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A Method of Solving the Inverse Kinematics of a Manipulator Arm

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This paper deals with a new approach to solve the inverse kinematic equations for a six-joint manipulator. As the mathematical methodology, the kinematic relationships are transformed into a high order polynomial of a joint angle variable at the end-point of the manipulator.

Using this algorithm proposed, the test calculations were performed to reproduce the individual joint angles. The accuracies of the solutions were quite satisfactory.

Keywords: Inverse Kinematic Problem, Homogeneous Transformation, Robot Manipulator

マニプレータ・アームの逆運動学方程式の一解法

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佐々木　　忍

（1986年1月30日受理）

本報告では，6関節のマニプレータに対する逆運動学方程式を解く新しい方法を取り扱っている。その数学的手法は，キネマテックな関係をマニプレータ先端部の関節に着目し，この関節角を変数とする高次の多項式に変換するものである。

このアルゴリズムを用いて各関節角の計算を実施したが，その解の精度は極めて満足のいくものである。

東海研究所：茨城県那珂郡東海村白方字白根2の4

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1. Introduction

In the study on an articulated mechanical system of robot manipulator, the so-called "arm kinematics" is concerned with the geometry of motion of a complex linkage mechanism relating to a fixed reference co-ordinate system (or absolute co-ordinates). In this case, the relationships between the position and orientation of the robot arm and its joint angle variables are the main subject of interest.

According to a mathematical description, the kinematic problem usually falls into two categories — the direct (or forward) and inverse kinematics problem. The direct problem is to determine the position and orientation of the manipulator in regard to a reference co-ordinate system, using the known data concerning the joint angle variables of the manipulator.

To this end, mathematical transformations from joint co-ordinates into Cartesian space co-ordinates are straight-forwardly obtained by successive multiplication of the matices.

For an n-degrees of freedom arm, the absolute co-ordinates at the final point of the open linkage must be represented in terms of n homogeneous transformation products composed of n joint variables of rotational and/or translational joints.

On the other hand, the inverse kinematic problem, the so-called " arm solution ", involves finding the required joint angle variable θ_i ,given the location and orientation at the final point (i.e., the finger tip of the manipulator) with respect to the reference co-ordinate frame located at the base of the manipulator.

Contrary to the direct problem, solution procedures are generally cumbersome and explicit analytical solutions can be derived only for some special arm configurations. The major reason for this is that the solution of the inverse problem is quite dependent on the manipulator structures such as joint configurations, redundancies, mechanical offset and mechanical constraints. As a matter of facts, there seem to be many articulated arms, for which the kinematic solutions cannot be represented in an explicit analytical form.

In order to accomplish precise position control of a robot manipulator, however, it is necessary to get correct solutions for the joint angle variables.

Out of such a necessity, some useful models have been developed up to date. Among them, a traditional method for the inverse kinematic problem is based on linearization of the kinematic equations in the neighborhood of a prescribed point in the joint variable space.
In that case, solution is highly dependent on the initial data (i.e., initial guess values) and a large deviation of the initial guess values from the actual position vector is not allowed due to the essential properties of linearization in determining a reliable joint solution.

In addition to this, the method involves the matrix inversion computation (called " inverse Jacobian ") in order to determine the increments of individual joint variables, with the consequence that the singularity points appearing frequently in the matrix will have significant influence on the execution of computation.

Apart from such features of linearization techniques for the inverse problem, our proposal starts from a different standpoint : Its underlying concept is to transform the kinematic relationships into a non-linear polynomial with a single angular variable. The real roots of its algebraic equation correspond to the angular solutions, and the determination of solutions is merely the numerical problem.

Compared with that the iterative methods based on the linearization approximation is limited to determination of a single set of joint variables ($\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, and θ_6), the present approach has an advantage in a way of finding out all possible groups of joint solutions corresponding to the motion of the robot arm end point.

The purpose of this paper is to demonstrate that the method developed here has the ability to successfully fulfill the non-linear mapping from the Cartesian co-ordinates into joint co-ordinates.

In the two sections that follow, the methodology is given that solves the inverse kinematics problem for a six-link manipulator. In the section 4, we show the numerical results of three sample problems in the graphical form. Together with the precision of joint solutions, calculations have clarified how each joint angle would behave while the end-effector moves towards goal position along the specified path. Input data requirements are given in Appendix 3.

2. Determination of Kinematics Based on Homogeneous Transformation

Given the geometrical structure parameters of a robot manipulator and the position co-ordinates and orientation at the end-point, the kinematic relationships are expressed as a function of the joint angle variables. For a mechanism that is constituted by a sequence of linkage with all joints controlled, the position and orientation of any link with respect to the base co-ordinate system (or world co-ordinate system) can be easily obtained by calculating the non-commutative product of successive co-ordinate transformation matrices up to the set of co-ordinate axes relating to that link. In the kinematic analysis of a manipulator, such a mathematical relationship between adjacent linkages can be described by a 4×4 homogeneous transformation matrix called "A matrix".

This method originated by J. Denavit and R.S. Hartenberg has been widely used as a convenient approach to derive the kinematic equations.(1)(2)*

In the case of the most general six-link manipulator, the situation of the end-point of the manipulator is specified by:

$$T_6 = \prod_{i=1}^{6} A_i$$

In order to determine the matrix T_6, we must calculate six geometrical co-ordinate transformations on the basis of the arm configuration.

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*) Number in bracket designates reference.

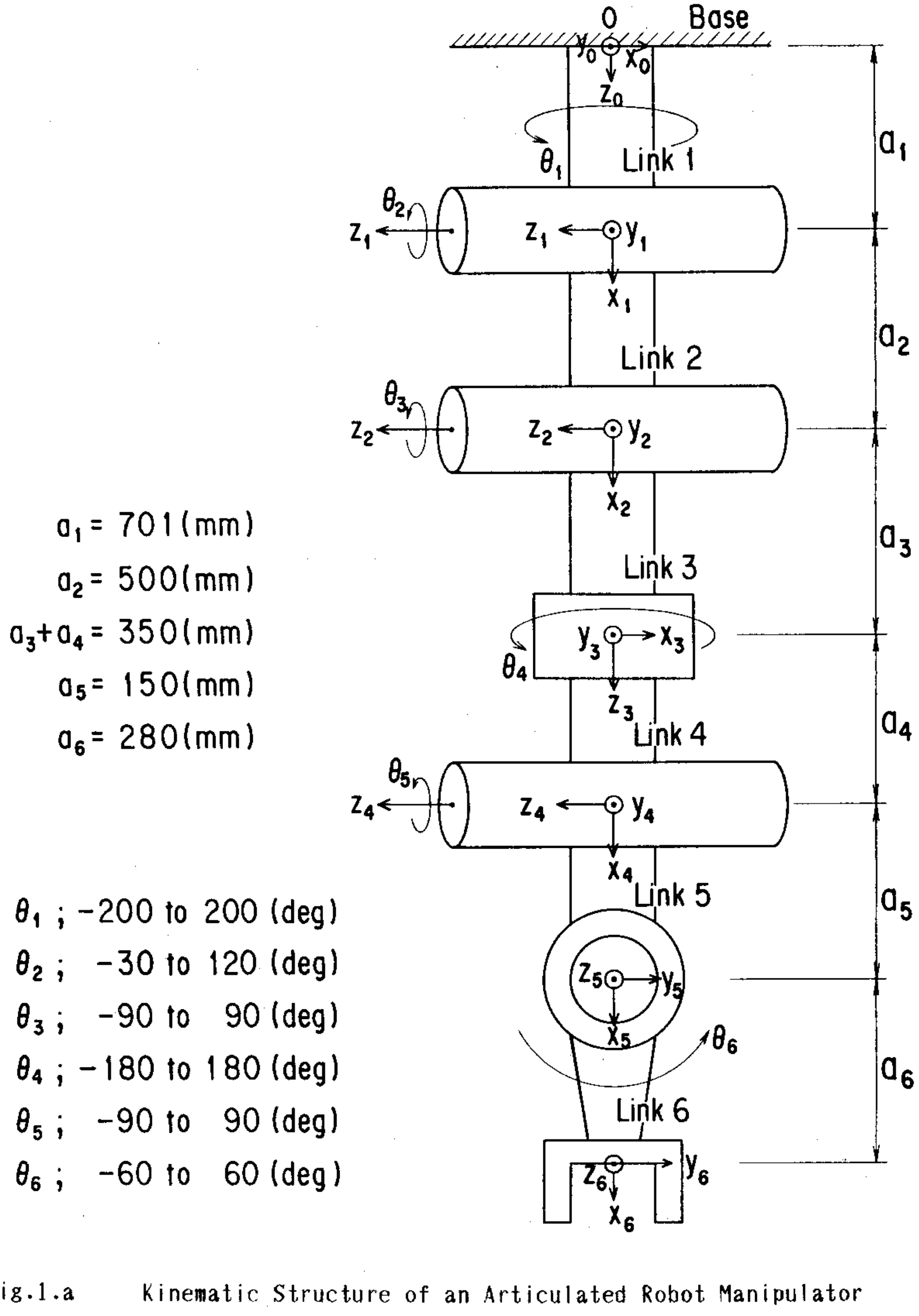


Fig.1.a Kinematic Structure of an Articulated Robot Manipulator

Now, we enter the concrete computation of A matrices to meet the specification of the existing articulated manipulator. The manipulator applied to the present study was selected from one of the robotic manipulator developed at JAERI. Fig. 1.a shows the schematics of this manipulator together with the co-ordinate systems. In this figure, the geometrical structure and dimension data such as the link length and operation ranges of joint angles are based on the specification of that manipulator.

Before the specification of A matrices and T_6 matrix, we make the following assumptions and notations.

(1) Each joint has only one degree of freedom, either rotational or translational.

(2) The motion of revolute joints about Z-axis follows the "right screw rule".

(3) The assignment of co-ordinate systems is not unique.

(4) The notations used throughout the derivation are:

(i) $c_i = \cos\theta_i$; $s_i = \sin\theta_i$; $c_{ij} = \cos(\theta_i + \theta_j)$; $s_{ij} = \sin(\theta_i + \theta_j)$

(ii)
$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}(p,q,r) = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1) A_1-matrix

Consider a right-handed orthonormal link co-ordinate system (x_0, y_0, z_0) defined at the supporting base, as shown in Fig. 2.

The origin 0 between link 0 and link 1 is set on the fixed base of the robot manipulator.

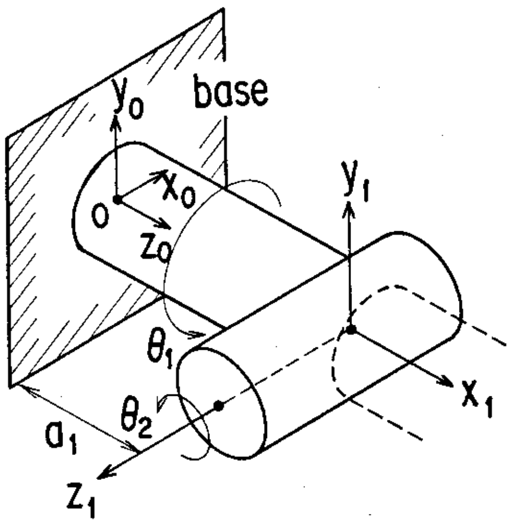


Fig.2 Schematics of Base to Link 1

We first rotate θ_1 about the Z_0-axis, then make the translation by a_1 along the Z_0-direction up to the shoulder of the robot arm. Further, producing a rotation of -90° about the y_0-axis, we establish a new co-ordinate system (x_1, y_1, z_1).

A_1-matrix representing this situation (rotational and translational transformation) is described as follows:

$$A_1 = Rot(z_0,\theta_1)Trans(0,0,a_1)Rot(y_0,-\frac{\pi}{2})$$

$$= \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -s_1 & -c_1 & 0 \\ 0 & c_1 & -s_1 & 0 \\ 1 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In other words, this matrix exhibits the position and orientation of the coordinate frame (x_1, y_1, z_1) with respect to the reference co-ordinate system (base).

(2) A_2-matrix

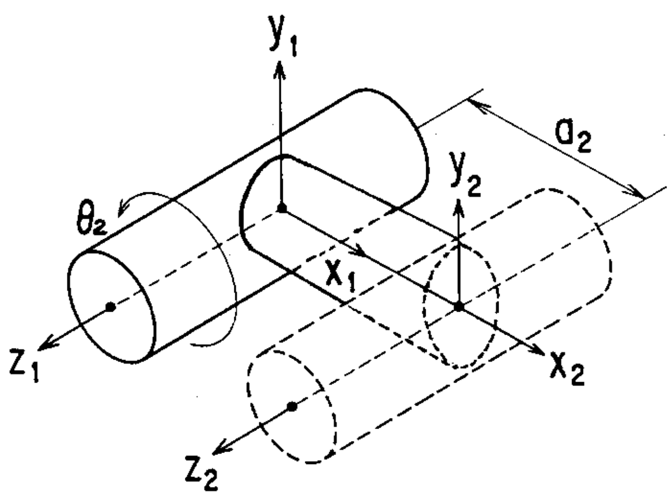


Fig.3 Schematics of Link 1 to Link 2

Referring to Fig. 3, we determine the second transformation matrix from the shoulder to the elbow of the robot body. It contains a rotation about the z_1-axis by θ_2 and translation by a_2 along the x_1-axis. After this transformation, the old co-ordinate system is replaced by (x_2, y_2, z_2).

$$A_2 = \mathrm{Rot}(z_1,\theta_2)\mathrm{Trans}(a_2,0,0)$$

$$= \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) A_3-matrix

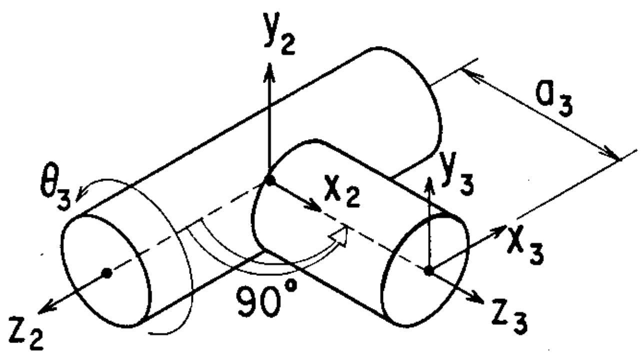


Fig.4 Schematics of Link 2 to link 3

Thirdly, we rotate θ_3 about the z_2-axis, slide a_3 along the x-axis, then rotate 90° in a clockwise direction about the y_2-axis so that the transformation is made from the elbow to forearm of the manipulator.

Thereby, we can establish the fourth co-ordinate frame (x_3, y_3, z_3) as indicated in Fig. 4.

$$A_3 = \mathrm{Rot}(Z_2,\theta_3)\mathrm{Trans}(a_3,0,0)\mathrm{Rot}(y_2,\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -s_3 & c_3 & a_3c_3 \\ 0 & c_3 & s_3 & a_3s_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(4) A_4-matrix

For the motion from the forearm to wrist parts, we make a rotation of angle θ_4 about the z_3-axis, followed by a translation of a_4 along the z_3-direction, then the second rotation about the y_3-axis by $-90°$.

With reference to Fig. 5, the transformation matrix A_4 up to the co-ordinate frame (x_4, y_4, z_4) assigned is:

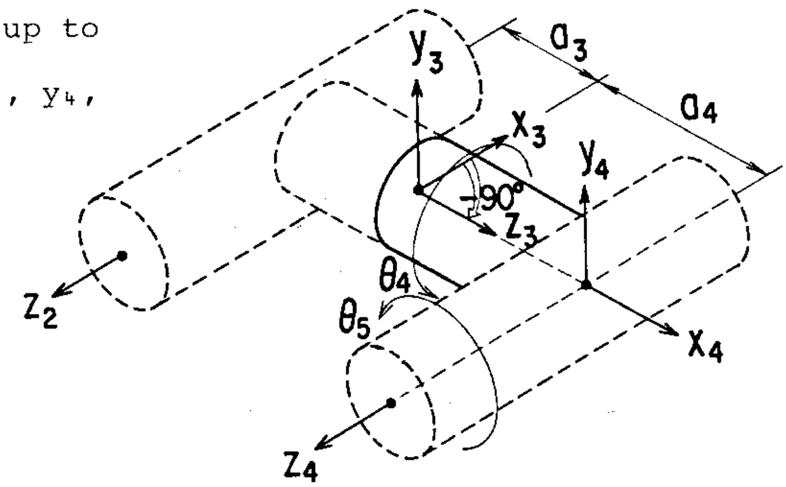


Fig.5 Schematics of Link 3 to Link 4

$$A_4 = \mathrm{Rot}(z_3,\theta_4)\mathrm{Trans}(0,0,a_4)\mathrm{Rot}(y_3,-\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -s_4 & -c_4 & 0 \\ 0 & c_4 & -s_4 & 0 \\ 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) A_5-matrix

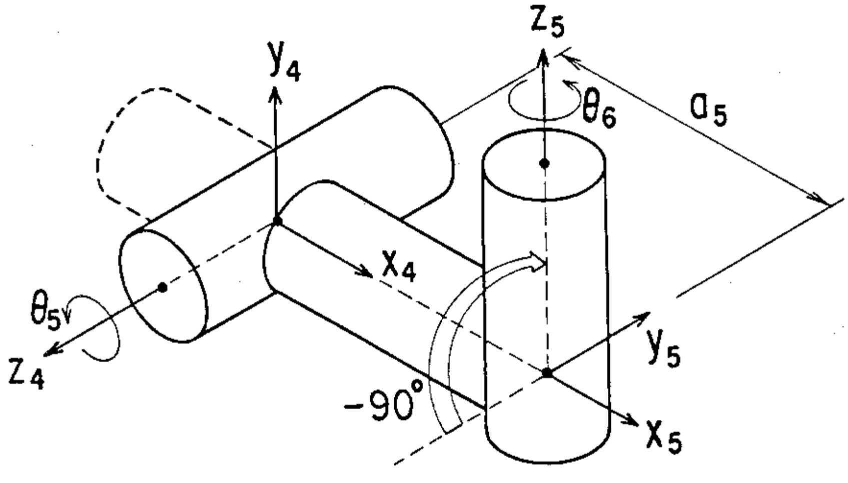


Fig.6 Schematics of Link 4 to Link 5

From the illustration of Fig. 6, the upward and downward movement of the wrist parts, are realized by the combination of: a rotation about the z_4-axis by the angle θ_5, a translation a_5, along the x_4-direction, and a rotation about the x_4-axis by - 90°. Hence, we obtain a new co-ordinate system (x_5, y_5, z_5) and A_5-matrix.

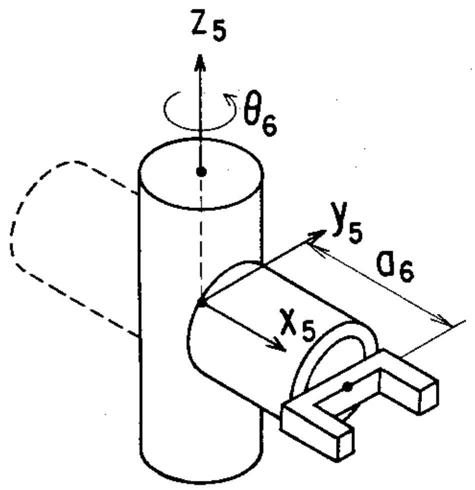
$$A_5 = \mathrm{Rot}(z_4,\theta_5)\mathrm{Trans}(a_5,0,0)\mathrm{Rot}(x_4,-\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_5 & -s_5 & 0 & a_5c_5 \\ s_5 & c_5 & 0 & a_5s_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_5 & 0 & -s_5 & a_5c_5 \\ s_5 & 0 & c_5 & a_5s_5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(6) A_6-matrix

Based on Fig. 7, left and right movement of the wrist with the end-effector (or hand) can be represented by the following transformation.



$$A_6 = Rot(z_5, \theta_6) Trans(a_6, 0, 0)$$

$$= \begin{pmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_6 & -s_6 & 0 & a_6 c_6 \\ s_6 & c_6 & 0 & a_6 s_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Fig.7 Schematics of Link 5 to Link 6

Now that we specified the individual A-matrices for a serial link manipulator, six chain products of these homogeneous transformations are postmultiplied successively.

