gives one of the output equations of our case, which correspond eqs.(1) and (2), respectively. Control parameters in those equations are $V_f(t)$ and $V_v(t)$, so we define

$$u'(t)=(u_1(t),\ u_2(t))=(V_f(t),\ V_v(t)). \qquad (42)$$

Variables in those equations are I_p, I_v, I_f, R_p, η_p, Λ_1, ℓ_i, r_p and B_z. Among them, R_p, η_p, Λ_1, ℓ_i and r_p characterize the plasma. R_p and I_p are quantities to be controlled. There are two ways in the treatment of the rest quantities, η_p, Λ_1, ℓ_i and r_p. The first is to regard them as time-varying parameters. The matrices A(t) and B(t) in the state equation (12) contain the parameters in this case. The second is to adopt them as one of the state variables or components of the state vector x in eq.(12). Henceforth, we select a compromizing way ; we regard η_p and r_p as parameters and Λ_1 and ℓ_i as state variables.

The adoption of Λ_1 is due to practical reasons: The plasma position depends on Λ_1 according to eq.(38) and, in addition, we can detect the magnetic probe signal which is a function of Λ_1. The quantity ℓ_i is included in Λ_1, so we introduce an assumption $\delta\ell_i \propto \delta\Lambda_1$ in their perturbations, which simplifies the following treatment. Contrary to Λ_1, η_p has little influence on our problem, at least explicitly. So we introduce another state equation in addition to eqs.(29) to (31) and (38):

$$\frac{d\Lambda_1(t)}{dt} + \frac{\Lambda_1(t)-\Lambda_{10}(t)}{\tau_\Lambda(t)} = 0. \tag{43}$$

Here $\tau_\Lambda(t)$ is the time constant of Λ_1, which is introduced for convinience sake. Discussion on $\tau_\Lambda(t)$ is given in section 5.

Let us examine the nature of our state variables ; I_p, I_f, I_v, R_p, Λ_1, and B_z. Because eq.(39) tells us B_z depends on I_v, we can get rid of B_z from the state variables. Equation

(38) tells I_p, R_p, Λ_1 and B_z(or I_v) are dependent each other, so three of the four can be independent variables. We take a standpoint that the plasma position is a function of I_p, B_z(or I_v) and Λ_1 so that we recognize I_p, I_v(instead of B_z) and Λ_1 as independent variables.

In short, we adopt the state vector

$$x'(t)=(x_1(t),\cdots,\ x_4(t))$$
$$=(I_p(t),\ I_f(t),\ I_v(t),\ \Lambda_1(t)), \tag{44}$$

and the output vector

$$y'(t)=(y_1(t),\cdots,\ y_4(t))$$
$$=(I_p(t),\ I_f(t),\ I_v(t),\ U(t)). \tag{45}$$

where U(t) is given by eq.(42).

4. Formalization

We start this section assuming we have obtained somehow the nominal time variation of the control vector $u_0(t)$ by off-line computations. One method to obtain $u_0(t)$ in our problem of controlling the plasma current and position is found in ref. 5. The next step is to deduce the standard state description of the system for the following perturbations

$$\delta x'(t)=(\delta x_1(t),\cdots,\ \delta x_4(t))$$
$$=(\delta I_p(t),\ \delta I_f(t),\ \delta I_v(t),\ \delta\Lambda_1(t)), \tag{46}$$

(38) tells I_p, R_p, Λ_1 and B_z(or I_v) are dependent each other, so three of the four can be independent variables. We take a standpoint that the plasma position is a function of I_p, B_z(or I_v) and Λ_1 so that we recognize I_p, I_v(instead of B_z) and Λ_1 as independent variables.

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$$\delta y'(t) = (\delta y_1(t), \cdots, \delta y_4(t))$$
$$= (\delta I_p(t),\ \delta I_f(t),\ \delta I_v(t),\ \delta U(t)), \qquad (47)$$

$$\delta u'(t) = (\delta u_1(t),\ \delta u_2(t))$$
$$= (\delta V_f(t),\ \delta V_v(t)). \qquad (48)$$

Discharge in a tokamak is of two stages : the first is the plasma current rising stage where the plasma radius $r_p(t)$ and plasma resistance $\eta_p(t)$ are time varying. We have to take into account the time variation of such parameters and derive the state description of the form given in eqs.(12) and (14). The second is the plasma current flat-top stage where an equilibrium is maintained. In other words the system is stationary. The parameters r_p, η_p are time-invariant to reduce the state description to the form given in eqs.(21) and (22). In the following we restrict our control object in this second equilibrium stage.