For instance,

$$A_1 A_2 = \begin{pmatrix} 0 & -s_1 & -c_1 & 0 \\ 0 & c_1 & -s_1 & 0 \\ 1 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -s_1 s_2 & -s_1 c_2 & -c_1 & -a_2 s_1 s_2 \\ s_2 c_1 & c_1 c_2 & -s_1 & a_2 c_1 s_2 \\ c_2 & -s_2 & 0 & a_2 c_2 + a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1)$$

$A_1 A_2 A_3$

$$= \begin{pmatrix} -s_1 s_2 & -s_1 c_2 & -c_1 & -a_2 s_1 s_2 \\ s_2 c_1 & c_1 c_2 & -s_1 & a_2 c_1 s_2 \\ 0 & -s_2 & 0 & a_2 c_2 + a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -s_3 & c_3 & a_3 c_3 \\ 0 & c_3 & s_3 & a_3 s_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
= \begin{pmatrix} c_1 & s_1 s_2 s_3 - s_1 c_2 c_3 & -s_1 s_2 c_3 - s_1 c_2 s_3 & -a_3 s_1 s_2 c_3 - a_3 s_1 c_2 s_3 - a_2 s_1 s_2 \\ s_1 & -s_2 s_3 c_1 + c_1 c_2 c_3 & s_2 c_1 c_3 + c_1 c_2 s_3 & a_3 s_2 c_1 c_3 + a_3 c_1 c_2 s_3 + a_2 c_1 s_2 \\ 0 & -c_2 s_3 - s_2 c_3 & c_2 c_3 - s_2 s_3 & a_3 c_2 c_3 - a_3 s_2 s_3 + a_2 c_2 + a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} c_1 & -s_1 c_{23} & -s_1 s_{23} & -a_3 s_1 s_{23} - a_2 s_1 s_2 \\ s_1 & c_1 c_{23} & c_1 s_{23} & a_3 c_1 s_{23} + a_2 c_1 s_2 \\ 0 & -s_{23} & c_{23} & a_3 c_{23} + a_2 c_2 + a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2)
$$

$A_1 A_2 A_3 A_4$

$$
= \begin{pmatrix} c_1 & -s_1 c_{23} & -s_1 s_{23} & -a_3 s_1 s_{23} - a_2 s_1 s_2 \\ s_1 & c_1 c_{23} & c_1 s_{23} & a_3 c_1 s_{23} + a_2 c_1 s_2 \\ 0 & -s_{23} & c_{23} & a_3 c_{23} + a_2 c_2 + a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -s_4 & -c_4 & 0 \\ 0 & c_4 & -s_4 & 0 \\ 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} -s_1 s_{23} & -s_4 c_1 - s_1 c_4 c_{23} & -c_1 c_4 + s_1 s_4 c_{23} & -a_4 s_1 s_{23} - a_3 s_1 s_{23} - a_2 s_1 s_2 \\ c_1 s_{23} & -s_1 s_4 + c_1 c_4 c_{23} & -s_1 c_4 - c_1 c_{23} s_4 & a_4 c_1 s_{23} + a_3 c_1 s_{23} + a_2 c_1 s_2 \\ c_{23} & -c_4 s_{23} & s_4 s_{23} & a_4 c_{23} + a_3 c_{23} + a_2 c_2 + a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
(3)
$$

$A_1 A_2 A_3 A_4 A_5$

$$
= \begin{pmatrix} -s_1 s_{23} & -s_4 c_1 - s_1 c_4 c_{23} & -c_1 c_4 + s_1 s_4 c_{23} & -a_4 s_1 s_{23} - a_3 s_1 s_{23} - a_2 s_1 s_2 \\ c_1 s_{23} & -s_1 s_4 + c_1 c_4 c_{23} & -s_1 c_4 - c_1 c_{23} s_4 & a_4 c_1 s_{23} + a_3 c_1 s_{23} + a_2 c_1 s_2 \\ c_{23} & -c_4 s_{23} & s_4 s_{23} & a_4 c_{23} + a_3 c_{23} + a_2 c_2 + a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\times \begin{pmatrix} c_5 & 0 & -s_5 & a_5 c_5 \\ s_5 & 0 & c_5 & a_5 s_5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix}
-s_1(s_{23}c_5+c_4c_{23}s_5)-s_4s_5c_1 & c_1c_4-s_1s_4c_{23} & s_1s_5s_{23}-s_4c_1c_5-s_1c_4c_{23}c_5 & -a_5(c_5s_{23}+s_5c_4c_{23})s_1-a_5s_5s_4c_1-(a_3+a_4)s_1s_{23}-a_2s_1s_2 \\
c_1(c_5s_{23}+c_4s_5c_{23})-s_1s_4s_5 & s_1c_4+c_1c_{23}s_4 & -c_1s_{23}s_5-s_1s_4c_5+c_1c_4c_5c_{23} & a_5(c_5s_{23}+s_5c_4c_{23})c_1-a_5s_5s_4s_1+(a_3+a_4)c_1s_{23}+a_2c_1s_2 \\
c_{23}c_5-c_4s_{23}s_5 & -s_4s_{23} & -c_{23}s_5-c_4s_{23}c_5 & a_5c_5c_{23}-a_5s_5c_4s_{23}+(a_3+a_4)c_{23}+a_2c_2+a_1 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (4)
$$

$$
T_6 = A_1A_2A_3A_4A_5A_6
$$

$$
= \begin{pmatrix}
t_{11} & t_{12} & t_{13} & t_{14} \\
t_{21} & t_{22} & t_{23} & t_{24} \\
t_{31} & t_{32} & t_{33} & t_{34} \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (5.a)
$$

where

$$t_{11} = -s_1(s_{23}c_5c_6+c_{23}s_5c_4c_6+c_{23}s_4s_6)+c_1(c_4s_6-c_6s_4s_5)$$

$$t_{12} = s_1(s_{23}s_6c_5+c_{23}s_5s_6c_4-c_{23}s_4c_6)+c_1(c_4c_6+s_4s_5s_6)$$

$$t_{13} = s_1(s_{23}s_5-c_{23}c_4c_5)-c_1s_4c_5$$

$$
\begin{aligned}
t_{14} = -s_1[&a_6(s_{23}c_5c_6+c_{23}s_5c_4c_6+c_{23}s_4s_6) \\
&+a_5(s_{23}c_5+c_{23}s_5c_4)+(a_3+a_4)s_{23}+a_2s_2] \\
&+c_1[a_6(c_4s_6-s_4s_5c_6)-a_5s_4s_5]
\end{aligned}
$$

$$t_{21} = s_1(c_4s_6-s_4s_5c_6)+c_1(s_{23}c_5c_6+c_{23}c_4c_6s_5+c_{23}s_4s_6)$$

$$t_{22} = s_1(c_4c_6+s_4s_5s_6)-c_1(s_{23}c_5s_6+c_{23}s_5s_6c_4-c_{23}s_4c_6)$$

$$t_{23} = -s_1s_4c_5-c_1(s_{23}s_5-c_{23}c_4c_5)$$

$$
\begin{aligned}
t_{24} = s_1[&a_6(c_4s_6-s_4s_5c_6)-a_5s_4s_5]+c_1[a_6(s_{23}c_5c_6 \\
&+c_{23}s_5c_4c_6+c_{23}s_4s_6)+a_5(s_{23}c_5+c_{23}s_5c_4) \\
&+(a_3+a_4)s_{23}+a_2s_2]
\end{aligned}
$$

$$t_{31} = c_5c_6c_{23}-c_4c_6s_5s_{23}-s_4s_6s_{23}$$

$$t_{32} = -c_5s_6c_{23}+s_5s_6c_4s_{23}-s_4c_6s_{23}$$

$$t_{33} = -s_5c_{23}-c_4c_5s_{23}$$

$$t_{34} = a_6(c_5c_6c_{23}-s_5c_6c_4s_{23}-s_4s_6s_{23})+a_5(c_5c_{23}-s_5c_4s_{23})$$
$$+(a_3+a_4)c_{23}+a_2c_2+a_1$$

From this repressentation, we can find the position and orientation of the end-point of a manipulator with reference to the base co-ordinate system.

3. Computational Algorithm

Following the fundamental computation of matrices in the preceding section, we proceed to the main subject of this report.

The first part is devoted to derive a non-linear equation from kinematic formulations. The latter section is concerned with a determination of joint angle variables.

3.1 Derivation of a Non-linear Equation with a Single Variable

Consider the arm matrix T_6 to be of the form:

$$T_6 = \begin{pmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5.b)$$

In Eq. (5.b), the left upper 3×3 matrix is a rotation matrix made with orthogonal unit vectors — "n", "o", and "a". "n" is the normal vector of the hand. Assuming a parallel-jaw hand, it is orthogonal to the fingers of the robot arm. "o" is the sliding vector of the hand, which indicates the direction of the finger motion as the gripper opens and closes. "a" is the approach vector of the hand, which indicates the direction normal to the palm of the hand. "p" is the position vector describing the location of the hand and also referred to as a "translation vector". These four column vectors are illustrated in Fig. 1.b.

In the present algorithm, the location of the finger tip was set at the point A and the length (d) of a tool attached

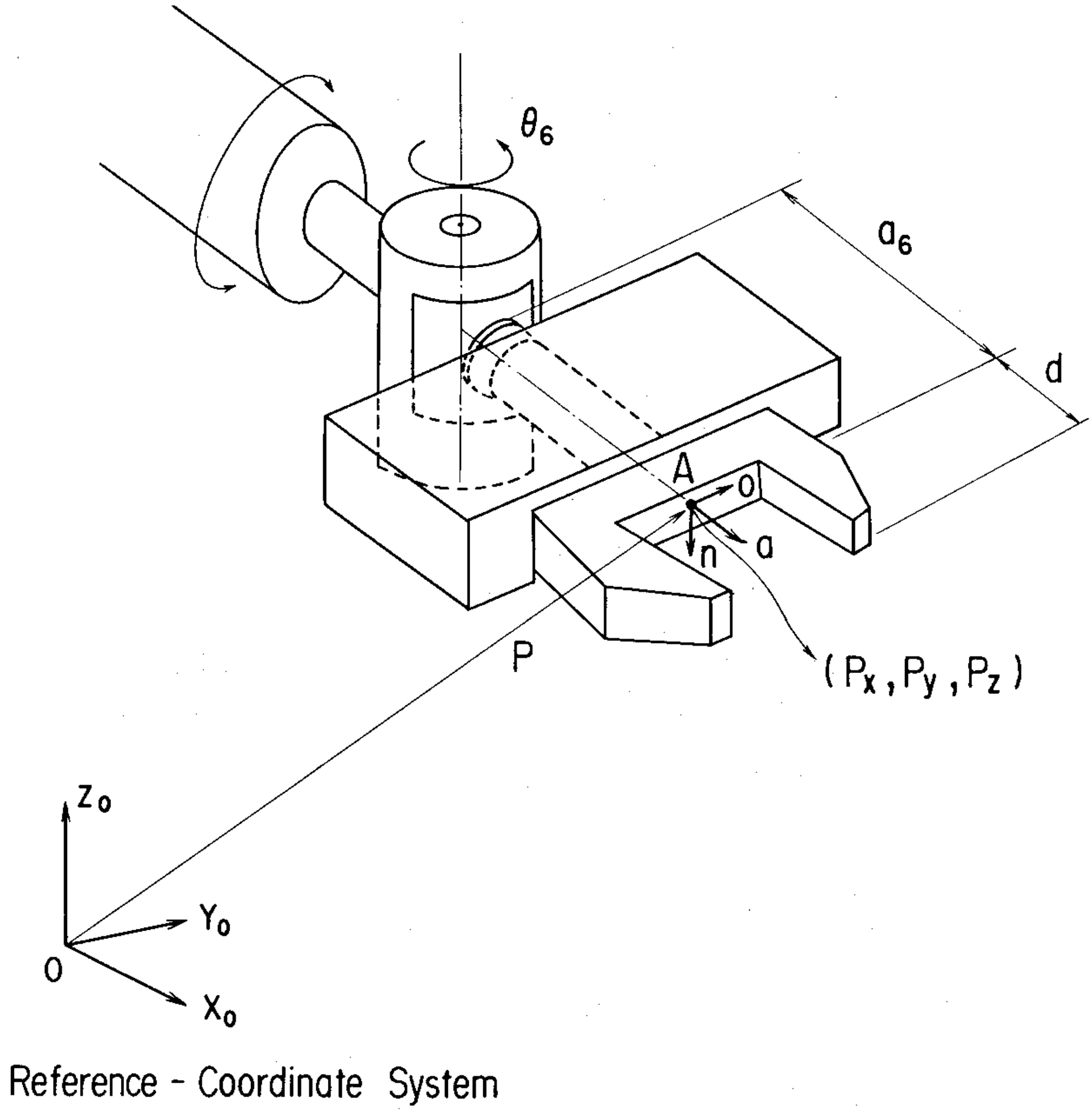


Fig.1.b General View of an End-Effector and Its Co-ordinate System

to the arm end-effector is neglected.

Equating the corresponding elements of the T_6 matrix in Eq. (5.a) and Eq. (5.b), we have kinematic expressions of the multi-joint arm shown in Fig. 1.a. In what follows, it is postulated that each component of T_6 in Eq. (5.b) is known.

$$n_x = -c_6(s_1s_{23}c_5 + s_4s_5c_1 + s_1s_5c_4c_{23}) + s_6(c_1c_4 - s_1s_4c_{23}) \quad (6)$$

$$n_y = c_6(c_1c_5s_{23} - s_1s_4s_5 + c_1c_4c_{23}s_5) + s_6(s_1c_4 + c_1c_{23}s_4) \quad (7)$$

$$n_z = c_6(c_5c_{23} - c_4s_5s_{23}) - s_4s_{23}s_6 \quad (8)$$

$$o_x = c_6(c_1c_4 - s_1s_4c_{23}) + s_6(s_1s_{23}c_5 + s_4s_5c_1 + s_1s_5c_4c_{23}) \quad (9)$$

$$o_y = c_6(s_1c_4 + c_1c_{23}s_4) - s_6(c_1c_5s_{23} - s_1s_4s_5 + c_1c_4c_{23}s_5) \quad (10)$$

$$o_z = -c_6(s_4s_{23}) + s_6(c_4s_5s_{23} - c_5c_{23}) \quad (11)$$

$$\begin{aligned} p_x = & -a_6c_6(s_1s_{23}c_5 + s_4s_5c_1 + s_1s_5c_4c_{23}) + a_6s_6(c_1c_4 - s_1s_4c_{23}) \\ & -a_5(c_5s_1s_{23} + s_4s_5c_1 + s_1s_5c_4c_{23}) - (a_3 + a_4)s_1s_{23} \\ & -a_2s_1s_2 \end{aligned} \quad (12)$$

$$\begin{aligned} p_y = & a_6c_6(c_1c_5s_{23} - s_1s_4s_5 + c_1c_4c_{23}s_5) + a_6s_6(s_1c_4 + c_1s_4c_{23}) \\ & +a_5(c_5c_1s_{23} - s_1s_4s_5 + c_1c_4c_{23}s_5) + (a_3 + a_4)c_1s_{23} \\ & +a_2s_2c_1 \end{aligned} \quad (13)$$

$$\begin{aligned} p_z = & a_6c_6(c_5c_{23} - c_4s_5s_{23}) + a_6(-s_4s_{23})s_6 + a_5(c_5c_{23} - s_5s_{23}c_4) \\ & + (a_3 + a_4)c_{23} + a_2c_2 + a_1 \end{aligned} \quad (14)$$

Here, each element of the third column of the T_6 matrix in Eq. (5) was not included in a group of Eqs. (6) to (11), because it is uniquely determined from the orthonormal co-ordinate relation, that is, $a = n \times o$. Now, we introduce the following parameters in view of the apparent features of the equations (6) through (11).

$$A = s_1 s_{23} c_5 + s_4 s_5 c_1 + s_1 s_5 c_4 c_{23} \quad (15)$$

$$B = c_1 c_4 - s_1 s_4 c_{23} \quad (16)$$

$$C = c_1 c_5 s_{23} - s_1 s_4 s_5 + c_1 c_4 c_{23} s_5 \quad (17)$$

$$D = s_1 c_4 + c_1 c_{23} s_4 \quad (18)$$

$$E = c_5 c_{23} - c_4 s_5 s_{23} \quad (19)$$

$$F = -s_4 s_{23} \quad (20)$$

$$a_3 + a_4 = a_{34}$$

Consequently, the original equations (6) to (11) are simplified as follows.

$$-A\ c_6 + B\ s_6 = n_x \quad (21)$$

$$C\ c_6 + D\ s_6 = n_y \quad (22)$$

$$E\ c_6 + F\ s_6 = n_z \quad (23)$$

$$B\ c_6 + A\ s_6 = o_x \quad (24)$$

$$D\ c_6 - C\ s_6 = o_y \quad (25)$$

$$F\ c_6 - E\ s_6 = o_z \quad (26)$$

Similarly, the position vector in Eqs. (12), (13) and (14) is represented by:

$$-a_6 c_6 A + a_6 s_6 B - a_5 A - a_{34} s_1 s_{23} - a_2 s_1 s_2 = P_x \quad (27)$$

$$a_6 c_6 C + a_6 s_6 D + a_5 C + a_{34} c_1 s_{23} + a_2 s_2 c_1 = P_y \quad (28)$$

$$a_6 c_6 E + a_6 s_6 F + a_5 E + a_{34} c_{23} + a_2 c_2 + a_1 = P_z \quad (29)$$

As can be noticed from Eqs. (21) through (26), new parameters defined in Eqs. (15) to (20) can be represented as a trigonometric function of only c_6 and s_6. Namely,

$$A = -n_x c_6 + o_x s_6 \quad (30)$$

$$B = n_x s_6 + o_x c_6 \quad (31)$$

$$C = n_y c_6 - o_y s_6 \quad (32)$$

$$D = n_y s_6 + o_y c_6 \quad (33)$$

$$E = n_z c_6 - o_z s_6 \quad (34)$$

$$F = o_z c_6 + n_z s_6 \quad (35)$$

From the above equations (27) and (28), we obtain

$$p_x c_1 + p_y s_1 = a_6(n_x c_1 + n_y s_1) + a_5(-Ac_1 + Cs_1) \tag{36}$$

On simplification, we have

$$(p_x - a_6 n_x + a_5 A)c_1 = (a_6 n_y + a_5 C - p_y)s_1 \ , \tag{37}$$

that is, $\tan\theta_1 = \dfrac{s_1}{c_1} = \dfrac{p_x - a_6 n_x + a_5 A}{a_6 n_y + a_5 C - p_y} = \dfrac{XX + a_5 A}{a_5 C - YY}$

$$= \frac{a_5 o_x s_6 - a_5 n_x c_6 + XX}{a_5 n_y c_6 - a_5 o_y s_6 - YY} \tag{38}$$

where

$$XX = p_x - a_6 n_x$$
$$YY = p_y - a_6 n_y \tag{39}$$

Suppose $\tan\frac{\theta_6}{2} = t$. Then $c_6 = \dfrac{1 - t^2}{1 + t^2}$, $s_6 = \dfrac{2t}{1 + t^2}$ and $\tan\theta_6 = \dfrac{2t}{1 - t^2}$.

Therefore, it follows that

$$\tan\theta_1 = \frac{a_5 o_x\left(\frac{2t}{1+t^2}\right) - a_5 n_x\left(\frac{1-t^2}{1+t^2}\right) + XX}{a_5 n_y\left(\frac{1-t^2}{1+t^2}\right) - a_5 o_y\left(\frac{2t}{1+t^2}\right) - YY}$$

$$= \frac{x_n + 2a_5 o_x t + x_p t^2}{y_n - 2a_5 o_y t + y_p t^2} \tag{40}$$

where

$$\left.\begin{aligned} x_p &= a_5 n_x + XX \\ x_n &= -a_5 n_x + XX \\ y_p &= -(a_5 n_y + YY) \\ y_n &= a_5 n_y - YY \end{aligned}\right\} \tag{41}$$

On the other hand, from Eq. (29) we have

$$p_z - a_6 n_z - a_1 = a_5 E + a_{34} c_{23} + a_2 c_2$$

$$= a_5 (n_z c_6 - o_z s_6) + a_{34} c_{23} + a_2 c_2$$

$$a_5 c_6 (n_z - o_z \tan\theta_6) - zz + a_{34} c_{23} = -a_2 c_2 \qquad (42)$$

where $zz = p_z - a_6 n_z - a_1$.

Additionally, Eq. (36) $\times (-s_1)$ + Eq. (37) $\times c_1$ reduces to

$$a_5 c_6 \{(n_y c_1 - n_x s_1) + (s_1 o_x - o_y c_1)\tan\theta_6\}$$

$$- \{-p_x s_1 + p_y c_1 + a_6 (n_x s_1 - n_y c_1)\} + a_{34} s_{23} = -a_2 s_2 \quad (43)$$

As described above, we obtained two principal equations (42) and (43) from kinematic formulations.

From here, we shall pay attention to a further simplification so that these equations can be unified in one single mathematical relationship.

Let $a_5 c_6 (n_z - o_z \tan\theta_6) - zz = \psi$, (44)

and $a_5 c_6 \{n_y c_1 - n_x s_1 + (s_1 o_x - o_y c_1)\tan\theta_6\}$

$$- \{-p_x s_1 + p_y c_1 + a_6 (n_x s_1 - n_y c_1)\} = \eta \qquad (45)$$

That is,

$$\psi + a_{34} c_{23} = -a_2 c_2 \qquad (46)$$

$$\eta + a_{34} s_{23} = -a_2 s_2 \qquad (47)$$

Adding the square of Eq. (46) and Eq. (47), we get the following simple form.

$$\psi^2 + \eta^2 - a = -2a_{34} (\psi c_{23} + \eta s_{23}) \qquad (48)$$

where $a = a_2{}^2 - a_{34}{}^2$

Calculation of η

Now, we perform the formulation of η using the Eqs. (45) and (40).

$$\eta = a_5 c_6 \{n_y c_1 - n_x s_1 + (s_1 o_x - o_y c_1)\tan\theta_6\}$$

$$- \{-p_x s_1 + p_y c_1 + a_6 (n_x s_1 - n_y c_1)\}$$

$$= a_5 c_6 c_1 \{n_y - n_x \tan\theta_1 + (o_x \tan\theta_1 - o_y)\tan\theta_6\}$$

$$+ c_1 \{p_x \tan\theta_1 - p_y - a_6 (n_x \tan\theta_1 - n_y)$$

$$= a_5 c_6 c_1 \{n_y - o_y \tan\theta_6 + (o_x \tan\theta_6 - n_x)\tan\theta_1\}$$

$$+ c_1 \{(p_x - a_6 n_x)\tan\theta_1 - (p_y - a_6 n_y)\}$$

$$= a_5 c_6 c_1 \{(n_y - o_y \tan\theta_6) + (o_x \tan\theta_6 - n_x)\tan\theta_1\}$$

$$+ c_1 (XX\tan\theta_1 - YY)$$

$$= a_5 c_6 c_1 \{n_y - o_y \frac{2t}{1-t^2} + (o_x \frac{2t}{1-t^2} - n_x)\tan\theta_1\}$$

$$+ c_1 (XX\tan\theta_1 - YY)$$

$$= a_5 c_1 \left(\frac{1-t^2}{1+t^2}\right)\left[\frac{n_y(1-t^2) - 2o_y t + \{2o_x t - n_x(1-t^2)\}\tan\theta_1}{1-t^2}\right]$$

$$+ c_1 (XX\tan\theta_1 - YY)$$

$$= a_5 c_1 \frac{1}{1+t^2} [n_y(1-t^2) - 2o_y t + (n_x t^2 + 2o_x t - n_x)\tan\theta_1]$$

$$+ c_1 (XX\tan\theta_1 - YY)$$

$$\eta = a_5 c_1 \left(\frac{1}{1+t^2}\right) [(n_y - 2o_y t - n_y t^2)(y_n - 2a_5 o_y t + y_p t^2)$$

$$+ (n_x t^2 + 2o_x t - n_x)(x_n + 2a_5 o_x t + x_p t^2)]\frac{1}{y_n - 2a_5 o_y t + y_p t^2}$$

$$+ c_1\left[\frac{XX(x_n + 2a_5o_xt + x_pt^2) - YY(y_n - 2a_5o_yt + y_pt^2)}{y_n - 2a_5o_yt + y_pt^2}\right]$$

$$= \frac{a_5c_1}{1+t^2}\cdot\frac{1}{y_n - 2a_5o_yt + y_pt^2}\ [n_yy_n - 2a_5n_yo_y + n_yy_pt^2$$

$$- 2o_yy_nt + 4a_5o_y{}^2t^2 - 2o_yy_pt^3 - n_yy_nt^2 + 2a_5n_yo_yt^3 - n_yy_pt^4$$

$$+ n_xx_nt^2 + 2a_5n_xo_xt^3 + n_xx_pt^4 + 2x_no_xt + 4a_5o_x{}^2t^2 + 2x_po_xt^3$$

$$- n_xx_n - 2a_5n_xo_xt - n_xx_pt^2] + \left(\frac{c_1}{y_n - 2a_5o_yt + y_pt^2}\right)[XX\cdot x_n$$

$$+ 2a_5o_x\cdot XXt + XX\cdot x_pt^2 - YYy_n + 2a_5o_yYYt - YYy_pt^2]$$

$$= \frac{c_1}{(1+t^2)(y_n - 2a_5o_yt + y_pt^2)}\left[\sum_{i=0}^{4} b_it^i \quad \sum_{i=5}^{9} b_it^{i-5}\right]$$

$$= \frac{c_1}{(1+t^2)(y_n - 2a_5o_yt + y_pt^2)}\ \sum_{i=0}^{4} \overline{cc}_i\ t^i \qquad (49)$$

(See the Appendix 3 as to the values of respective coefficients in a power series.)

Calculation of ψ

Based on the ψ-description in Eq. (44), we will try to express as a function of variable t.

$$\psi = a_5c_6(n_z - o_z\tan\theta_6) - zz$$

$$= a_5\left(\frac{1-t^2}{1+t^2}\right)\left(n_z - o_z\cdot\frac{2t}{1-t^2}\right) - zz$$

$$= \frac{1}{1+t^2}\ [a_5(n_z - 2o_zt - n_zt^2) - zz(1+t^2)]$$

$$= \frac{1}{1+t^2}\ [(a_5n_z - zz) - 2a_5o_zt - (a_5n_z + zz)t^2]$$

$$= \frac{1}{1+t^2}\ \sum_{i=1}^{3} do_i\ t^{i-1} \qquad (50)$$

Calculation of $c_1{}^2$

Using the value of $\tan\theta_1$ derived from Eq. (40), it follows that

$$\frac{1}{c_1^2} = 1+\tan^2\theta_1 = 1+\left(\frac{x_n+2a_5o_x t+x_p t^2}{y_n-2a_5o_y t+y_p t^2}\right)^2$$

thus,

$$c_1{}^2 = \frac{(y_n-2a_5o_y t+y_p t^2)^2}{(x_n+2a_5o_x t+x_p t^2)^2+(y_n-2a_5o_y t+y_p t^2)^2}$$

$$= \frac{(y_n-2a_5o_y t+y_p t^2)^2}{\sum_{i=0}^{4} f_i t^i}. \qquad (51)$$

Calculation of ψ^2

The square calculation of Eq. (50) results in

$$\psi^2 = \frac{1}{(1+t^2)^2}\left(\sum_{i=1}^{3} do_i\ t^{i-1}\right)^2 = \frac{1}{(1+t^2)^2}\sum_{i=0}^{4} d_i\ t^i. \qquad (52)$$

Calculation of $(\eta^2+\psi^2)$

Referring to Eqs. (50) and (52), the square sum of η and ψ is:

$$\eta^2+\psi^2 = \frac{c_1^2}{(1+t^2)^2}\cdot\frac{\left(\sum_{i=0}^{4}\overline{cc}_i\ t^i\right)^2}{(y_n-2a_5o_y t+y_p t^2)^2} + \frac{1}{(1+t^2)^2}\sum_{i=0}^{4} d_i\ t^i$$

$$= \frac{1}{(1+t^2)^2}\cdot\frac{\cancel{(y_n-2a_5o_y t+y_p t^2)^2}}{\sum_{i=0}^{4} f_i\ t^i}\cdot\frac{\left(\sum_{i=0}^{4}\overline{cc}_i\ t^i\right)^2}{\cancel{(y_n-2a_5o_y t+y_p t^2)^2}}$$

$$+ \frac{1}{(1+t^2)^2} \sum_{i=0}^{4} d_i \; t^i$$

$$= \frac{1}{(1+t^2)^2} \left\{ \frac{\left(\left(\sum_{i=0}^{4} \overline{cc}_i \right) t^i \right)^2}{\sum_{i=0}^{4} f_i \; t^i} + \sum_{i=0}^{4} d_i \; t^i \right\}$$

$$= \frac{1}{(1+t^2)^2} \cdot \frac{1}{\sum_{i=0}^{4} f_i \; t^i} \left\{ \sum_{i=0}^{8} e_i \; t^i + \left(\sum_{i=0}^{4} d_i \; t^i \right) \left(\sum_{i=0}^{4} f_i \; t^i \right) \right\}$$

$$= \frac{1}{(1+t^2)^2} \cdot \frac{1}{\sum_{i=0}^{4} f_i \; t^i} \left\{ \sum_{i=0}^{8} e_i \; t^i + \sum_{i=0}^{8} g_i \; t^i \right\} . \qquad (53)$$

Thus,

$$\eta^2 + \psi^2 - a = \frac{1}{(1+t^2)^2 \sum_{i=0}^{4} f_i \; t^i} \left\{ \sum_{i=0}^{8} e_i \; t_i + \sum_{i=0}^{8} g_i \; t^i - a \sum_{i=0}^{8} h_i \; t^i \right\}$$

$$= \frac{1}{(1+t^2)^2 \sum_{i=0}^{4} f_i \; t^i} \sum_{i=0}^{8} J_i \; t^i \qquad (54)$$

where $\quad (1+t^2)^2 \sum_{i=0}^{8} f_i \; t^i = \sum_{i=0}^{8} h_i \; t^i$

In this way, we could express $(\eta^2 + \psi^2 - a)$ as a rational function of t. Let us return to the original equation (48)

$$\eta^2 + \psi^2 - a = -2a_{34}(\psi c_{23} + \eta s_{23}) .$$

After taking the square of both sides, and putting $\tan\theta_{23} = k$ in this equation, we obtain

$$(\psi^2 + \eta^2 - a)^2 = \frac{4a_{34}{}^2}{1 + k^2} (\psi + k\eta)^2 . \tag{55}$$

As the next step, we consider the θ_6-expression of $\tan\theta_{23}$. To this end, provided that we solve Eqs. (8) and (16) for s_4c_{23} and s_4s_{23}, we obtain

$$\left.\begin{aligned} s_4c_{23} &= (-n_x s_1 + n_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6 \\ s_4s_{23} &= -(n_z s_6 + o_z c_6) \end{aligned}\right\} . \tag{56}$$

Accordingly,

$$k = \tan\theta_{23} = \frac{-(n_z s_6 + o_z c_6)}{(-n_x s_1 + n_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6} \tag{57}$$

Using the value of $\tan\theta_6$, $\tan\theta_1$ and Eq. (57), $k\eta$ can be represented as follows:

Calculation of $k\eta$

$$k\eta = \frac{c_1 \sum_{i=0}^{4} \overline{cc}_i \, t^i}{(1 + t^2)(y_n - 2a_5 o_y t + y_p t^2)} \cdot \frac{-(n_z \tan\theta_6 + o_z)}{(-n_x s_1 + n_y c_1)\tan\theta_6 + (o_y c_1 - o_x s_1)}$$

$$= \frac{c_1 \left(\sum_{i=0}^{4} \overline{cc}_i \, t^i \right)}{(1 + t^2)(y_n - 2a_5 o_y t + y_p t^2)} \cdot \frac{-(o_z + 2n_z t - o_z t^2)(y_n - 2a_5 o_y t + y_p t^2)}{c_1 \left(\sum_{i=0}^{4} I_i \, t^i \right)}$$

$$= \frac{-\left(\sum_{i=0}^{4} \overline{cc}_i \ t^i\right)(o_z + 2n_z t - o_z t^2)}{(1+t^2)\left(\sum_{i=0}^{4} I_i \ t^i\right)} \tag{58}$$

Calculation of $c_{23}{}^2$

With the aid of Eq. (57), it is possible to represent $c_{23}{}^2$ as a function of t.

$$c_{23}{}^2 = \frac{1}{1+\tan^2\theta_{23}} = \frac{1}{1+k^2} = \frac{1}{1+\left(\dfrac{n_z s_6 + o_z c_6}{(-n_x s_1 + n_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6}\right)^2}$$

$$= \frac{[(-n_x s_1 + n_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6]^2}{[(-n_x s_1 + n_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6]^2 + (n_z s_6 + o_z c_6)^2}$$

$$= \frac{[(-n_x s_1 + n_y c_1)\tan\theta_6 + (o_y c_1 - o_x s_1)]^2}{[(-n_x s_1 + n_y c_1)\tan\theta_6 + (o_y c_1 - o_x s_1)]^2 + (o_z + n_z \tan\theta_6)^2}$$

$$= \frac{c_1{}^2[(-n_x \tan\theta_1 + n_y)\tan\theta_6 + (o_y - o_x \tan\theta_1)]^2}{c_1{}^2[(-n_x \tan\theta_1 + n_y)\tan\theta_6 + (o_y - o_x \tan\theta_1)]^2 + (o_z + n_z \tan\theta_6)^2}$$

$$= \frac{c_1{}^2[(-n_x \tan\theta_1 + n_y)\frac{2t}{1-t^2} + (o_y - o_x \tan\theta_1)]^2}{c_1{}^2[(-n_x \tan\theta_1 + n_y)\frac{2t}{1-t^2} + (o_y - o_x \tan\theta_1)]^2 + \left(o_z + n_z \frac{2t}{1-t^2}\right)^2}$$

$$= \frac{c_1{}^2[2t(-n_x \tan\theta_1 + n_y) + (o_y - o_x \tan\theta_1)(1-t^2)]^2}{c_1{}^2[2t(-n_x \tan\theta_1 + n_y) + (o_y - o_x \tan\theta_1)(1-t^2)]^2 + (o_z(1-t^2) + 2n_z t)^2} \tag{59}$$

$$c_{23}{}^2 = \frac{c_1{}^2\left(\sum_{i=0}^{4} I_i \ t^i\right)^2}{c_1{}^2\left(\sum_{i=0}^{4} I_i \ t^i\right)^2 + (2n_z t + o_z - o_z t^2)(y_n - 2a_5 o_y t + y_p t^2)^2}$$

$$= \frac{\left[(y_n - 2a_5 o_y t - y_p t^2)^2 \Big/ \sum_{i=0}^{4} f_i \ t^i\right]\left(\sum_{i=0}^{4} I_i \ t^i\right)^2}{\left[(y_n - 2a_5 o_y t + y_p t^2)^2 \Big/ \sum_{i=0}^{4} f_i \ t^i\right]\left(\sum_{i=0}^{4} I_i \ t^i\right)^2 + \Delta}$$

where, $\Delta = (2n_z t + o_z - o_z t^2)^2 (y_n - 2a_5 o_y t + y_p t^2)^2$

Thus, $$c_{23}^2 = \frac{\left(\sum_{i=0}^{4} I_i\ t^i\right)^2}{\left(\sum_{i=0}^{4} I_i\ t^i\right)^2 + (2n_z + o_z - o_z t^2)^2 \left(\sum_{i=0}^{4} f_i\ t^i\right)}$$

$$= \frac{\sum_{i=0}^{8} I_{2i}\ t^i}{\sum_{i=0}^{8} I_{2i}\ t^i + (2n_z t + o_z - o_z t^2)^2 \left(\sum_{i=0}^{4} f_i\ t^i\right)} \tag{60}$$

$$= \frac{\left(\sum_{i=0}^{8} I_{2i}\ t^i\right)}{\sum_{i=0}^{8} I_{2i}\ t^i + \left(\sum_{i=0}^{4} LL_i\ t^i\right)\left(\sum_{i=0}^{4} f_i\ t^i\right)}$$

$$= \frac{\left(\sum_{i=0}^{8} I_{2i}\ t^i\right)}{\sum_{i=0}^{8} I_{2i}\ t^i + \sum_{i=0}^{8} PL_i\ t^i}$$

$$= \frac{\sum_{i=0}^{8} I_{2i}\ t^i}{\sum_{i=0}^{8} L_i\ t^i} \tag{60'}$$

Calculation of $(k\eta + \psi)$

With reference to Eqs. (50) and (58), we have

$$k\eta + \psi = \frac{-(o_z + 2n_z t - o_z t^2)\left(\sum_{i=0}^{4} \overline{cc}_i\ t^i\right)}{(1 + t^2)\left(\sum_{i=0}^{4} I_i\ t^i\right)} + \frac{\sum_{i=0}^{3} do_i\ t^{i-1}}{1 + t^2}$$

$$
= \frac{1}{(1+t^2)\left(\sum_{i=0}^{4} I_i \; t^i\right)} \left\{ -(o_z + 2n_z t - o_z t^2)\left(\sum_{i=0}^{4} \overline{cc}_i \; t^i\right) + \left(\sum_{i=1}^{3} do_i \; t^{i-1}\right)\left(\sum_{i=0}^{4} I_i \; t^i\right) \right\}
$$

$$
= \frac{1}{(1+t^2)\left(\sum_{i=0}^{4} I_i \; t^i\right)} \left\{ \sum_{i=0}^{6} K_i \; t^i + \sum_{i=0}^{6} II_i \; t^i \right\}
$$

$$
= \frac{1}{(1+t^2)\left(\sum_{i=0}^{4} I_i \; t_i\right)} \left\{ \sum_{i=0}^{6} PI_i \; t^i \right\} \tag{61}
$$

Taking the square of both sides in Eq. (61) and inserting it into the right side of Eq. (55), then we have

$$
4a_{34}{}^2 c_{23}{}^2 (k\eta + \psi)^2 = 4a_{34}{}^2 \frac{\cancel{\left(\sum_{i=0}^{8} I_{2i} \; t^i\right)}}{\sum_{i=0}^{8} L_i \; t^i} \cdot \frac{\left(\sum_{i=0}^{6} PI_i \; t^i\right)^2}{(1+t^2)^2 \cancel{\left(\sum_{i=0}^{8} I_{2i} \; t^i\right)}}
$$

$$
= 4a_{34}{}^2 \frac{\left(\sum_{i=0}^{6} PI_i \; t^i\right)^2}{\left(\sum_{i=0}^{8} L_i \; t^i\right) \cdot (1+t^2)^2}
$$

$$
= \frac{\left(\sum_{i=0}^{12} m_i \; t^i\right)}{\left(\sum_{i=0}^{8} L_i \; t^i\right)(1+t^2)^2} \tag{62}
$$

On the other hand, using the result of Eq. (54), the left-side of Eq. (55) gives:

$$(\eta^2 + \psi^2 - a)^2 = \frac{1}{(1+t^2)^4 \left(\sum_{i=0}^{4} f_i\ t^i\right)^2} \left(\sum_{i=0}^{8} J_i\ t^i\right)^2$$

$$= \frac{1}{(1+t^2)^4 \left(\sum_{i=0}^{4} f_i\ t^i\right)^2} \left(\sum_{i=0}^{16} N_i\ t^i\right) \qquad (63)$$

When equating Eqs. (62) and (63), we obtain

$$\frac{\left(\sum_{i=0}^{12} m_i\ t^i\right)}{\left(\sum_{i=0}^{8} L_i\ t^i\right)(1+t^2)^2} = \frac{\left(\sum_{i=0}^{16} N_i\ t^i\right)}{(1+t^2)^4 \left(\sum_{i=0}^{4} f_i\ t^i\right)^2} = \frac{\sum_{i=0}^{16} N_i\ t^i}{\left(\sum_{i=0}^{12} f_i\ t^i\right)(1+t^2)^2}$$

or
$$\frac{\left(\sum_{i=0}^{12} m_i\ t^i\right)}{\left(\sum_{i=0}^{8} L_i\ t^i\right)} = \frac{\left(\sum_{i=0}^{16} N_i\ t^i\right)}{(1+t^2)^2 \left(\sum_{i=0}^{4} f_i\ t^i\right)^2} = \frac{\left(\sum_{i=0}^{16} N_i\ t^i\right)}{\left(\sum_{i=0}^{12} f_i\ t^i\right)} \qquad (64)$$

Following the cancellation of the denominator in Eq. (64), we define:

$$\left(\sum_{i=0}^{12} m_i\ t^i\right)\left(\sum_{i=0}^{12} f_i\ t^i\right) = \sum_{i=0}^{24} P_i\ t^i \qquad (65)$$

and
$$\left(\sum_{i=0}^{8} L_i\ t^i\right)\left(\sum_{i=0}^{16} N_i\ t^i\right) = \sum_{i=0}^{24} q_i\ t^i\ . \qquad (66)$$

As a final description, we get

$$\sum_{i=0}^{24} (p_i - q_i)t^i = \sum_{i=0}^{24} r_i\ t^i = 0\ . \qquad (67)$$

As derived above, the problem of finding the articular angles of a manipulator was reduced to a non-linear higher-order algebraic equation $f(t) = \sum_{i=0}^{24} r_i t^i = 0$ with respect to $\tan(\theta_6/2)$.

In order to compute roots of this polynomial as exactly as possible, we used the Bairstow's method,[3],[4] which is outlined in Appendix. Since $|\theta_6| \leq 60°$ is specified as the operation domain of the end-effector, it is required for us to find the feasible solutions of f(t) on the interval $|t| \leq \frac{1}{\sqrt{3}}$.

3.2 Determination of Individual Joint Angles

(1) <u>Calculation of θ_6</u>

Once the desired solutions t are found from the algebraic equation (67), a joint angle θ_6 can be easily calculated.

That is, $\tan\frac{\theta_6}{2} = t$

thus, we have

$$\theta_6 = 2\tan^{-1} t \ . \qquad (68)$$

Corresponding trigonometric function is

$$s_6 = \sin\theta_6$$

$$c_6 = \cos\theta_6 \ .$$

Using these values, we can define Eqs. (30) through (35).

(2) <u>Calculation of θ_1</u>

Let

$$X_1 = XX + a_5 A$$

and $Y_1 = a_5 C - YY$,

then Eq. (38) reduces to

$$X_1 \cdot c_1 = Y_1 \cdot s_1 \quad \text{(See Eq. (39) for XX and YY)}$$

Thus,
$$\theta_1 = \tan^{-1}\left(\frac{X_1}{Y_1}\right) \qquad (69)$$
$$s_1 = \sin\theta_1$$
$$c_1 = \cos\theta_1$$

(3) Calculation of θ_{23}

From Eq. (48), we have

$$\psi^2 + \eta^2 - a = -2a_{34}(\psi c_{23} + \eta s_{23})$$
$$= -2a_{34}\sqrt{\psi^2 + \eta^2}\ \sin(\theta_{23} + \varepsilon) \quad .$$

where $\varepsilon = \tan^{-1}(\psi/\eta)$, $a = a_2{}^2 - a_{34}{}^2$

Thus, we have

$$\sin(\theta_{23} + \varepsilon) = (\psi^2 + \eta^2 - a)/-2a_{34}\sqrt{\psi^2 + \eta^2}$$
$$\cos(\theta_{23} + \varepsilon) = \pm\sqrt{1 - \sin^2(\theta_{23} + \varepsilon)} \quad .$$

Hence,
$$\tan(\theta_{23} + \varepsilon) = \frac{\sin(\theta_{23} + \varepsilon)}{\cos(\theta_{23} + \varepsilon)}$$

or
$$\theta_{23} = \tan^{-1}\left(\frac{\pm(\psi^2 + \eta^2 - a)/(-2a_{34}\sqrt{\psi^2 + \eta^2})}{\sqrt{1 - \{(\psi^2 + \eta^2 - a)/(-2a_{34}\sqrt{\psi^2 + \eta^2})\}^2}}\right) - \tan^{-1}\left(\frac{\psi}{\eta}\right) \qquad (70)$$

$$s_{23} = \sin\theta_{23}$$
$$c_{23} = \cos\theta_{23}$$

(4) Calculation of θ_4

Making use of Eqs. (16) and (18), c_4 and s_4 are described as follows.

$$c_4 = B\ c_1 + D\ s_1$$
$$s_4 = (-B\ s_1 + D\ c_1)/c_{23} \quad (c_{23} \neq 0)$$

if $c_{23} = 0$, we use Eq. (20) to determine s_4.

or $$s_4 = -F/s_{23}$$

Thus, we have

$$\theta_4 = \tan^{-1}\left(\frac{s_4}{c_4}\right) . \qquad (71)$$

(5) Calculation of θ_5

By Eq. (15) × c_1 - Eq. (17) × s_1, we obtain

$$\begin{aligned} s_5 &= (Ac_1 - Cs_1)/s_4 \\ &= \{c_1(-n_x c_6 + o_x s_6) - s_1(n_y c_6 - o_y s_6)\}/s_4 \qquad (72) \end{aligned}$$

$$(s_4 \neq 0)$$

On the other hand, Eq. (15) × s_1 + Eq. (17) × c_1 leads to

$$c_5 s_{23} + s_5 c_4 c_{23} = A\ s_1 + C\ c_1 . \qquad (73)$$

In the case of $s_4 = 0$, Eq. (73) × c_{23} - Eq. (17) × s_{23} results in

$$s_5 = \{c_{23}(A\ s_1 + C\ c_1) - E\ s_{23}\}/c_4 . \qquad (74)$$

From the computation Eq. (73) × s_{23} + Eq. (17) × c_{23}, it holds that

$$c_5 = (A\ s_1 + C\ c_1)s_{23} + E\ c_{23} . \qquad (75)$$

Hence,

$$\theta_5 = \tan^{-1}\left(\frac{s_5}{c_5}\right) \qquad (76)$$

(6) Calculation of θ_2

Eqs. (43) and (44) give:

$$c_2 = \{p_z - a_6 n_z - a_1 - a_5(n_z c_6 - o_z s_6) - a_{34} c_{23}\}/a_2$$

$$s_2 = \{-p_x s_1 + p_y c_1 + a_6(n_x s_1 - n_y c_1) - a_5(n_y c_1 - n_x s_1)c_6 - a_5(s_1 o_x - o_y c_1)s_6 - a_{34}\ s_{23}\}/a_2$$

Therefore, we have

$$\theta_2 = \tan^{-1}\frac{s_2}{c_2}\ . \tag{77}$$

(7) Calculation of θ_3

From the calculation of θ_{23} and θ_2, we can get

$$\theta_3 = \theta_{23} - \theta_2 \tag{78}$$

In the computer code, the solutions for joint angles obtained in this manner are fed into the direct kinematic routine (Subroutine CHK) in order to re-calculate the individual elements of the arm matrix T_6, where the evaluation is performed as to what degree the prediction of location and orientation based on these joint angles solutions could agree with the input data given to solve the inverse kinematic problem.

When the above error checks are completed under the appropriate convergence criteria, desired solutions are determined within their operation limits.

4. Test Calculations

The following three sample problems were tested to verify the validity of computer code for the inverse kinematic solutions. They were restricted to a fundamental simulation such that the finger-tip of the manipulator moves along a simple straight line trajectory in the Cartesian space. As one approximation of this trajectory from the initial point A to terminal point B, we divided the path segment $\overline{AB}$ into n-equidistant points, named "position numbers". They start at the initial setpoint (N=0) and end at the (n+1)th point — goal position (N=n).
The location and orientation at each point are calculated by means of a linear interpolation method.

In every case tried here, the orientation of the end-effector was kept unvaried during a movement.
The calculated result of each joint is represented as the angular displacement from the home position ($\theta_i = 0$), of the manipulator shown in Fig.1.a.
As a basic point to keep in mind here, we restricted attention only as to how to solve the inverse kinematics problem. Thus, no consideration is given relative to arm dynamics and compliant motion while the manipulator moves to destination along the trajectory.