The procedure of the deduction is given in section 2. Applying eq.(5) on eqs.(29) to (31) and eq.(43), we have

$$L_{po}\frac{d\delta I_p(t)}{dt} + I_{po}\frac{d\delta L_p(t)}{dt} + M_{fp}\frac{d\delta I_f(t)}{dt} + M_{pvo}\frac{d\delta I_v(t)}{dt}$$
$$+ I_{vo}\frac{d\delta M_{pv}(t)}{dt} + \eta_p \delta I_p(t) = 0 \quad , \qquad (49)$$

$$M_{fp}\frac{d\delta I_p(t)}{dt} + L_f\frac{d\delta I_f(t)}{dt} + \eta_f \delta I_f(t) - \delta V_f(t) = 0 \quad , \qquad (50)$$

$$M_{pvo}\frac{d\delta I_p(t)}{dt}+I_{po}\frac{d\delta M_{pv}(t)}{dt}+L_v\frac{d\delta I_v(t)}{dt}$$

$$+\eta_v\delta I_v(t)-\delta V_v(t)=0 \quad , \tag{51}$$

$$\frac{d\delta\Lambda_1}{dt}+\frac{\delta\Lambda_1}{\tau_\Lambda}=0 \quad . \tag{52}$$

The equilibrium equation becomes

$$\Delta(t)=\frac{\mu_o}{4\pi\nu I_{vo}}(\Lambda_o\delta I_p(t)+I_{po}\delta\Lambda_1(t)-\frac{I_{po}}{I_{vo}}\Lambda_o\delta I_v(t)), \tag{53}$$

where $\Lambda_o=\ell n(8R_v/r_p)+\Lambda_{10}-\frac{1}{2}$. This gives the explicit dependence of the plasma displacement on $\delta I_p(t)$, $\delta\Lambda_1(t)$ and $\delta I_v(t)$. As $\ell_i(t)$ is related to $\Lambda_1(t)$ by eq.(37), we make an assumption below to simplify the treatment:

$$\delta\ell_i(t)=2\alpha\delta\Lambda_1(t) \quad , \tag{54}$$

where α is a constant. Substitution of eqs.(53) and (54) into eq.(32) leads us to the expression:

$$L_p(t)=\mu_o(R_v+\Delta(t))(\ell n\frac{8R_p(t)}{r_p}+\frac{\ell_{io}+\delta\ell_i(t)}{2}-2)$$

$$=\mu_o(R_v+\Delta(t))(\Gamma_o+\alpha\delta\Lambda_1(t))$$

$$\sim\mu_o\{\Gamma_oR_v+(\alpha R_v+\frac{\mu_o\Gamma_oI_{po}}{4\pi\nu I_{vo}}\delta\Lambda_1(t)$$

$$+\frac{\mu_o\Gamma_o\Lambda_o}{4\pi\nu I_{vo}}\delta I_p(t)-\frac{\mu_o\Gamma_o\Lambda_o I_{po}}{4\pi\nu I_{vo}^2}\delta I_v(t)\} \ , \tag{55}$$

where

$$\Gamma_o=\ell n(\frac{8R_v}{r_p})+\frac{\ell_{io}}{2}-2 \ . \tag{56}$$

Similarly, eq.(35) becomes

$$M_{pv}=M_o+M_{o1}\delta\Lambda_1(t)+M_{o2}\frac{\Delta(t)}{r_v}$$

$$=M_o+(M_{o1}+\frac{\mu_o M_{o2} I_{po}}{4\pi\nu r_v I_{vo}})\delta\Lambda_1(t)$$

$$+\frac{\mu_o M_{o2}\Lambda_o}{4\pi\nu r_v I_{vo}}\delta I_p(t)-\frac{\mu_o M_{o2}\Lambda_o I_{po}}{4\pi\nu r_v I_{vo}^2}\delta I_v(t) \ , \tag{57}$$

where

$$M_o=\frac{\mu_o\pi r_v N_v}{4}\{\ell n\frac{8R_v}{r_v}-1+\frac{r_p^2}{r_v^2}(\Lambda_{1o}+\frac{1}{2})\} \ ,$$

$$M_{o1}=\frac{\mu_o\pi r_p^2 N_v}{4r_v} \ , \qquad M_{o2}=\frac{\mu_o\pi R_v}{2} \ . \tag{58}$$

Substituting eqs.(55) and (57) into eqs.(49) to (51), we have the state description of the form eq.(20). These matrices are given in eqs.(59) to (61):

$$D=(d_{ij}), \quad (i=1,\ldots, 4, \ j=1,\ldots, 4) \tag{59}$$

$$d_{11}=\mu_o\Gamma_o R_o+\frac{\mu_o\Lambda_o}{4\pi\nu}(\frac{\mu_0\Gamma_0 I_{po}}{I_{vo}}+\frac{M_{o2}}{r_v}) \ ,$$

$d_{12}=M_{fp}$,

$$d_{13}=M_o-\frac{\mu_o\Lambda_oI_{po}}{4\pi\nu I_{vo}}\left(\frac{\mu_o\Gamma_oI_{po}}{I_{vo}}+\frac{M_{o2}}{r_v}\right) ,$$

$$d_{14}=\mu_oI_{po}\left(\alpha R_o+\frac{\mu_o\Gamma_oI_{po}}{4\pi\nu I_{vo}}\right)+I_{vo}\left(M_{o1}+\frac{\mu_oM_{o2}I_{po}}{4\pi\nu r_vI_{vo}}\right) ,$$

$d_{21}=M_{fp}$,

$d_{22}=L_f$,

$d_{23}=0$,

$d_{24}=0$,

$$d_{31}=M_o+\frac{\mu_oM_{02}\Lambda_oI_{po}}{4\pi\nu r_vI_{vo}} ,$$

$d_{32}=0$,

$$d_{33}=L_v-\frac{\mu_oM_{o2}\Lambda_oI_{po}^2}{4\pi\nu r_vI_{vo}^2} .$$

$$d_{34}=I_{po}\left(M_{o1}+\frac{\mu_oM_{o2}I_{po}}{4\pi\nu r_vI_{vo}}\right) ,$$

$d_{41}=0$,

$d_{42}=0$,

$d_{43}=0$,

$d_{44}=1$,

$$E=\begin{pmatrix} -\eta_p & & & \\ & -\eta_f & & 0 \\ 0 & & -\eta_v & \\ & & & -\frac{1}{\tau_\Lambda} \end{pmatrix} \qquad (60)$$

$$H=\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{61}$$

Next, we modify the output equation given in eq.(41). Substituting eq.(53), we have

$$\begin{aligned} U(t) &= U_o + \delta U(t) \\ &= U_o + \delta I_p(t)\frac{\mu_o}{2\pi R_v}\Big[\ell n\frac{8R_v}{r_p} + \Lambda_{10} - \frac{1}{2} \\ &\quad - \ell n\frac{8R_v}{r_m} - \frac{r_p^2}{r_m^2}(\Lambda_{10} + \frac{1}{2})\Big] \\ &\quad + \delta\Lambda_1(t)\frac{\mu_o I_{po}}{2\pi R_v}(1 - \frac{r_p^2}{r_m^2}) - \Delta(t)\frac{\mu_o I_{po}}{\pi r_m^2} \\ &= U_o + \delta I_p(t)\frac{\mu_o}{2\pi R_o}\Big[\ell n\frac{8R_v}{r_p} + \Lambda_{10} - \frac{1}{2} - \ell n\frac{8R_v}{r_m} \\ &\quad - \frac{r_p^2}{r_m^2}(\Lambda_{10} + \frac{1}{2}) - \frac{\mu_o R_v I_{po}\Lambda_o}{2\pi r_m^2 \nu I_{vo}}\Big] \\ &\quad + \delta I_v(t)\frac{\mu_o^2 I_{po}\Lambda_o}{4\pi^2 r_m^2 \nu I_{vo}^2} \\ &\quad + \delta\Lambda_1(t)\frac{\mu_o I_{po}}{2\pi R_v}\Big[1 - \frac{r_p^2}{r_m^2} - \frac{\mu_o R_v I_{po}}{2\pi r_m^2 \nu I_{vo}}\Big] \quad , \end{aligned} \tag{62}$$

where

$$\begin{aligned} U_o &= -\frac{\mu_o I_{po}}{2\pi R_v}\ell n\frac{8R_v}{r_m} + \frac{1}{\pi R_v}\Big\{\frac{\mu_o I_{po}}{2}(\ell n\frac{8R_v}{r_p} + \Lambda_{1o} - \frac{1}{2}) \\ &\quad - \frac{\mu_o I_{po}}{2}\frac{r_p^2}{r_m^2}(\Lambda_{1o} + \frac{1}{2})\Big\} \quad . \end{aligned} \tag{63}$$