(1) Horizontal movement in the x-direction

(see Fig.8)

. Position co-ordinate of the initinal point A =

(-250,780,1201) (mm in unit)

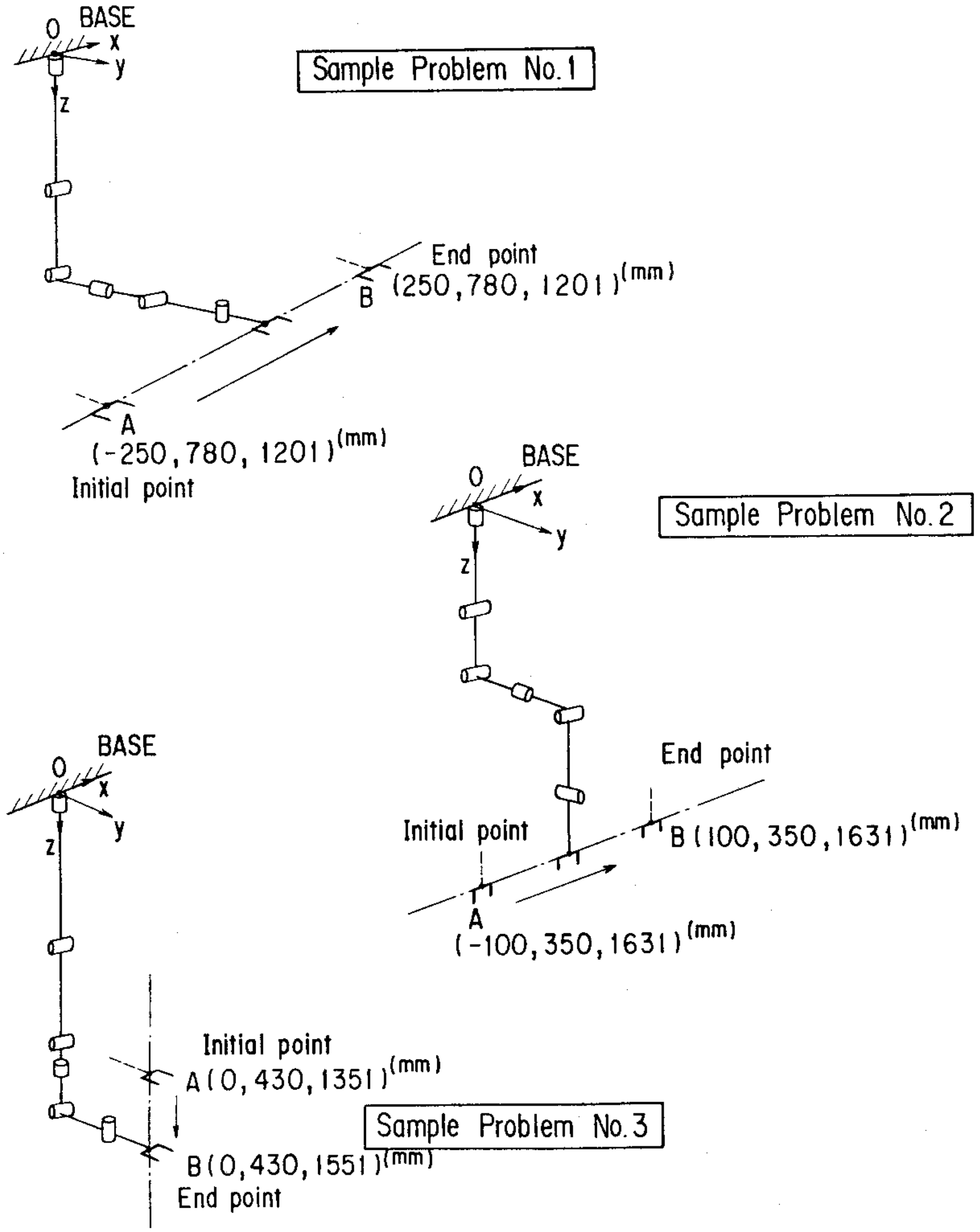


Fig.8 Motions between Positions

Position co-ordinate of the terminal point B =

(250,780,1201) (mm in unit)

- Number of points (position numbers) = 100
- Direction cosines

NX = 0.0 : OX = 1.0

NY = 1.0 : OY = 0.0

NZ = 0.0 : OZ = 0.0

(The same data is used at the terminal point because a constant direction is intended.)

- Convergence condition

$EPS = 10^{-4}$, $EPS1 = 10^{-3}$

Under the above motion scheme, the calculated joint angles are presented in Figs. 9.1 through 9.6.
As the first example applied, 101 sampling points are assigned between the initial position and destination.
For a single solution of θ_6 and θ_5 , the remaining joint angle solutions possess three possible combinations as designated by the symbols G_1,G_2,and G_3,respectively.

In these figures, the graph G_2 shows the complete solution curve corresponding to the motion of the end-effector from the starting point to the end.

On the other hand, the graphs G_1 and G_3 exhibit the partial solutions feasible in restricted intervals.

As can be noticed, the graph G_1 terminates at the position 86 for the above manipulator motion scheme, while the graph G_3 starts at the point 14 and the first portion of the path is not feasible. On examination of computed results, it was confirmed that the existence of joints angle solutions for restricted path intervals was merely caused by the lower or upper limits, respectively, of the operation conditions with respect to the joint angle θ_1 . The accuracy of solutions was of the order of less than 10^{-8} on the average.

(2) Horizontal movement of a downward directed end-effector (see Fig.8)

. Position co-ordinate of the initial point A =
(-100,350,1631) (mm in unit)
Position co-ordinate of the terminal point B =
(100,350,1631) (mm in unit)

. Number of points (position numbers) = 41

. Direction cosines

NX = 0.0 : OX = 1.0
NY = 0.0 : OY = 0.0
NZ = 1.0 : OZ = 0.0

. Convergence condition

$EPS = 10^{-4}$, $EPS1 = 10^{-3}$

Prior to the present calculation, we firstly set the values of x-coordinate to be -250 and 250, similar to the sample problem 1, and then tried to compute.
Since this input data corresponded to a position in-accessible by any state of the manipulator, however, no

feasible solutions were found. Hence, the new data given above was used as the second trial.

The details of calculated results are given in Figs. 10-1-1 to 10-6-1 and Figs.10-1-2 to 10-6-2.

In this computation, we obtained two feasible solution groups — case 1 and case 2. This implies that a polynomial of 24-th order had two different roots, compared to the one of the sample problem 1.

Fig. 10.a exhibits the solution behavior of this polynomial f(tan(θ_6/2)) at the initial point and terminal point, respectively. As shown, the equation has undoubtedly two real roots under the constraint condition of the end-point of the manipulator arm ,that is, $|\theta_6| \leq 60$ (deg).

In this way, the detailed behaviors on this function are presented in the form of a graph at each step of position.

Seeing case 1, five solution curves G_1 through G_5 for joint angle variables θ_1,θ_2,θ_3,and θ_4 were also present corresponding to each value of the angle θ_6 .

Excluding the graph G_2, all the rest of computed curves are considered, from the viewpoint of physical realizability, to give partially possible solutions found in the intermediate interval while the end-effector is moving toward the goal position.

For instance, when we see the graphs G_3 and G_4 for θ_1-plots, the continuity of both curves is seemingly kept at the position 21 as if they were of a single unified solution curve. However,as can be seen in θ_4 -plots, these two graphs (G_3 and G_4) are cut off in the neighborhood of the upper or lower limits due to the specified range condition of the joint angle θ_4 .

Therefore, in this case, the graphs G_3 and G_4 must be regarded as different solution groups, respectively, although the graph G_4 is mathematically valid as the solution curve following the graph G_3.

As exemplified in these figures, it should be particularly noted that all sequence of mathematical exact solutions does not represent realizable solution curves corresponding to the restricted manipulator motion. Some of them are partial feasible solutions coming from the mechanical constraint conditions.

On the other hand, the calculation (i.e., case 2) for another root of θ_6 displays that individual joint solutions were uniquely determined. (see Figs. 10-1-2 to 10-6-2)

(3) Vertical movement of the end-effector

(see Fig. 8)

- Position co-ordinate of the initial point A = (0,430,1351) (mm in unit)

  Position co-ordinate of the terminal point B = (0,430,1551) (mm in unit)

- Number of points (position numbers) = 40
- Direction cosines

NX = 0.0	OX = 1.0
NY = 1.0	OY = 0.0
NZ = 0.0	OZ = 0.0

. Convergence condition

EPS = 10^{-4} , EPS1 = 10^{-3}

Finally, the analytical results of individual joint angles for the sample problem 3 are shown in Figs. 11-1 to 11-6. For the movement from the point A to B, no rotation at the finger tip was computed throughout the whole durations. Like the above two problems, graphs G_1, G_2 and G_3 were adopted as the feasible solutions relating to the angles θ_1, θ_2, θ_3, and θ_4 . The cause that the graph G_1 did not provide the solution at the first stage of computation, is due to the constrained condition of joint angle θ_2.

As demonstated in the above sample problems, the values of six joint co-ordinates (θ_1, θ_2, θ_3, θ_4, θ_5 and θ_6) were determined which will bring the robot hand to a specified position and orientation.

Admitting that the original kinematic relationships usually cannot be satisfied strictly due to round-off errors and limited capabilities in the numerical application, the accuracies of the present solutions, as exemplified in Fig.12, was proved to be sufficiently high for a given tolerance (EPS1 in the code). As the future problem, however, we must check in detail the characteristics of solutions for the current approach because our numerical experience, so far, was limited only to several test cases.

Input data used for the computation are listed in Appendix.

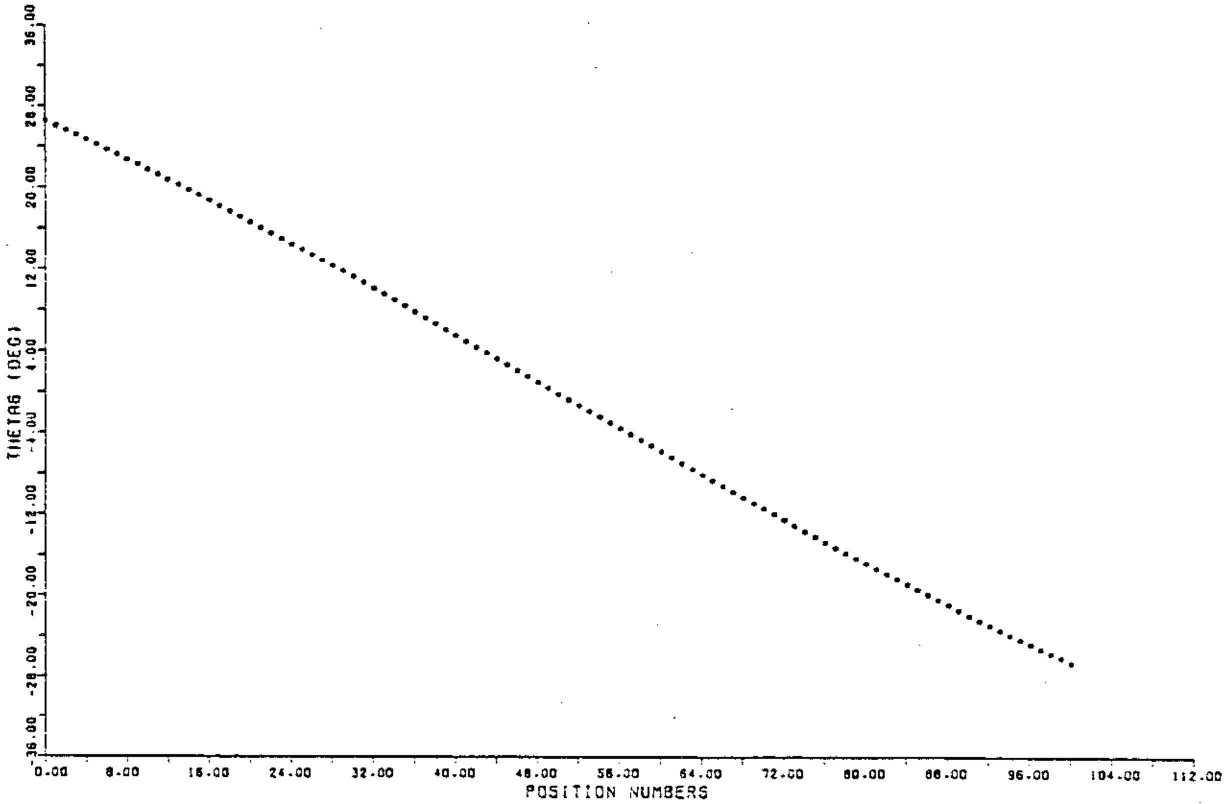


Fig.9-1 Behavior of Joint Angle θ_6 (Sample Problem #1)

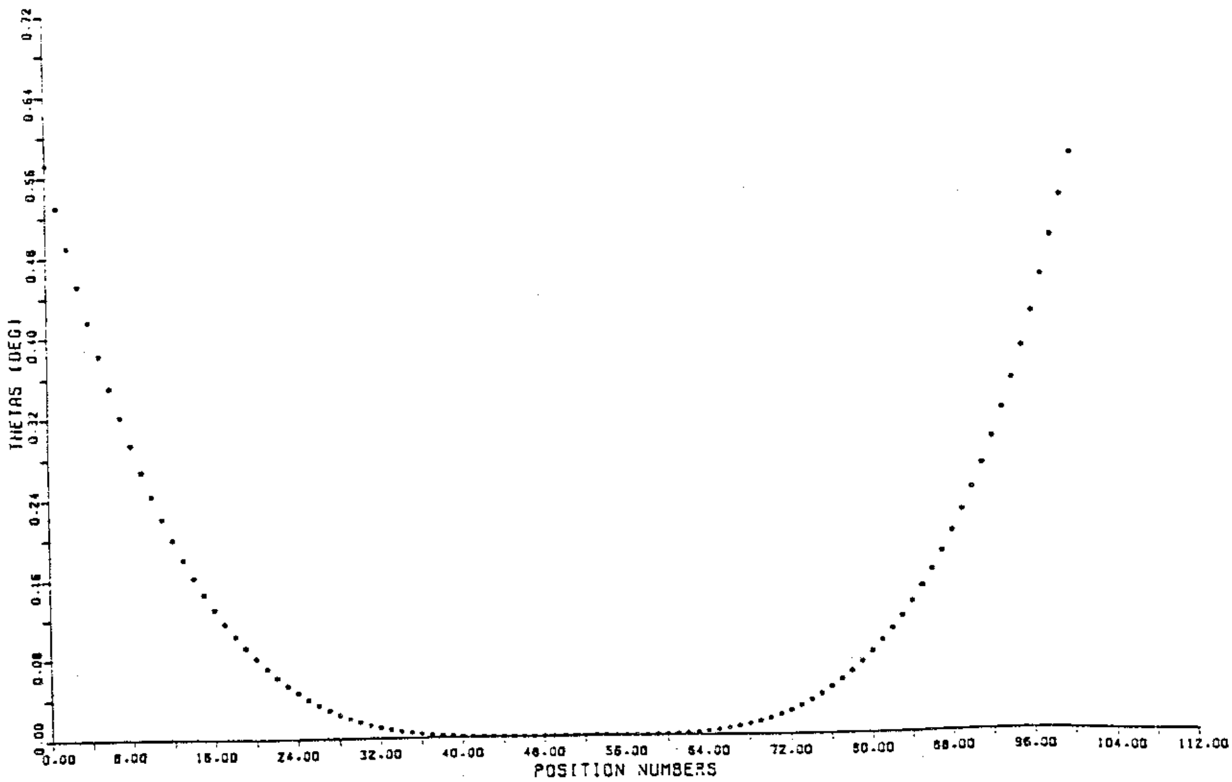


Fig.9-2 Behavior of Joint Angle θ_5 (Sample Problem #1)

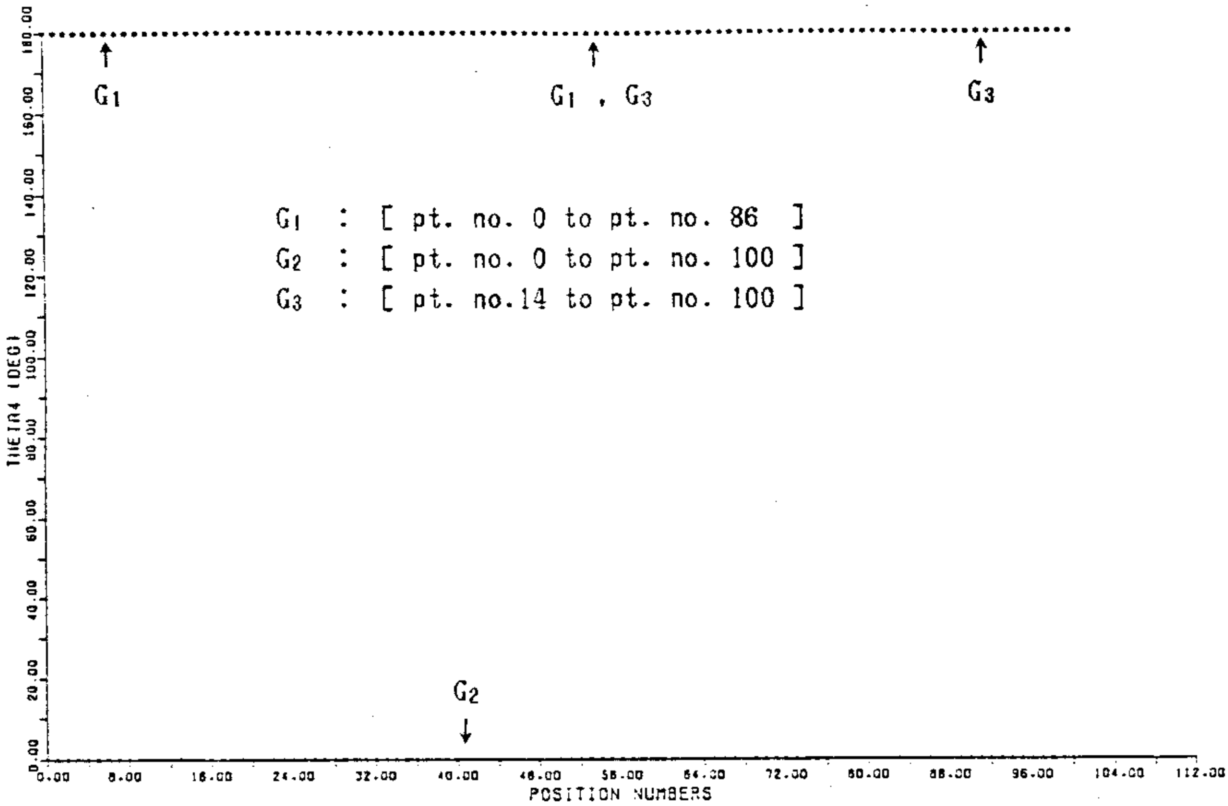


Fig.9-3 Behavior of Joint Angle θ_4 (Sample Problem #1)

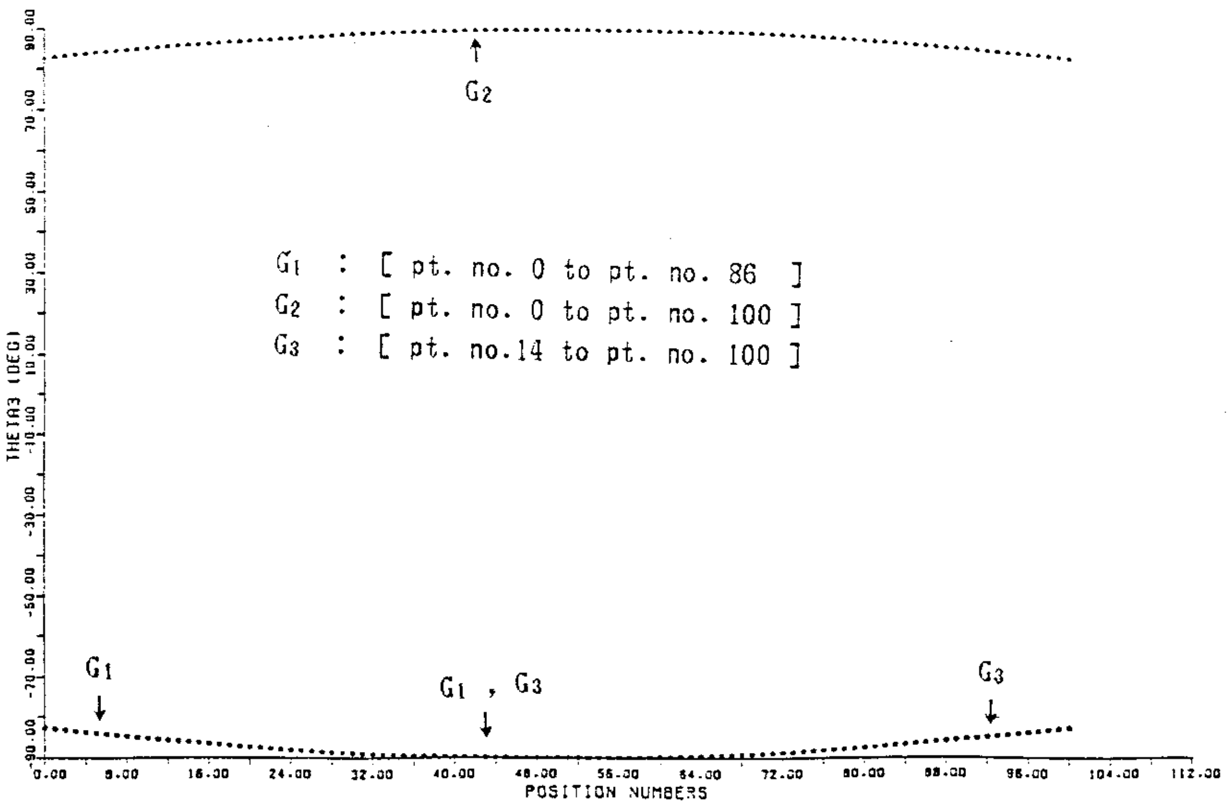


Fig.9-4 Behavior of Joint Angle θ_3 (Sample Problem #1)

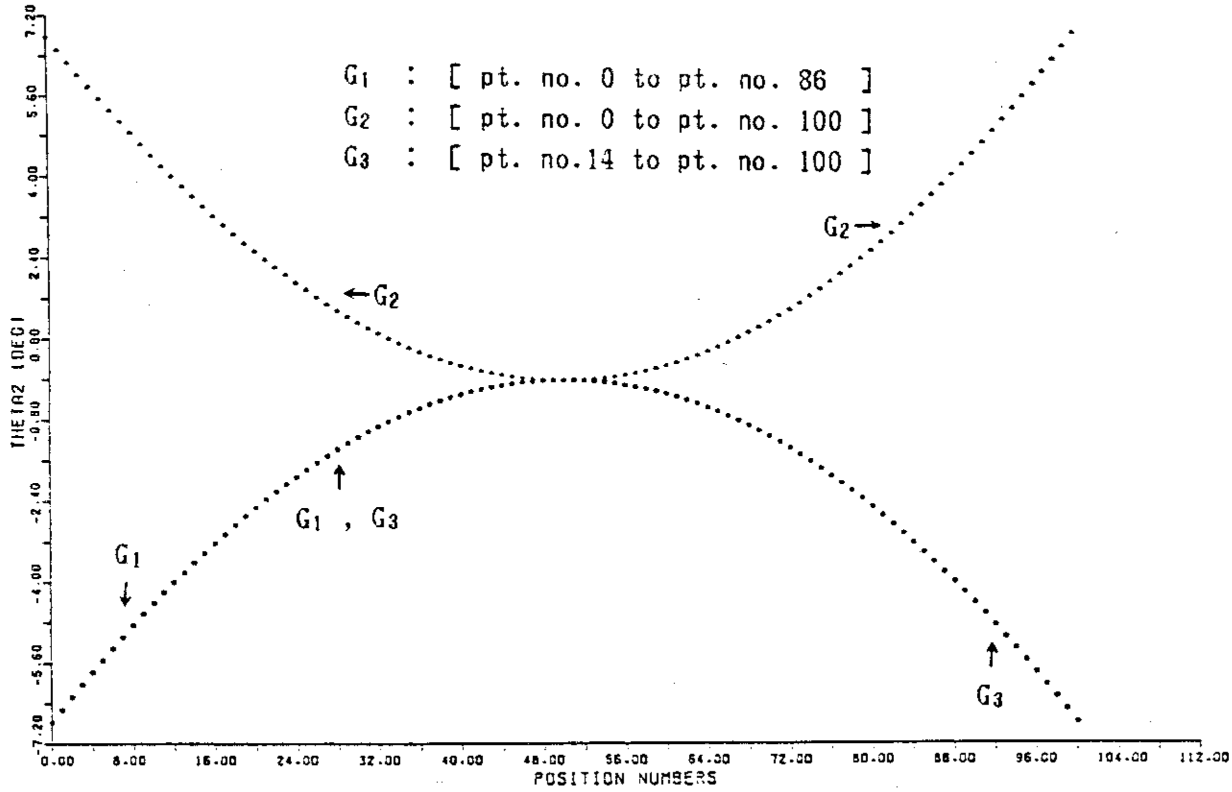


Fig.9-5 Behavior of Joint Angle θ_2 (Sample Problem #1)