We can now make the relation between δx and δy in the form of eq.(22), where C is given by eq.(64):

$$C=(c_{ij}), \qquad (i=1,\ldots,4,\ j=1,\ldots,4) \tag{64}$$

$$c_{11}=c_{22}=c_{33}=1 \quad ,$$
$$c_{12}=c_{13}=c_{14}=0 \quad ,$$
$$c_{21}=c_{23}=c_{24}=0 \quad ,$$
$$c_{31}=c_{32}=c_{34}=0 \quad ,$$

$$c_{41}=\frac{\mu_o}{2\pi R_o}\Big[\ell n\frac{8R_v}{r_p}+\Lambda_{o1}-\frac{1}{2}-\ell n\frac{8R_v}{r_m} -\frac{r_p^2}{r_m^2}(\Lambda_{o1}+\frac{1}{2})-\frac{\mu_o R_v I_{po}\Lambda_o}{2\pi r_m^2 \nu I_{vo}}\Big] \quad ,$$

$$c_{42}=0 \quad ,$$

$$c_{43}=\frac{\mu_o I_{po}\Lambda_o}{4\pi^2 r_m^2 \nu I_{vo}^2} \quad ,$$

$$c_{44}=\frac{\mu_o I_{po}}{2\pi R_v}\Big[1-\frac{r_p^2}{r_m^2}-\frac{\mu_o R_v I_{po}}{2\pi r_m^2 \nu I_{vo}}\Big] \quad .$$

Now that we have arrived at the standard state description, eqs.(21) and (22), the next step to be taken is the selection of the cost function given in eq.(23). The first term in the integral of eq.(23) should express the squared error of the state. Remembering our control object is to keep $R_p=R_v$ and $I_p=I_{po}$, or $\Delta=0$ and $\delta I_p=0$, we write

$$\delta x'(t)Q\delta x(t)=q_1\Delta^2(t)+q_2\delta I_p^2(t) \quad . \tag{65}$$

We square eq.(53) and express in the following form:

$$\Delta^2(t)=\delta x'(t)Q\delta x(t) \quad , \tag{66}$$

where

$$Q_1=\left(\frac{\mu_o}{4\pi\nu I_{vo}}\right)^2\begin{bmatrix} \Lambda_o^2 & 0 & \Lambda_o I_{po} & \frac{I_{po}\Lambda_o^2}{I_{vo}} \\ 0 & 0 & 0 & 0 \\ \Lambda_o I_{po} & 0 & I_{po}^2 & \frac{I_{po}^2\Lambda_o}{I_{vo}} \\ \frac{I_{po}\Lambda_o^2}{I_{vo}} & 0 & \frac{I_{po}^2\Lambda_o}{I_{vo}} & \left(\frac{I_{po}\Lambda_o}{I_{vo}}\right)^2 \end{bmatrix} . \tag{67}$$

The second term of the righthand side of eq.(65) is readily expressed as follows:

$$\delta I_p^2(t)=\delta x'(t)Q_2\delta x \quad , \tag{68}$$

where

$$Q_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{69}$$

The matrix Q then becomes

$$Q=q_1Q_1+q_2Q_2 \quad , \tag{70}$$

where parameters q_1, q_2 should be selected appropriately.

The second term in the integral of eq.(23) should reflect the ranges of the control inputs given by eq.(42): The matrix R is selected according to the realizable voltage range of the power supply.

The hoped-for contribution of this paper will be given in the hitherto description: Our problem has been formalized into eqs.(21) to (23). It is apparent eqs.(27) and (28) hold. The rest procedures will be readily accomplished. Main part of the necessary on-line calculations is multiplication of a 2×4 matrix and 4 dimensional vector. Note that the calculation of plasma position $\Delta(t)$ is not required in our formalization.

5. Discussion

In this section some complements are mentioned briefly under appropriate head-lines.