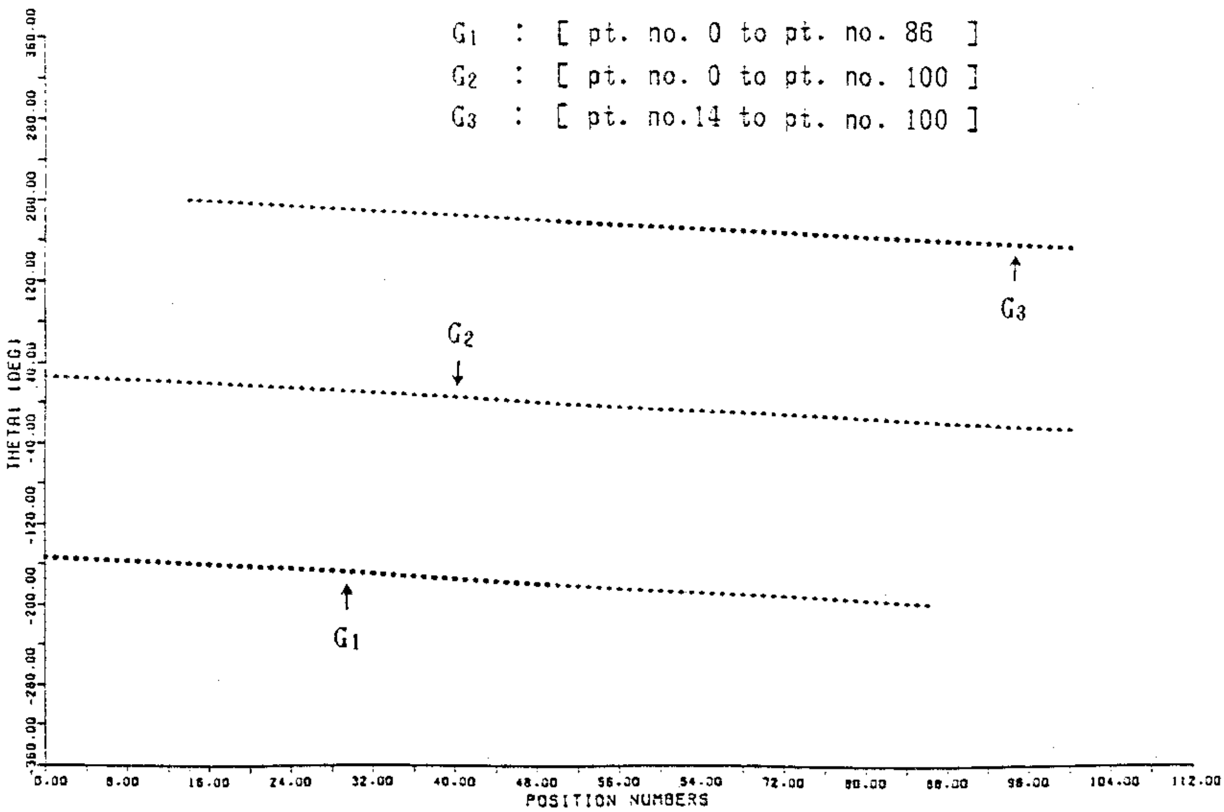


Fig.9-6 Behavior of Joint Angle θ_1 (Sample Problem #1)

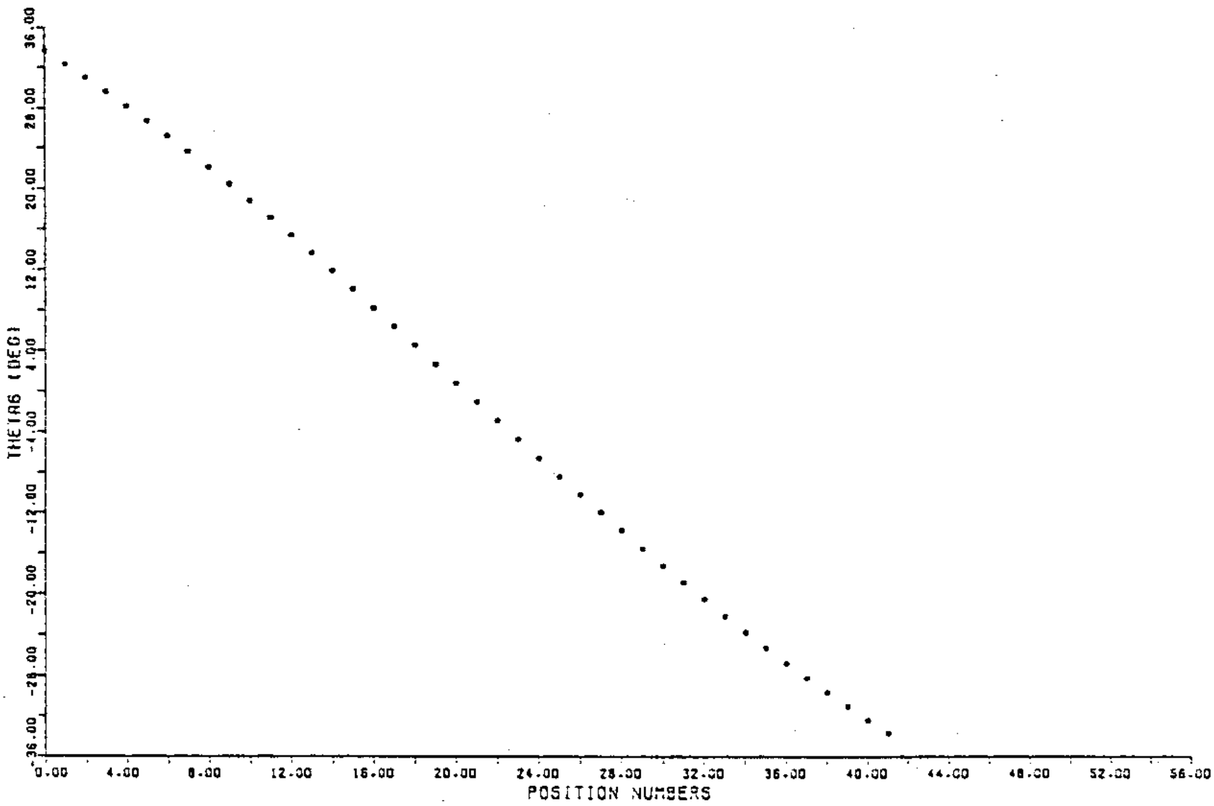


Fig.10-1-1 Behavior of Joint Angle θ_6 (Sample Problem #2-case 1)

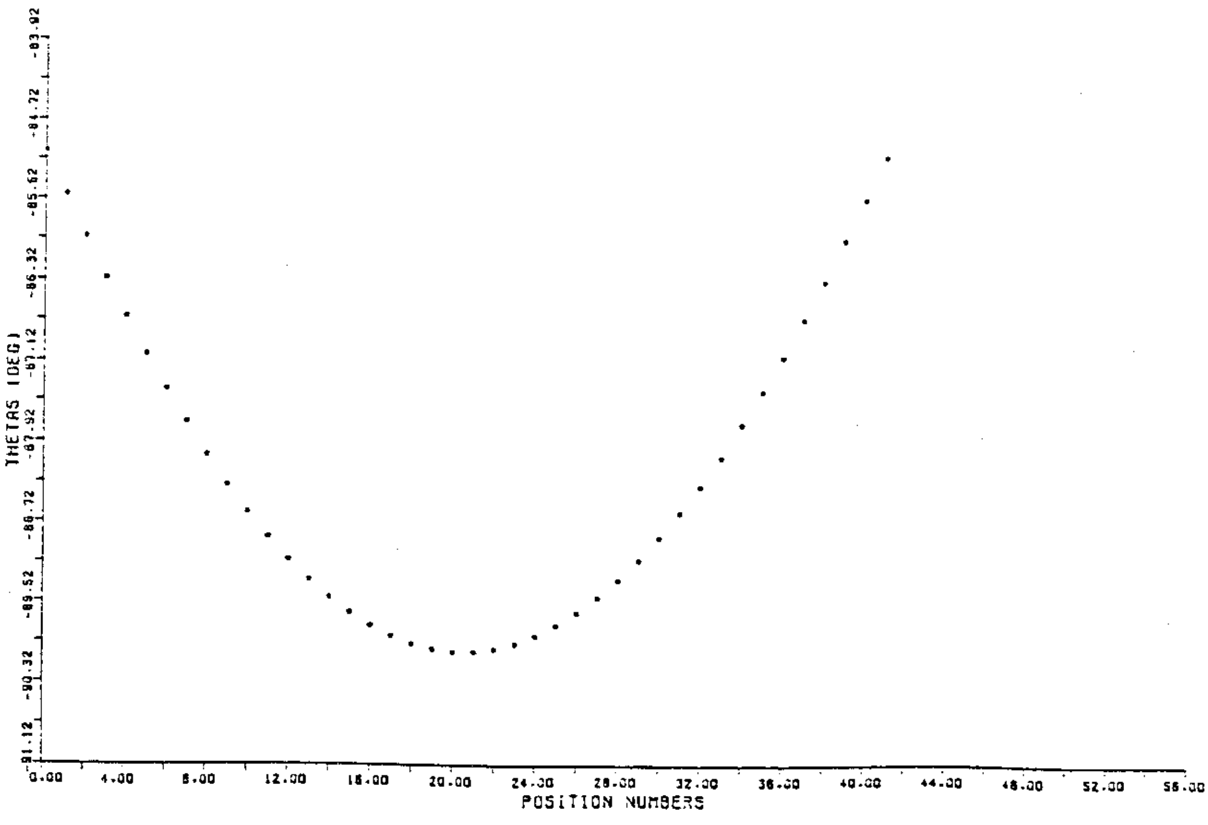


Fig.10-2-1 Behavior of Joint Angle θ_5 (Sample Problem #2-case 1)

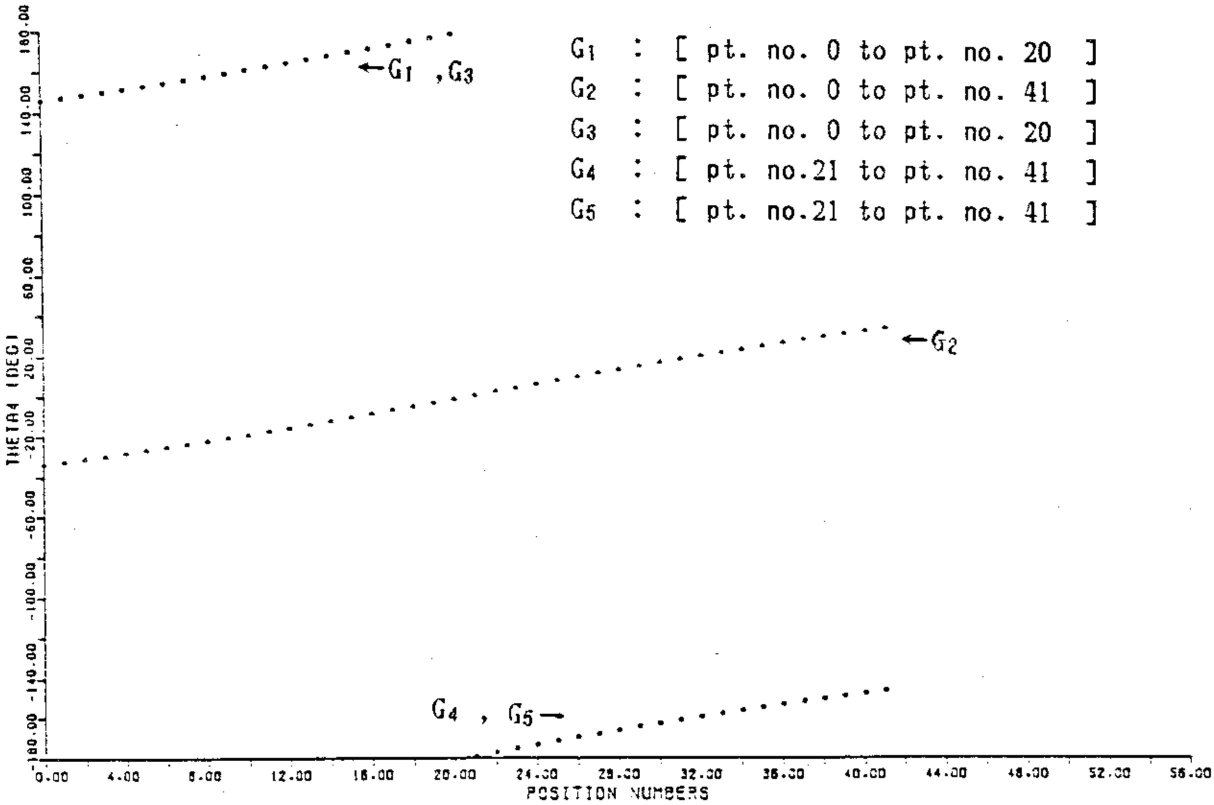


Fig.10-3-1 Behavior of Joint Angle θ_4 (Sample Problem #2-case 1)

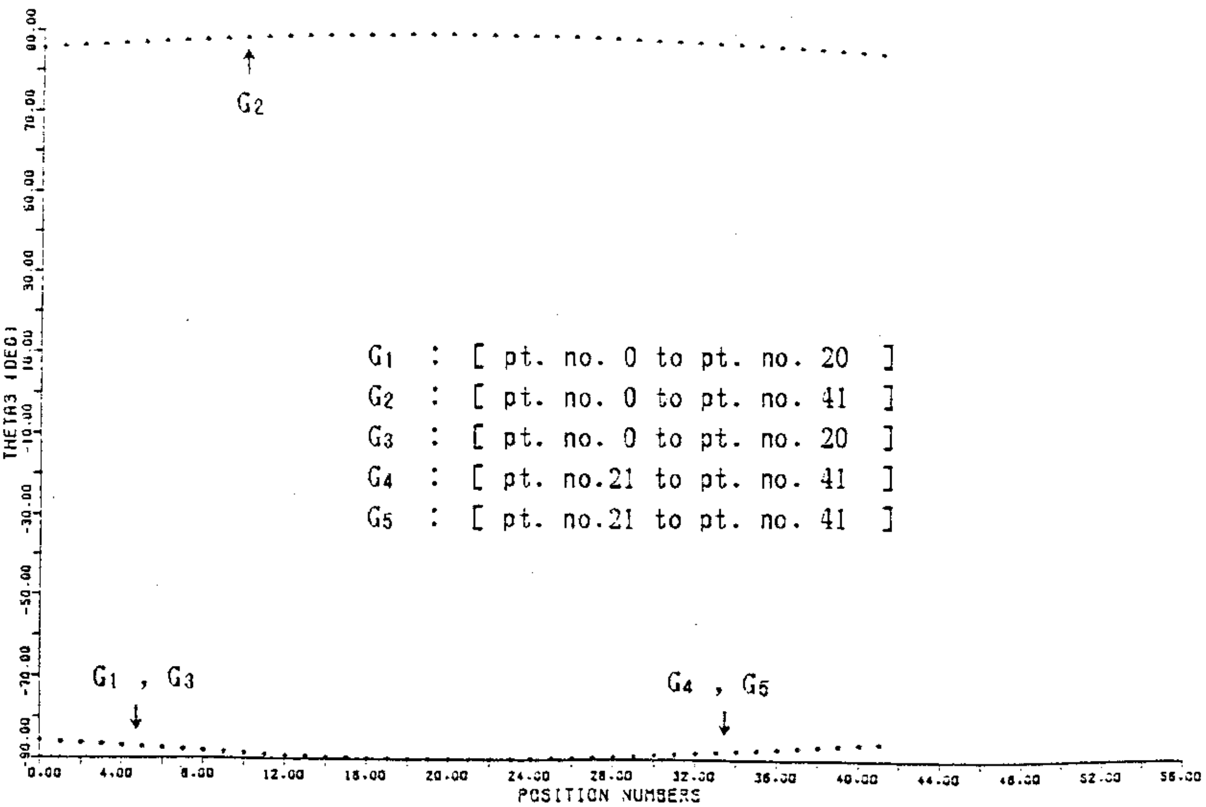


Fig.10-4-1 Behavior of Joint Angle θ_3 (Sample Problem #2-case 1)

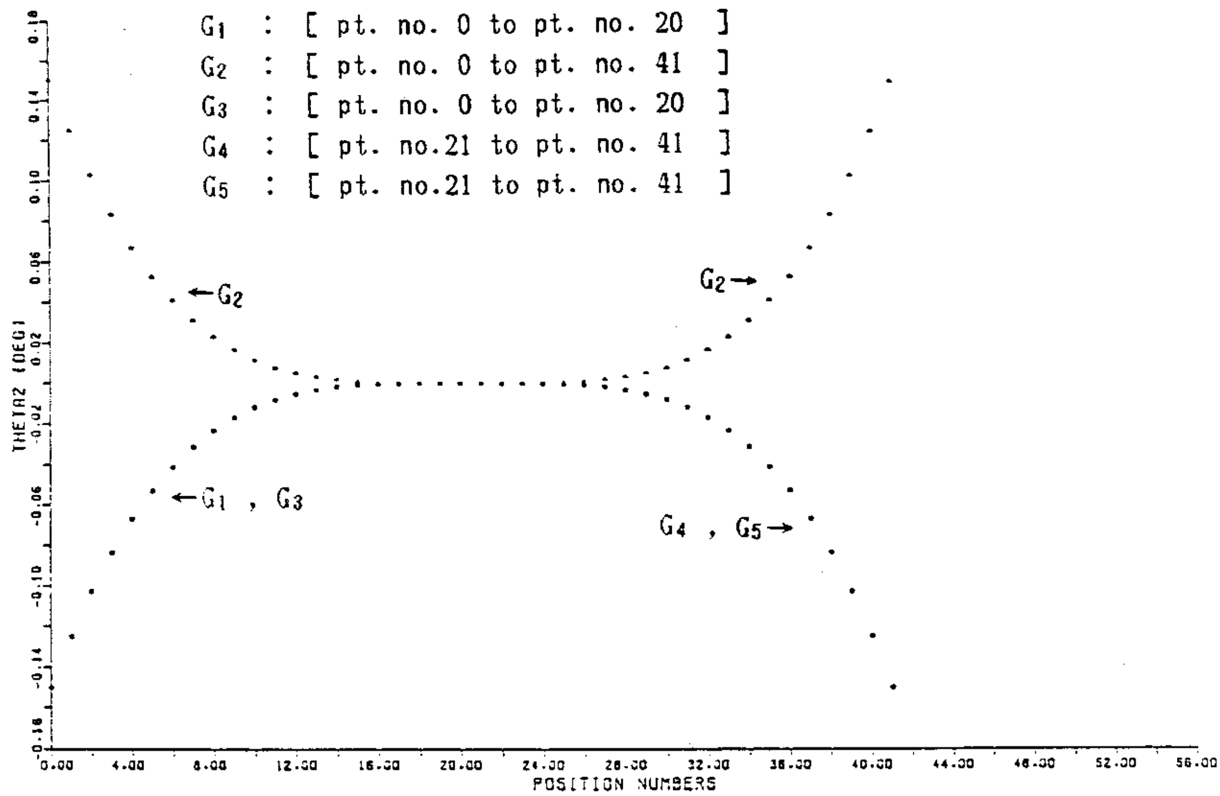


Fig.10-5-1 Behavior of Joint Angle θ_2 (Sample Problem #2-case 1)

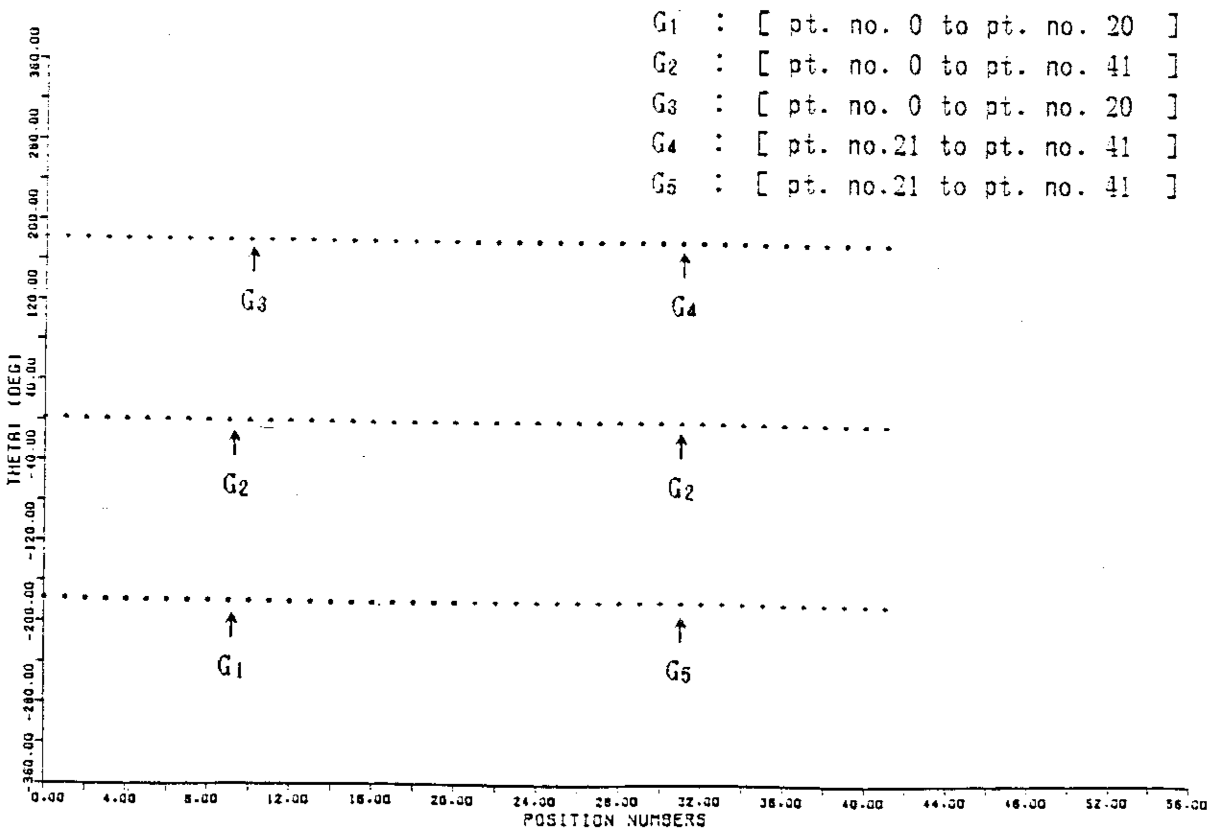


Fig.10-6-1 Behavior of Joint Angle θ_1 (Sample Problem #2-case 1)

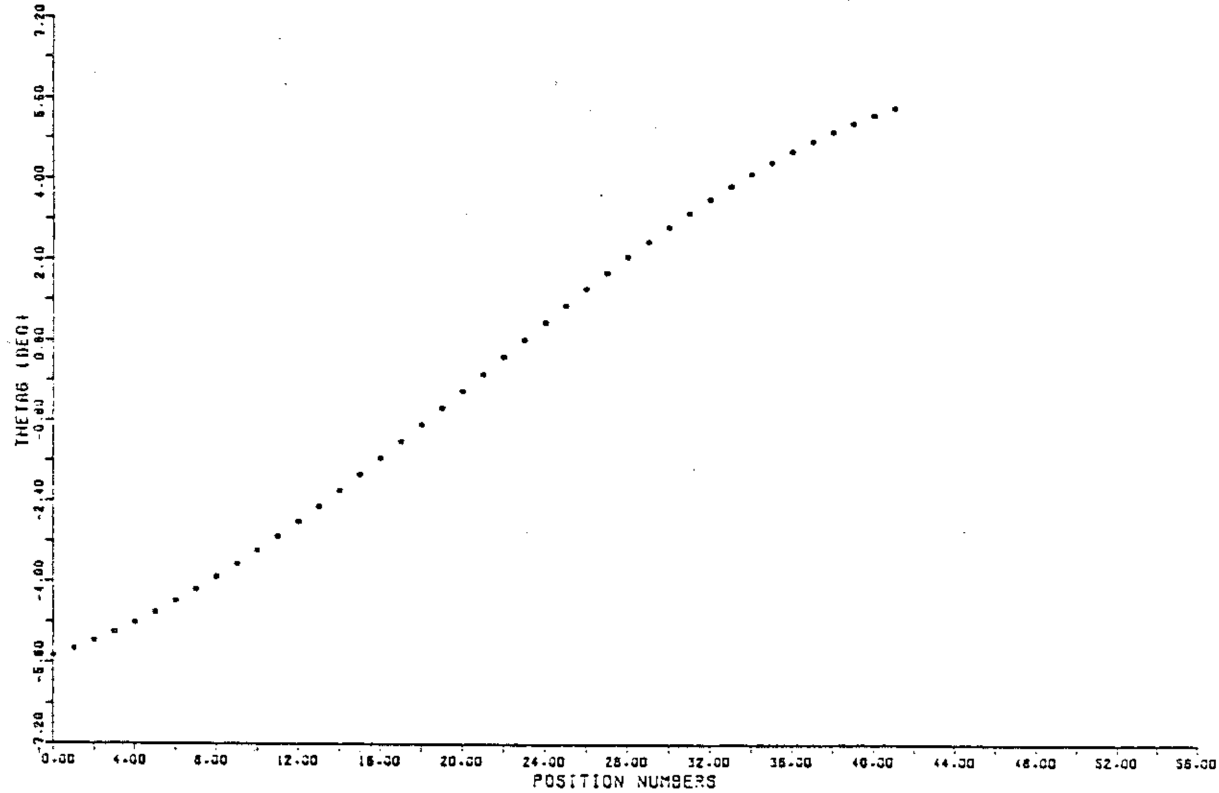


Fig.10-1-2 Behavior of Joint Angle θ_6 (Sample Problem #2-case 2)

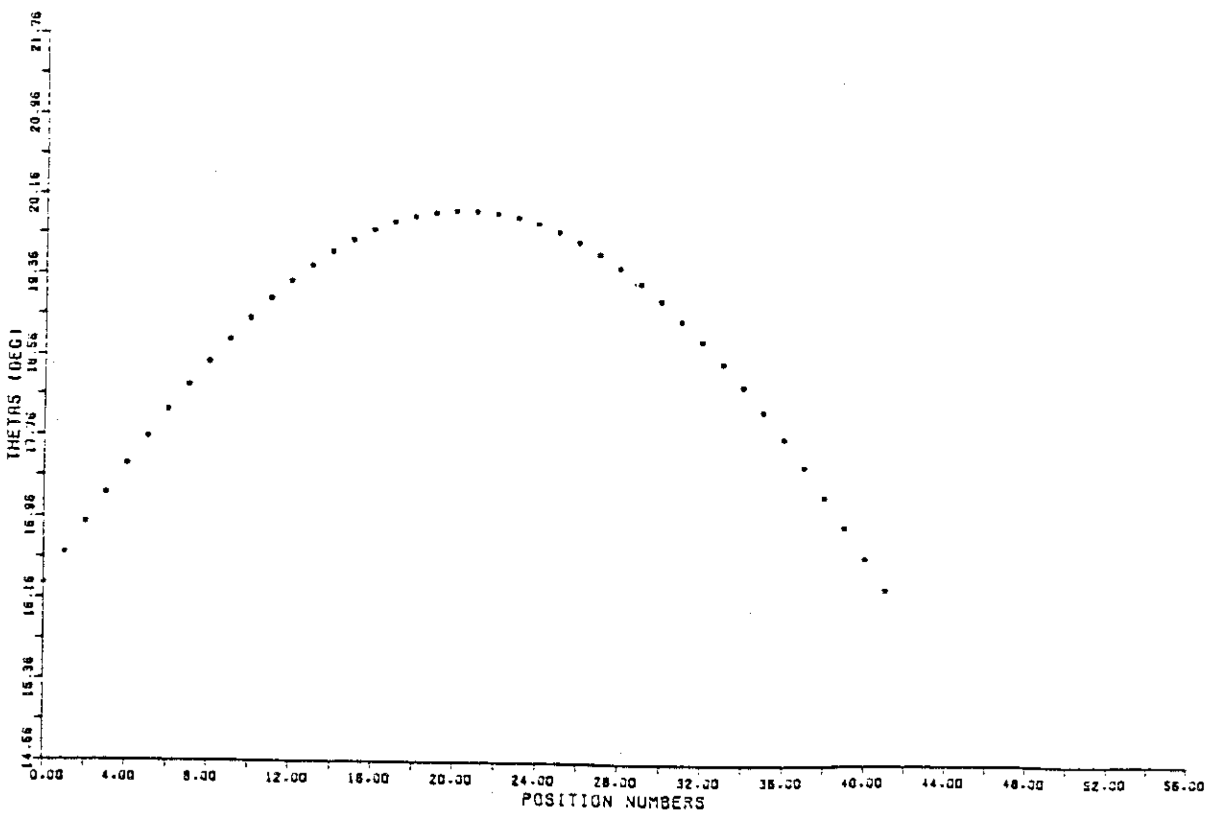


Fig.10-2-2 Behavior of Joint Angle θ_5 (Sample Problem #2-case 2)

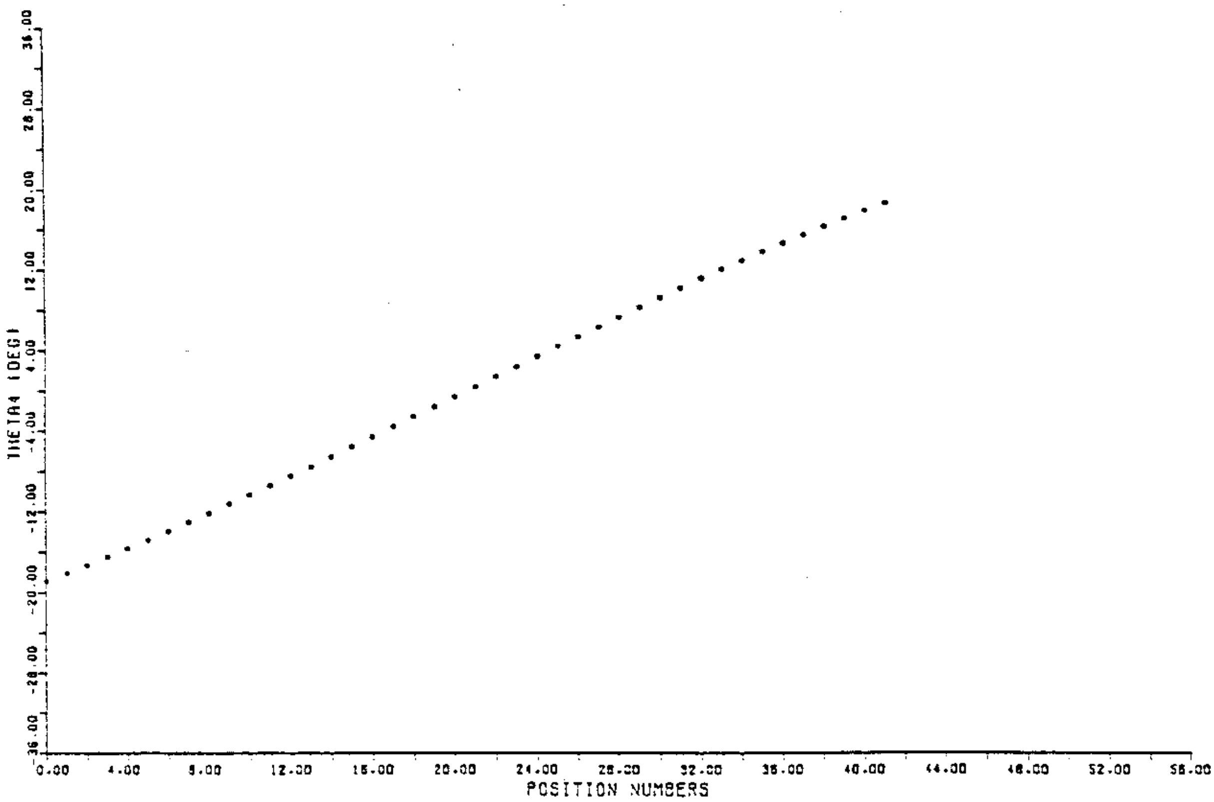


Fig.10-3-2 Behavior of Joint Angle θ_4 (Sample Problem #2-case 2)

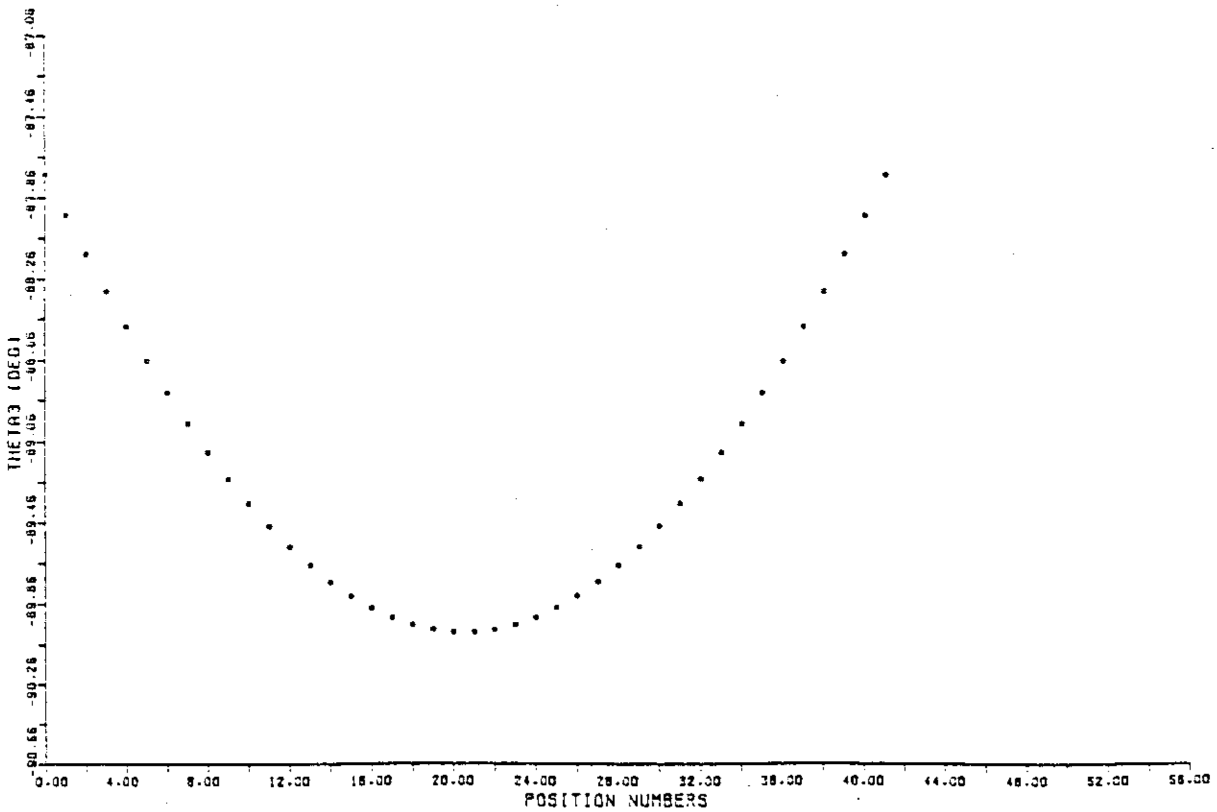


Fig.10-4-2 Behavior of Joint Angle θ_3 (Sample Problem #2-case 2)

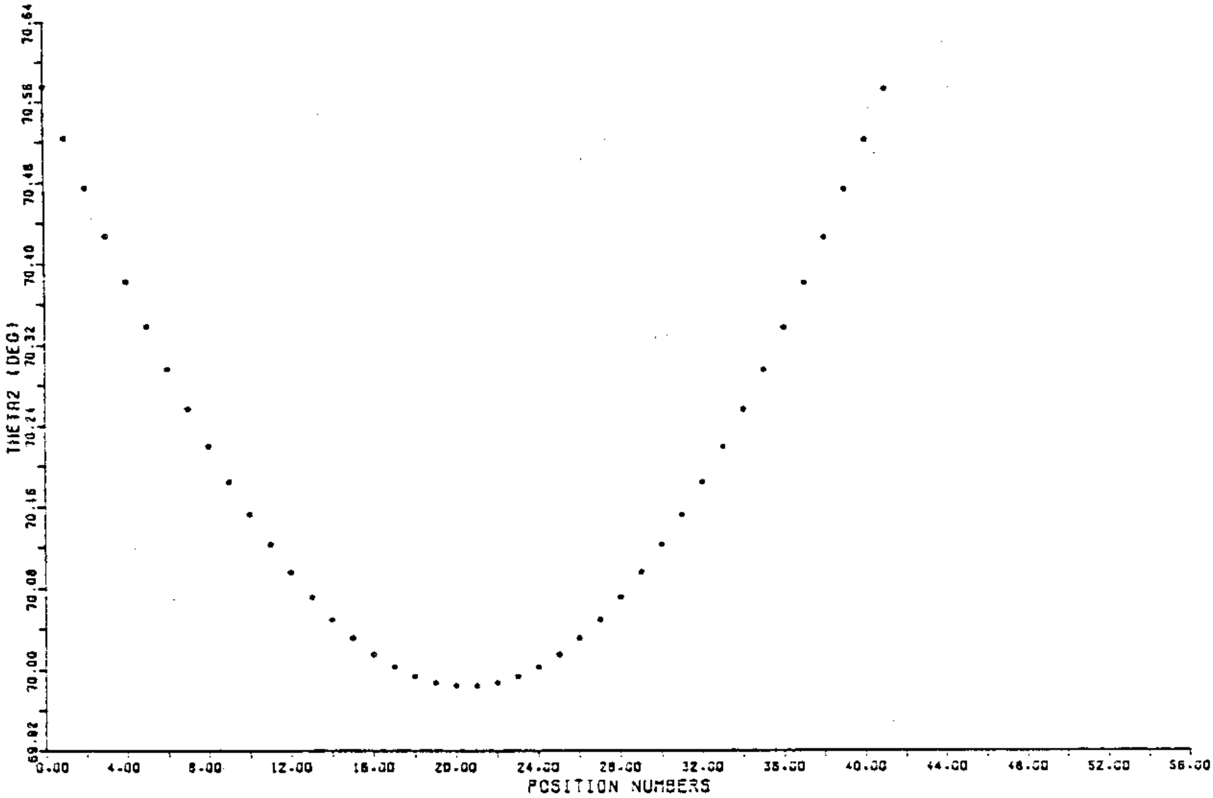


Fig.10-5-2 Behavior of Joint Angle θ_2 (Sample Problem #2-case 2)

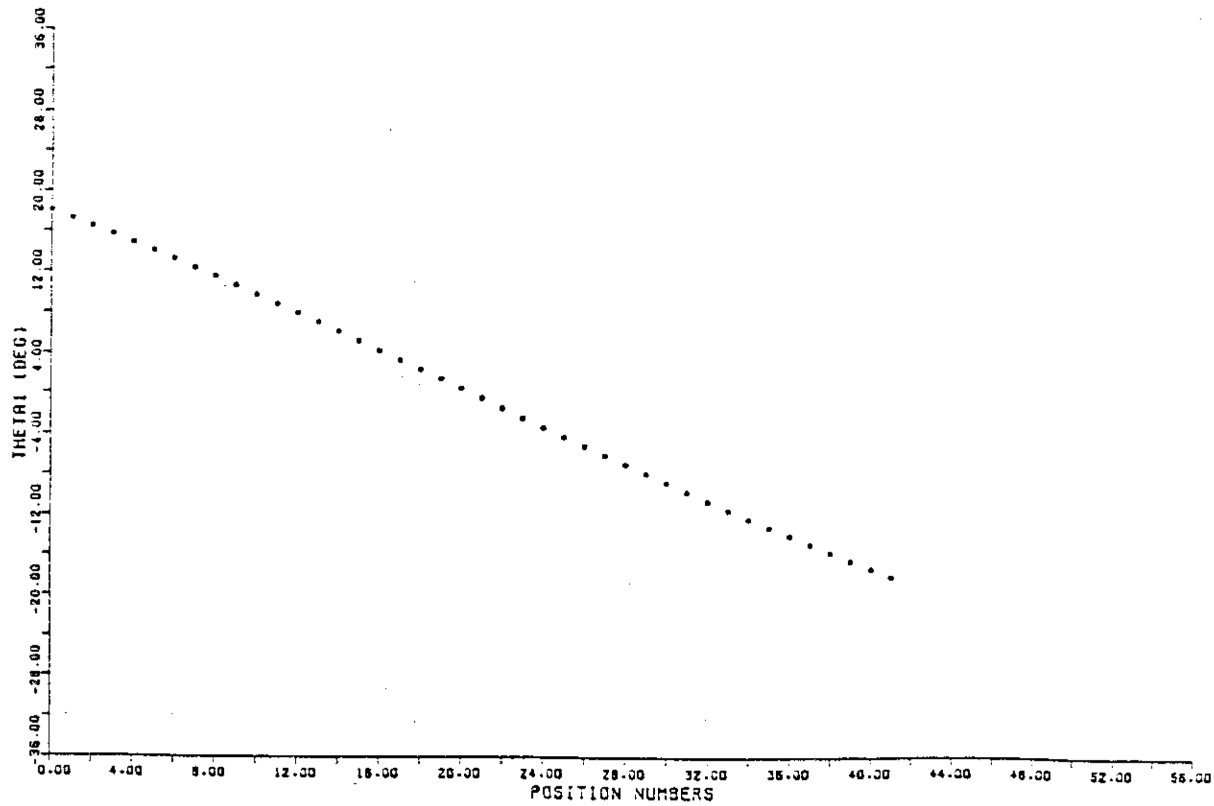
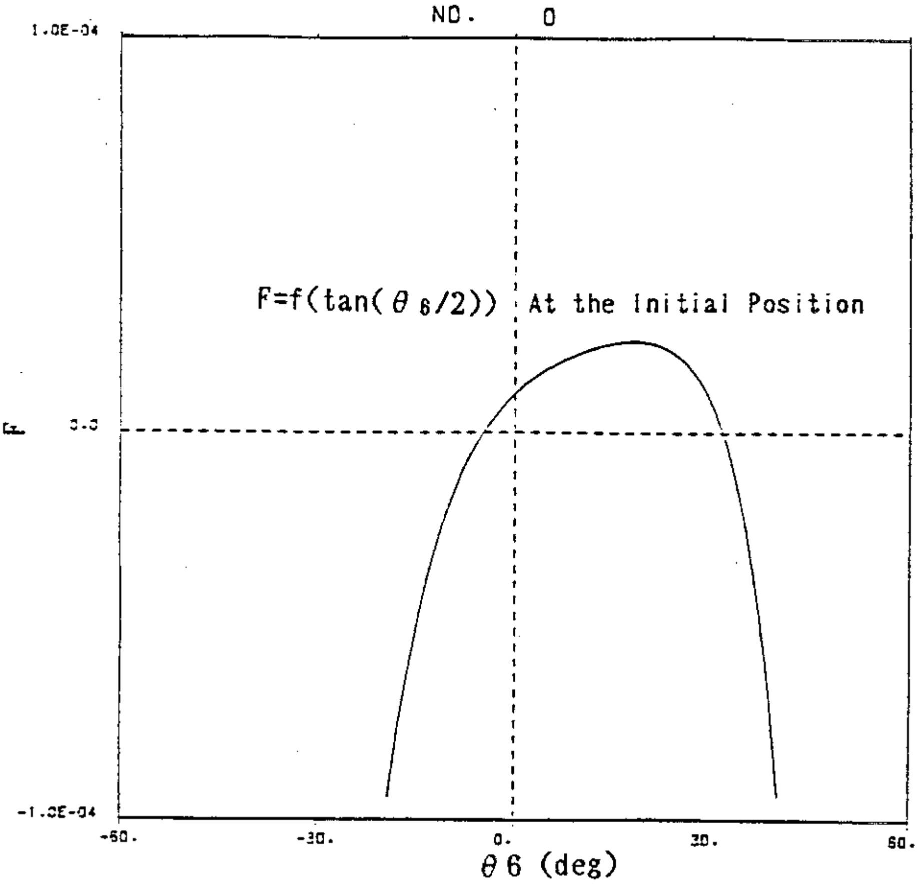


Fig.10-6-2 Behavior of Joint Angle θ_1 (Sample Problem #2-case 2)