<u>Value of the time constant τ_Λ</u>

In eq.(43) the time constant τ_Λ is introduced. Using eq.(37) we can write

$$1/\tau_\Lambda=1/\tau_{\beta_p}+1/(2\tau_{\ell_i}) \quad . \tag{71}$$

According to the definition, we have

$$\beta_p(t)=\frac{4N(t)T(t)}{\mu_0 I_p^2(t)R_p(t)} \quad , \tag{72}$$

$$Q=q_1Q_1+q_2Q_2 \quad , \tag{70}$$

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so that the time constant of β_p depends particle confinement time governing the total particle number N, energy confinement time governing the plasma temperature T, and I_p, R_p. The other quantity $\ell_i(t)$ is related to the current distribution. It cannot be defined so clearly as $\beta_p(t)$. Its time constant τ_{ℓ_i} must be either estimated by some simulation studies or represented by an appropriate gross quantity such as the skin time.

It is possible to formalize somehow the problem on Λ_1 according to such physical concept as described above. However, it is not n(t), T(t) or current distribution but $\Lambda_1(t)$ that is measurable in real-time. So we have adopted $\Lambda_1(t)$ as one of the state variables in our formalization. Practically, τ_Λ value should be identified on the basis of experiments.

System identification

In this paper we have started from a physical model to obtain the state description. There exists, however, another method to obtain the state description, which pays attention to a statistical nature of input-output characteristics of the system, such as cross-correlation or cross spectral density.[8] This method of identification suits the on-line data processing system.

Expansion to a more complicated system

The tokamak dealt with in this paper has only two coils: ohmic heating coil and vertical field coil. A tokamak of the next generation under design has, however, more poloidal coils. The JT-60 has horizontal and quadruple field coils to be controlled by the feedback method in addition to the two dealt with in this paper[9]. Because the vertical field coil has the n-value, its mutual inductances between the horizontal and quadruple field coils cannot be neglected. The method developed in this paper is suitably expanded for the design of the feedback control system of the JT-60.

The JET has a poloidal field coil system different from usual ones. It has several coils whose currents can be controlled separately so that one system functions as transformer, position-control and shaping coils.[10] To control such a system by the feedback, the method developed in this paper is essential.

Another assumption that eddy current effect is negligible has been introduced in the treatment of this paper. An actual tokamak has distributed parameter components such as a vacuum chamber. Eddy currents are induced on the components and resultant fields affect the controllability. Such currents can be expressed by circuit equations with lumped parameters similar to eqs.(30) to (32). The procedure for the expression can be found in ref. 5. Once the expression is made, the method described in this paper can be readily applied.

Cost function

Just a casual description is given in section 3 on the selection of the weighting matrices Q, R of the cost function (eq.(24)). The selection of the matrices is, however, not a simple matter. There is no universal agreement on precisely how these are to be selected for any given applications.

We have considered only physical requirements in selection of Q, R previously. It is also said that the appropriate selection of these matrices can minimize the errors due to the stochastic nature of the control object. Further descriptions on this problem are found in ref. 8.

Expansion to the discrete time system

In the control of plasma position and current in a large tokamak, combination of digital computer and phase-controlled thyristor power supply is usually employed. These devices can change the coil voltage only discretely, so the concept of discrete time system or sampled data system should be introduced in the problem. A control rule parallel to that given in section 2 is available for the discrete time system, where difference equations substitute for the differential equations.[8)]

Computing time

In this paragraph a practical aspect of the control, computing ability of a mini-computer, is examined and application of the control rule on a large tokamak such as the JT-60 is considered. The form of the necessary on-line computation is given by eq.(24), which is multiplication of a 2×4 matrix and 4 dimensional vector containing at least 8 multiplications and 6 additions. An example of a computer proposed to be used in the JT-60 poloidal field control is HIDIC 80, whose floating point computing time is 2.4~4.9 μsec for addition and 3.8~4.4 μsec for multiplication. The computing time of 8 multiplications and 6 additions in HIDIC 80 is 44.8~64.6 μsec.

As a power supply to drive the coils of the JT-60, 12-or 24-phase controlled thyristors are proposed whose averaged controllable time interval is 1,666~833 μsec. Compared with this time interval, we can regard the computation time in the computer is small enough.

For a more complicated system, such has additional position and shape control coils, use of an array transform processor must be helpful. An example is AP-210B, which has an ability to perform multiplication of two 10×10 matrices in 630 μsec.

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