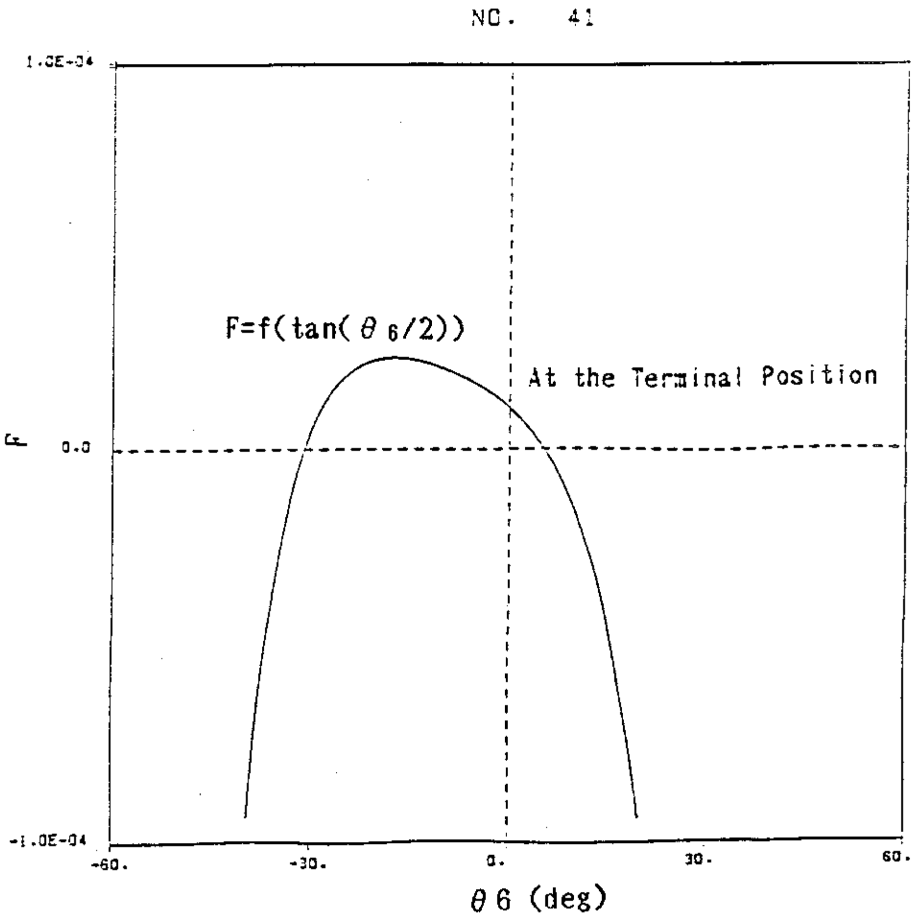


Fig.10-a Solution Behavior of a Polynomial
(At the Initial Position (N=0) and Terminal Position (N=41))

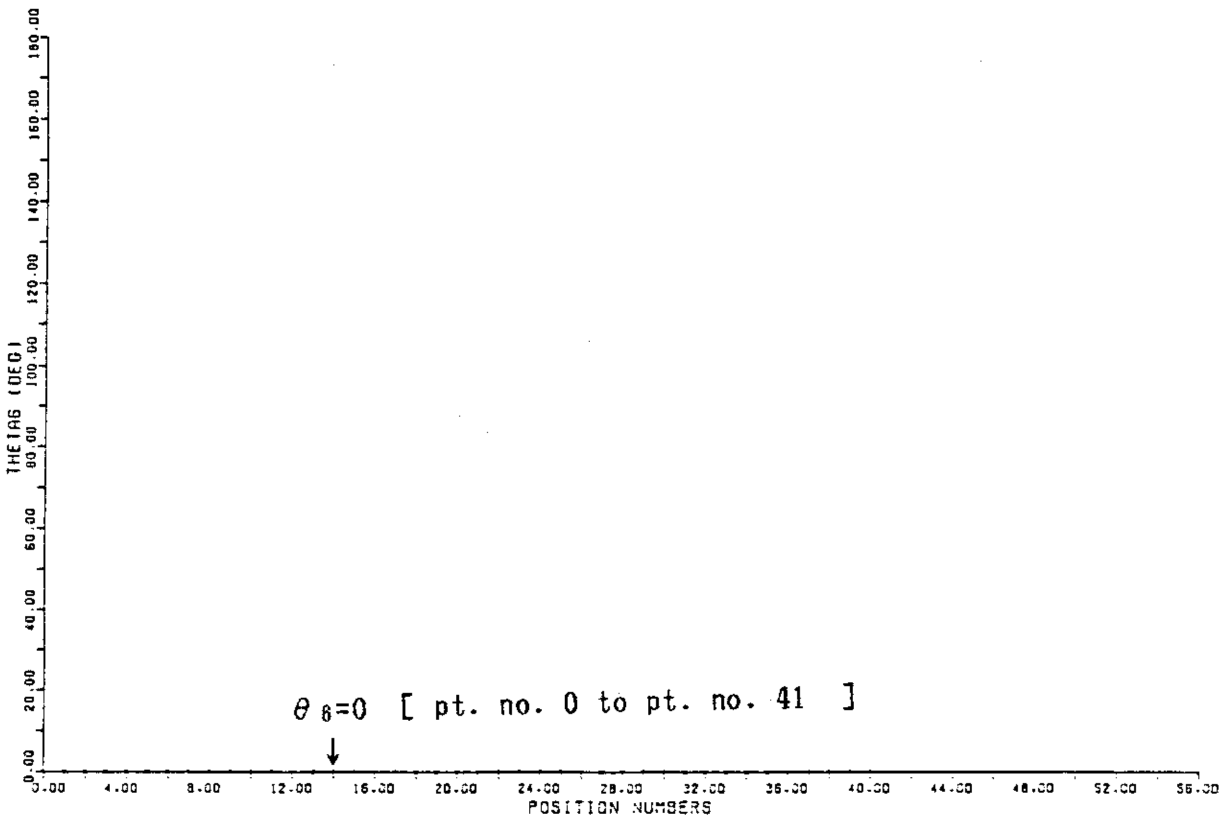


Fig.11-1 Behavior of Joint Angle θ_6 (Sample Problem #3)

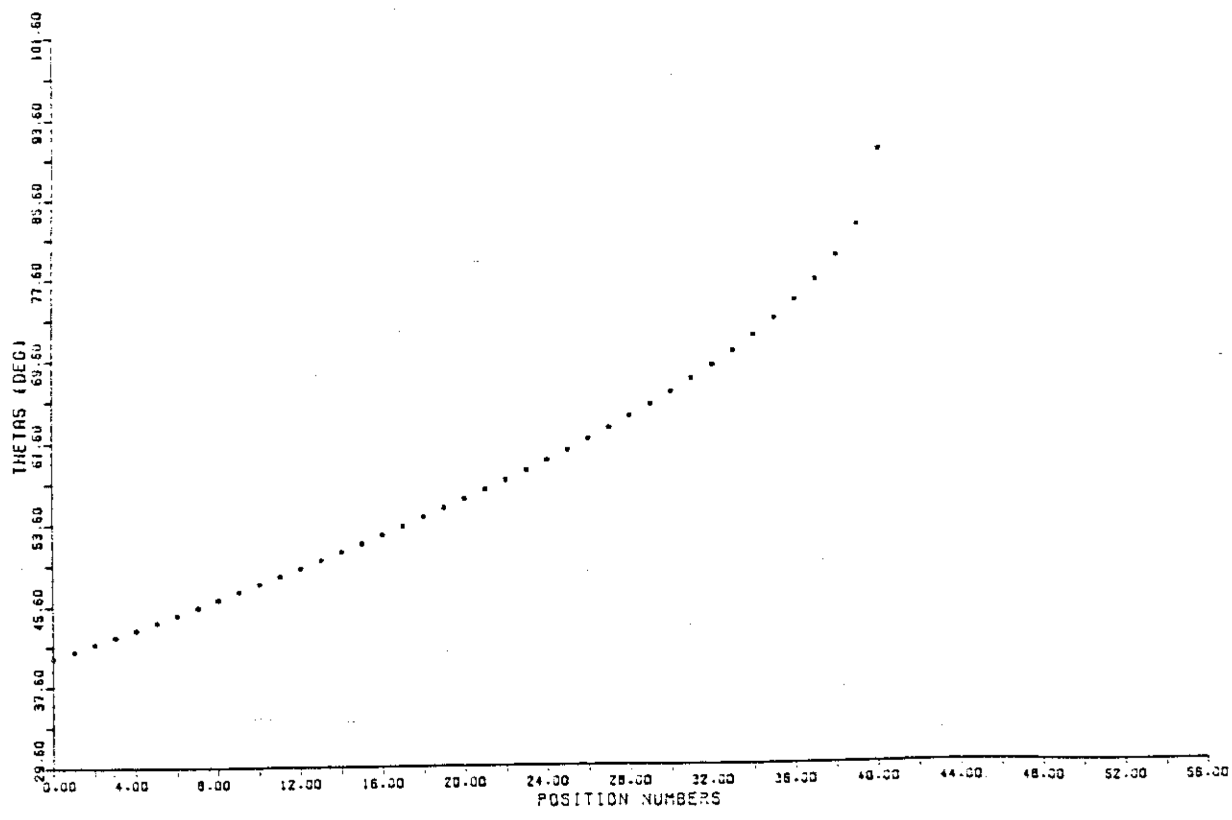


Fig.11-2 Behavior of Joint Angle θ_5 (Sample Problem #3)

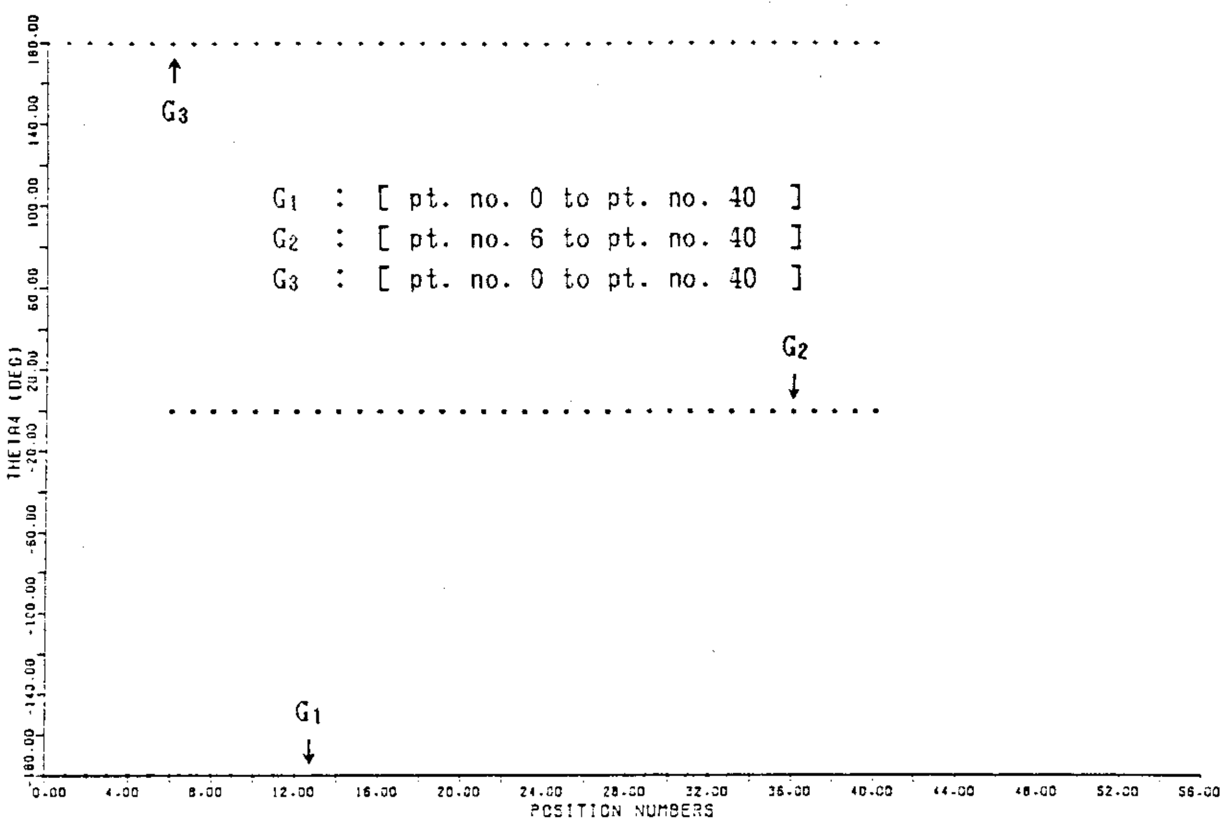


Fig.11-3 Behavior of Joint Angle θ_4 (Sample Problem #3)

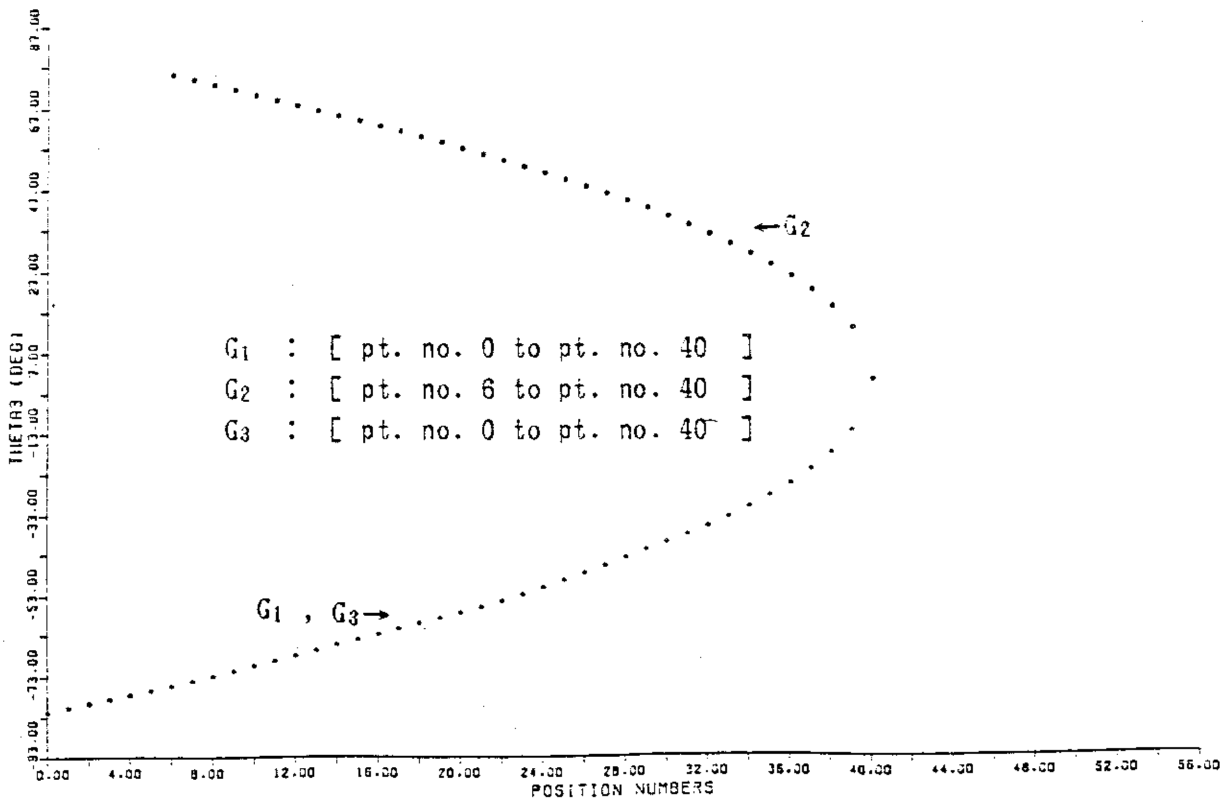


Fig.11-4 Behavior of Joint Angle θ_3 (Sample Problem #3)

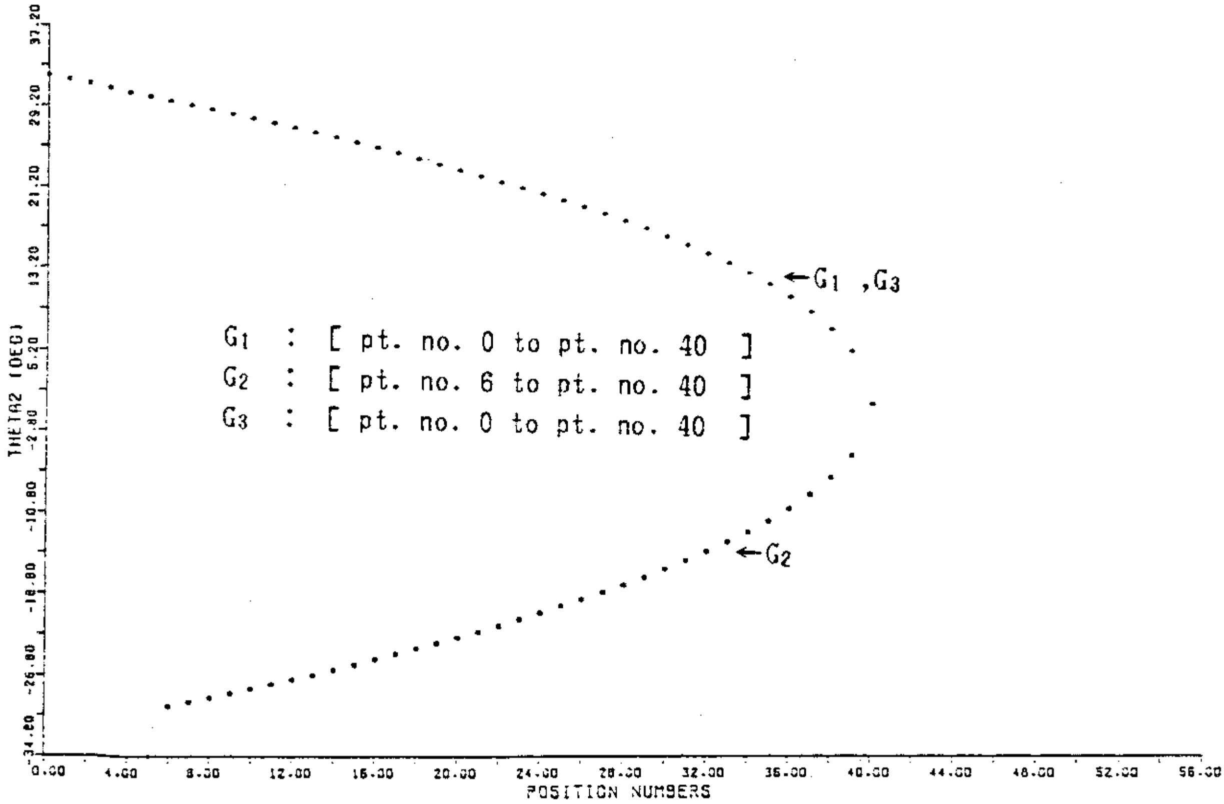


Fig.11-5 Behavior of Joint Angle θ_2 (Sample Problem #3)

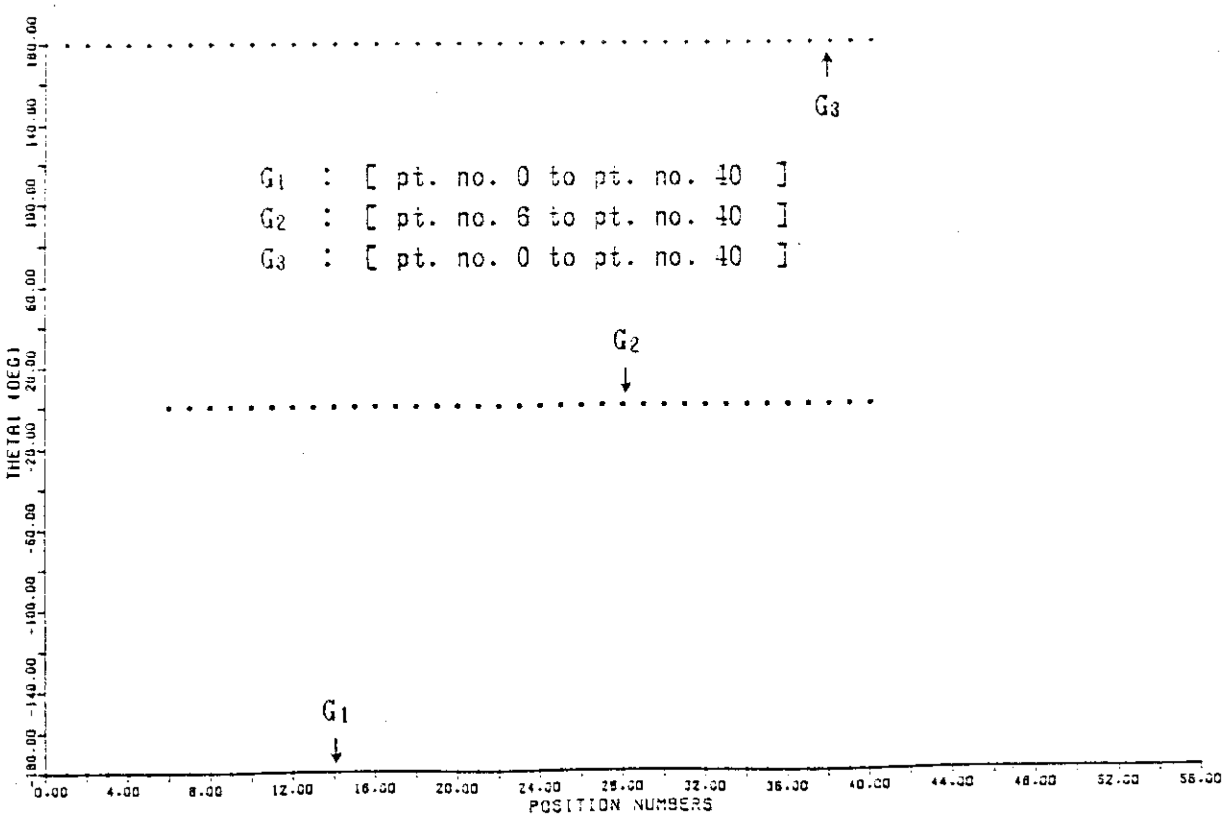


Fig.11-6 Behavior of Joint Angle θ_1 (Sample Problem #3)

```
*****************************************************************************
                                      <------ Position number ( "0" means the initial position number )
NO. --------------------------  0  ---  1
                                          <------ Group number of obtained solutions t
ANSWER OF POLYNOMIAL ---------  T   =  0.30261D+00   < T = TAN(THETA6/2) >      <------ Root of a polynomial f(t)=0

 ADOPT ANSWERS (DEG) ----------

   THETA1 = -177.2467    THETA2 =    -0.1502    THETA3 =   -85.7259           <------ Solutions of joint angles
   THETA4 =  146.2277    THETA5 =   -85.0428    THETA6 =    33.6731                   ( Deg )

                                                                                      Input data to the
                                                                                      inverse problem
INPUT                   0.0          :  1.00000D+00 :  0.0          : -1.00000D-01  <------ ( nx ox ax px )
        VALUES ---      0.0          :  0.0         :  1.00000D+00  :  3.50000D-01          ( ny oy ay py )
                        1.00000D+00  :  0.0         :  0.0          :  1.63100D+00          ( nz oz az pz )

CALCULATED              1.09379D-08  :  1.00000D+00 : -7.83633D-10  : -1.00000D-01  <------ Te matrix reproduced by
        VALUES ---     -1.17620D-09  :  7.83633D-10 :  1.00000D+00  :  3.50000D-01          joint angle solutions
                        1.00000D+00  : -1.09379D-08 :  1.17620D-09  :  1.63100D+00

ABSOLUTE                1.09379D-08  :  1.11022D-16 :  7.83633D-10  :  4.42800D-09  <------ Absolute errors between the
        ERRORS ---      1.17620D-09  :  7.83633D-10 :  0.0          :  5.41339D-10          above two matrix components
                        8.32667D-17  :  1.09379D-08 :  1.17620D-09  :  9.09681D-10

*****************************************************************************

NO. --------------------------  0  ---  1

ANSWER OF POLYNOMIAL ---------  T   =  0.30261D+00   < T = TAN(THETA6/2) >

 ADOPT ANSWERS (DEG) ----------
                                                                                      Different kinds of joint
   THETA1 =    2.7533    THETA2 =     0.1502    THETA3 =    85.7259           <------ angle solutions derived
   THETA4 =  -33.7722    THETA5 =   -85.0428    THETA6 =    33.6731                   from the same value of t

INPUT                   0.0          :  1.00000D+00 :  0.0          : -1.00000D-01
        VALUES ---      0.0          :  0.0         :  1.00000D+00  :  3.50000D-01
                        1.00000D+00  :  0.0         :  0.0          :  1.63100D+00

CALCULATED              1.09379D-08  :  1.00000D+00 : -7.83633D-10  : -1.00000D-01
        VALUES ---     -1.17620D-09  :  7.83633D-10 :  1.00000D+00  :  3.50000D-01
                        1.00000D+00  : -1.09379D-08 :  1.17620D-09  :  1.63100D+00

ABSOLUTE                1.09379D-08  :  9.71445D-17 :  7.83633D-10  :  4.42800D-09
        ERRORS ---      1.17620D-09  :  7.83633D-10 :  0.0          :  5.41339D-10
                        1.11022D-16  :  1.09379D-08 :  1.17620D-09  :  9.09682D-10

*****************************************************************************

NO. --------------------------  0  ---  1

ANSWER OF POLYNOMIAL ---------  T   =  0.30261D+00   < T = TAN(THETA6/2) >

 ADOPT ANSWERS (DEG) ----------
                                                                                      Different kinds of joint
   THETA1 =  182.7533    THETA2 =    -0.1502    THETA3 =   -85.7259           <------ angle solutions derived
   THETA4 =  146.2277    THETA5 =   -85.0428    THETA6 =    33.6731                   from the same value of t
                                      .
                                      .
                                      .    <------ Terminal point number
*****************************************************************************
NO. --------------------------  41  ---  1

ANSWER OF POLYNOMIAL ---------  T   = -0.30261D+00   < T = TAN(THETA6/2) >

 ADOPT ANSWERS (DEG) ----------

   THETA1 = -182.7533    THETA2 =    -0.1502    THETA3 =   -85.7259
   THETA4 = -146.2277    THETA5 =   -85.0428    THETA6 =   -33.6731

INPUT                   0.0          :  1.00000D+00 :  0.0          :  1.00000D-01
        VALUES ---      0.0          :  0.0         :  1.00000D+00  :  3.50000D-01
                        1.00000D+00  :  0.0         :  0.0          :  1.63100D+00

CALCULATED             -1.09379D-08  :  1.00000D+00 :  7.83633D-10  :  1.00000D-01
        VALUES ---     -1.17620D-09  : -7.83633D-10 :  1.00000D+00  :  3.50000D-01
                        1.00000D+00  :  1.09379D-08 :  1.17620D-09  :  1.63100D+00

ABSOLUTE                1.09379D-08  :  9.71445D-17 :  7.83633D-10  :  4.42800D-09
        ERRORS ---      1.17620D-09  :  7.83633D-10 :  0.0          :  5.41339D-10
                        1.11022D-16  :  1.09379D-08 :  1.17620D-09  :  9.09681D-10
```

Fig.12 Detailed of Computed List (an Example of Sample Problem #2)

5. Conclusions

In order to solve the inverse problem for an articulated robot manipulator, a non-linear equation model was introduced here. As illustrated in the test calculations, the numerical results for individual joint angles were obtained with sufficient high accuracies and reliability.

Therefore, the present approach is useful for obtaining accurate solutions of the inverse kinematics.

The major features of this method are summarized in the following.

(1) Compared to the conventional iterative methodologies due to linearization, the proposed method is possible to find out, at the same time, all possible solutions (under the mechanical constraints on the range of joint angles), which depend on the location and orientation of end-effector of the manipulator. These solutions will contribute to identify all possible orientation of the arm.

(2) No special restrictions on input data or operation are needed in our algorithm, compared with those of the linearization techniques stated already in section 1.

(3) Based on the evaluation of numerical errors incorporated in the code, accuracy of solutions obtained is quite high.

As the future problem of interest, the generalization of the present algorithm is still left to be done.

Acknowledgment

The author would like to express his gratitude to Mr. Y. Shinohara, chief of Reactor Control Laboratory, who critically read a draft of the paper and made useful suggestions.

References

(1) Denavit, J. and Hartenberg, R.S. "A Kinematic Notation for Low-Pair Mechanisms Based on Matrices", Trans. ASME, Journal of Applied Mechanics, June 1955, pp 215-221.

(2) Paul, R.P., "Robot Manipulators: Mathematics, Programming and Control", MIT Press, 1981

(3) Isoda, K. et al., "Handbook of Numerical Calculation", ohmu-sha, 1971

(4) Togawa, H, "Numerical Calculation of Matrix", ohmu-sha, 1973

Appendix 1 Outline of Bairstow's method

We use Bairstow's iterative method to find an approximation to a quadratic factor of a given polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = 0 \tag{1}$$

Let $Q(x) = x^2 + px + q$ be any quadratic polynomial with real coefficients. Then

$$\left.\begin{array}{l} f(x) = (x^2 + px + q)Q_1(x) + Rx + S \\ \text{where a quotient } Q_1(x) = b_0x^{n-2} + b_1x^{n-3} + \dots b_{n-2} . \end{array}\right\} \tag{2}$$

$$\left.\begin{array}{l} \text{Additionally, assume } Q_1(x) = (x^2 + px + q)Q_2(x) + \bar{R}x + \bar{S}, \\ \text{where } Q_2(x) = c_ox^{n-4} + c_1x^{n-5} + \dots + c_{n-4} \end{array}\right\} \tag{3}$$

Thus, we get:

$$\left.\begin{array}{l} b_o = a_o,\ b_1 = a_1 - pb_o \\ b_k = a_k - pb_{k-1} - qb_{k-2} \quad (k = 2,3, \dots, n) \\ c_o = b_o,\ c_1 = b_1 - pb_o \\ c_k = b_k - pc_{k-1} - qc_{k-2} \quad (k = 2,3, \dots, n-1) \end{array}\right\} \tag{4}$$

Using the coefficient relations in Eqs. (1) to (4), we have

$$R = b_{n-1},\quad S = b_n + pb_{n-1} \tag{5}$$

$$\bar{R} = C_{n-3},\quad \bar{S} = C_{n-2} + pC_{n-3} . \tag{6}$$

Suppose that $XQ_1(x)$ divided by (x^2+px+q) gives $R^*x + S^*$ as the remainder.

or

$$
\begin{aligned}
xQ_1(x) &= Q_3(x)(x^2+px+q) + R^*x + S^* \\
&= (x^2+px+q)xQ_2(x) + \bar{R}x^2 + \bar{S}x \\
&= (x^2+px+q)xQ_2(x) + c_{n-3}x^2 + (c_{n-2} + pc_{n-3})x \\
&= (x^2+px+q)(xQ_2(x)+c_{n-3}) + c_{n-2}x - qc_{n-3} \qquad (7)
\end{aligned}
$$

Hence, we obtain

$$
\left.\begin{aligned}
R^* &= c_{n-2} \\
S^* &= -c_{n-3}\, q = c_{n-1} - b_{n-1} + p\, c_{n-2}
\end{aligned}\right\} \qquad (8)
$$

On the other hand, if we differentiate Eq. (2) with respect to p and q, we have

$$
\left.\begin{aligned}
xQ_1(x) &= -(x^2 + px + q)\frac{\partial Q_1}{\partial p} - x\frac{\partial R}{\partial p} - \frac{\partial S}{\partial p} \\
Q_1(x) &= -(x^2 + px + q)\frac{\partial Q_1}{\partial q} - x\frac{\partial R}{\partial q} - \frac{\partial S}{\partial q}
\end{aligned}\right\} \qquad (9)
$$

From these Eqs. (3),(6),(7) and (8),

$$
\left.\begin{aligned}
\frac{\partial R}{\partial p} &= -R^* = -c_{n-2} \\
\frac{\partial S}{\partial p} &= -S^* = -(c_{n-1} - b_{n-1} + p\, c_{n-2} \\
\frac{\partial R}{\partial q} &= -\bar{R} = -c_{n-3} \\
\frac{\partial S}{\partial q} &= -\bar{S} = -(c_{n-2} + p\, c_{n-3})
\end{aligned}\right\} \qquad (10)
$$

If $Q(x) = Q'(x) = (x^2+p'x+q')$ is a factor of $f(x)$, R and S in Eq. (2) must be zero.

i.e.,
$$
\left.\begin{aligned}
R(p,q) &= 0 \\
S(p,q) &= 0
\end{aligned}\right\} \qquad (11)
$$

Suppose that $Q(x)$ is an approximation to $Q'(x)$; then, p and q are assumed to have approximations to p' and q', respectively.

or
$$p' = p + \Delta p$$
$$q' = q + \Delta q$$

Therefore

$$\left.\begin{aligned} R(p + \Delta p,\ q + \Delta q) &= 0 \\ S(p + \Delta p,\ q + \Delta q) &= 0 \end{aligned}\right\} \qquad (12)$$

Expanding in Taylor series and truncating after the first-order terms, we get:

$$\left.\begin{aligned} R(p,q) + \frac{\partial R}{\partial p}\Delta p + \frac{\partial R}{\partial q}\Delta q &= 0 \\ S(p,q) + \frac{\partial S}{\partial p}\Delta p + \frac{\partial S}{\partial q}\Delta q &= 0 \end{aligned}\right\} \qquad (13)$$

Solving Eq. (13), we find that:

$$\left.\begin{aligned} \Delta p &= \frac{R\cdot c_{n-2} - S\cdot c_{n-3}}{c_{n-2}^{2} - \bar{c}_{n-1}\ c_{n-3}} = \frac{b_{n-1}c_{n-2} - b_{n}c_{n-3}}{c_{n-2}^{2} - \bar{c}_{n-1}\cdot c_{n-3}} \\ \Delta q &= \frac{S\cdot c_{n-2} - R\cdot c_{n-1}}{c_{n-2}^{2} - \bar{c}_{n-1}\ c_{n-3}} = \frac{b_{n}c_{n-2} - b_{n-1}\bar{c}_{n-1}}{c_{n-2}^{2} - \bar{c}_{n-1}\cdot c_{n-3}} \end{aligned}\right\} \qquad (14)$$

, where
$$c_{n-2}^{2} - \bar{c}_{n-1}c_{n-3} \neq 0$$
$$\bar{c}_{n-1} = c_{n-1} - b_{n-1} \qquad (15)$$

Now we can replace p by $(p+\Delta p)$, and q by $(q+\Delta q)$, and repeat the above procedure.

Description of the code for the computation of equation

For the polynomial $f(x)=0$ to be solved, which was derived from the kinematic relationships, individual coefficients a_0, a_1, ... , a_{24} were, at large, of an extremely complicated form. On further examination of the characteristics, in some cases, they are too large and in other cases too small depending on given orientation and position. Prior to entering the concrete solution procedures, therefore, the consideration on treatments of these coefficients was needed so as to suppress propagation of numerical errors as low as possible.

To this end, we first take the absolute values of respective coefficients of decision equation and assume that C be the logarithm of their geometric mean. That is,

$$C = \frac{1}{n} \log_{10} |a_0||a_1| \ \ldots \ |a_n|$$

where n is a degree of polynomial $f(x)$.

If $|a_i| = 0$, we set $C = C' + 1.0$ (i.e., we use 1.0 for $\log_{10}|a_i|$ in the above formulation)

Using the value of C obtained, each coefficient is defined as:

$$a_i = \frac{a_i}{10^C} \qquad (i = 0, \cdots, n)$$

If the absolute value of new coefficient made through this procedure is less than 10^{-30}, that term is regarded as to be zero, and omitted from the present equation system.

After establishment of the algebraic equation, the next thing we must do is to find a quadratic factor $(x^2 + px + q)$ as

explained in the outline of the Bairstow's method.
Concering the initial values p and q, the programming was specified as follows:

(1) Initialization of p and q

(A) First case (IR = 0)

- if $a_n = 0$, then we assume P = 1 and q = 0.
- if $a_n \neq 0$,

$$
\text{then} \begin{cases}
\text{(i)} \left|\dfrac{a_{n-2}}{a_n}\right| \geq 0.2, & \begin{cases} p = 0.01 & \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| < 10^{-5}\right) \\ q = \dfrac{a_n}{a_{n-2}} \end{cases} \\
\qquad\qquad \text{or} & \begin{cases} p = \dfrac{a_{n-1}}{a_{n-2}} & \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| \geq 10^{-5}\right) \\ q = \dfrac{a_n}{a_{n-2}} \end{cases} \\
\text{(ii)} \left|\dfrac{a_{n-2}}{a_n}\right| < 0.2, & p = 0.5 \text{ and } q = -1.0
\end{cases}
$$

(B) Second case (IR = 1)

When the starting values of case (A) are bad and the resulting convergence of solution is not accomplished within iteration numbers required, the second option (B) is used for the initial values p and q. (see Fig. 13)

- if $a_n = 0$, then we assume p = -1.0 and q = 0
- if $a_n \neq 0$,

$$
\text{then}
\begin{cases}
\text{(i)} \quad \left|\dfrac{a_{n-2}}{a_n}\right| \geq 0.2, &
\begin{cases}
p = -0.01 & \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| < 10^{-5}\right) \\
q = \dfrac{a_n}{a_{n-2}}
\end{cases} \\[2ex]
 &
\begin{cases}
p = -\dfrac{a_{n-1}}{a_{n-2}} & \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| \geq 10^{-5}\right) \\
q = \dfrac{a_n}{a_{n-2}}
\end{cases} \\[2ex]
\text{(ii)} \quad \left|\dfrac{a_{n-2}}{a_n}\right| < 0.2, & p = -0.5 \text{ and } q = -1.0
\end{cases}
$$

(2) Computation of Δp and Δq

Increments Δp and Δq in Bairstow's routine (Subroutine BEA) are usually solved by the elimination method. If it is unsuccessul, however, the other approach is prepared which allows the logarithmic treatment of coefficients in the linearized system, as previously mentioned.

(3) Convergence criteria

Final determination of Δp and Δq are made in accordance with the convergence criteria in the flow chart given in Fig. 13.

(4) Determination of a root x of equation f(x)=0

Together with the conditions of $|x| \leq \frac{1}{\sqrt{3}}$, when the absolute value of imaginary part of x is less than 10^{-3}, that root is treated as to be real.

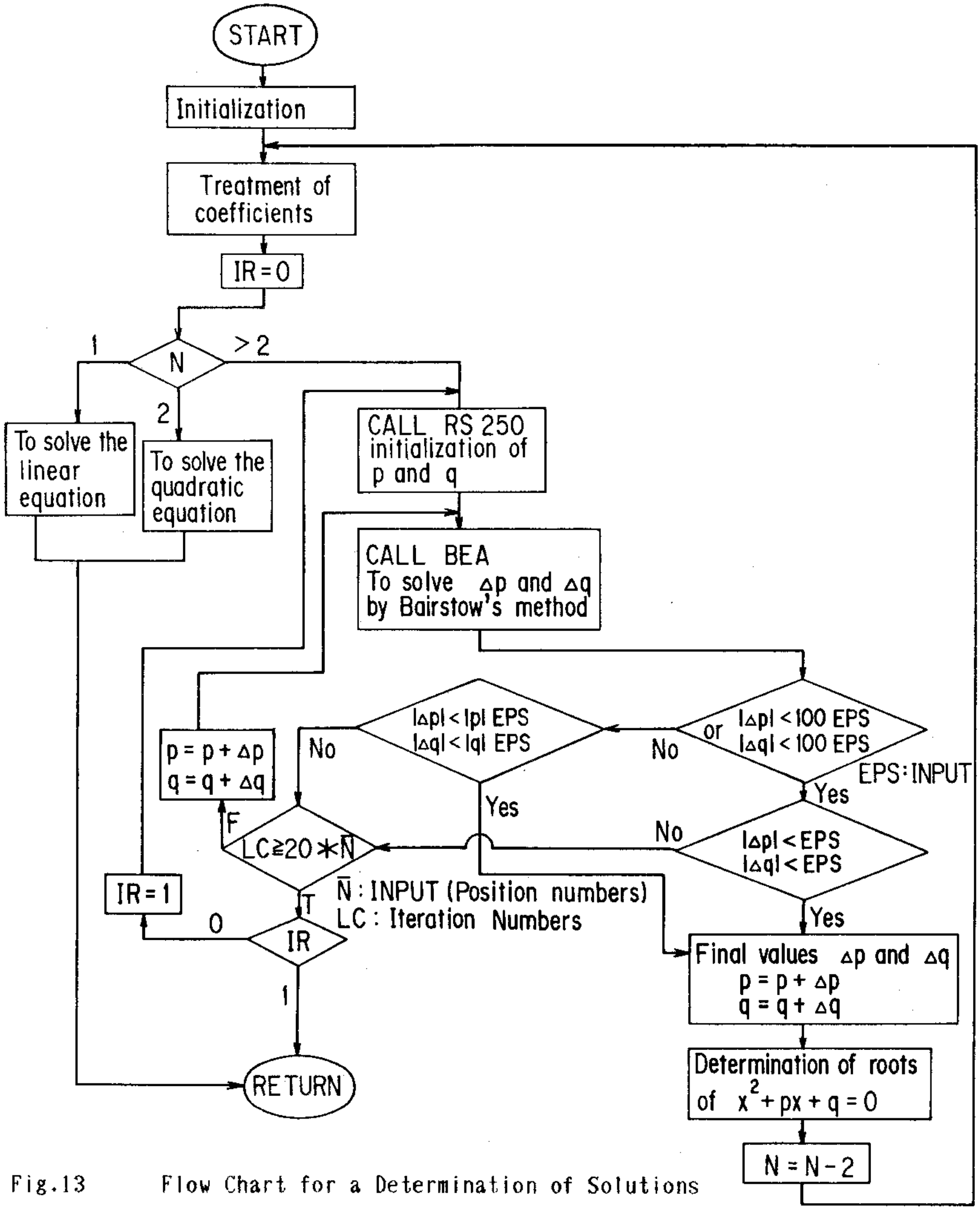


Fig.13 Flow Chart for a Determination of Solutions

Appendix 2

Input Data Requirements

The current code written in FORTRAN 77 is operational on the FACOM-380 computer. In this section, the input data requirements are presented in the form necessary for computer execution from TSS terminals.

Record 1	TITLE(A50)*	= Title of Problem
Record 2	NDEL(I*4) (NDEL $\geq$ 0)	= Position numbers between initial and terminal position. If NDEL = 0, Record No.8 through No.11 are neglected, because a point interpolation is not used.
Record 3	EPS(R*8)	= Convergence condition related to the determination of a quadratic factor $(x^2 + px + q)$
	EPS1(R*8)	= Check of the validity of the calculated articular angles. Joint angles solutions are substituted into the components of the T_6 matrix and compared with the given data.

Record 4 ~ Record 6 ---- Initial point data ----

Record 4	DATA(1,4,1)	= P_x(R*8)	= Initial position of x-direction of the manipulator hand (m)
	DATA(2,4,1)	= P_y(R*8)	= Initial position of y-direction of the manipulator hand (m)

---

*) ( ) denotes the type of variable.

DATA(3,4,1) = P_z(R*8) = Initial position of z-direction of the manipulator hand

Record 5 GET('A1') = Option of orientation calculation

= '0': user specified

= '1': Roll-Pitch-Yaw transformation

= 'Z': Return to the initial stage Record 1 (Re-trial of input data).

= excepting the above letter (default)

: Euler transformation

Record 6

(i) If GET = '0', then

DATA(1,1,1) = NX(R*8) = x-component of normal vector n

DATA(2,1,1) = NY(R*8) = y-component of normal vector n

DATA(3,1,1) = NZ(R*8) = z-component of normal vector n

(ii) If GET = '1', then

w_1(R*8) = Rotation angle about the z-axis (Deg)

w_2(R*8) = Rotation angle about the y-axis (Deg)

w_3(R*8) = Rotation angle about the x-axis (Deg)

The direction cosines are calculated as follows.

DATA(1,1,1) = $\cos w_1 \cdot \cos w_2$

DATA(2,1,1) = $\sin w_1 \cdot \cos w_2$

DATA(3,1,1) = $-\sin w_2$

DATA(1,2,1) = $\cos w_1 \cdot \sin w_2 \cdot \sin w_3 - \sin w_1 \cdot \cos w_3$

DATA(2,2,1) = $\sin w_1 \cdot \sin w_2 \cdot \sin w_3 + \cos w_1 \cdot \cos w_3$

DATA(3,2,1) = $\cos w_2 \cdot \sin w_3$

(iii) If GET = Euler option,

w_1(R*8) = Rotation angle about the z-axis (Deg)

w_2(R*8) = Rotation angle about the y-axis (Deg)

w_3(R*8) = Rotation angle about the z-axis (Deg)

The direction cosines are:

DATA(1,1,1) = $\cos w_1 \cdot \cos w_2 \cdot \cos w_3 - \sin w_1 \cdot \sin w_3$

DATA(2,1,1) = $\sin w_1 \cdot \cos w_2 \cdot \cos w_3 + \cos w_1 \cdot \sin w_3$

DATA(3,1,1) = $-\sin W_2 \cdot \cos W_3$

DATA(1.2,1) = $-\cos w_1 \cdot \cos w_2 \cdot \sin w_3 - \sin w_1 \cdot \cos w_3$

DATA(2,2,1) = $-\sin w_1 \cdot \cos W_2 \cdot \sin W_3 + \cos w_1 \cdot \cos w_3$

DATA(3,2,1) = $\sin w_2 \cdot \sin w_3$

Record 7 If GET = '0', then

DATA(1,2,1) = OX(R*8) = x-component of sliding vector O

DATA(2,2,1) = OY(R*8) = y-component of sliding vector O

DATA(3,2,1) = OZ(R*8) = z-component of sliding vector O

Record 8 ~ Record 11 ---- Terminal point data

Record 8 DATA(1,4,2) = P_x(R*8) = Terminal position of x-direction of the manipulator hand (m)

DATA(2,4,2) = P_y(R*8) = Terminal position of y-direction of the manipulator hand (m)

DATA(3,4,2) = P_z(R*8) = Terminal position of z-direction of the manipulator hand (m)

Record 9 GET('A1') = Option of orientation calculation

= '0': user specified

= '1': Roll-Pitch-Yaw transformation

= 'Z': Return to Record 4

(Retrial of end-point data)

= excepting the above letter (default)

: Euler transformation

Record 10

(i) If GET = '0', then

DATA(1,1,2) = NX(R*8) = x-component of normal vector n

DATA(2,1,2) = NY(R*8) = y-component of normal vector n

DATA(3,1,2) = NZ(R*8) = z-component of normal vector n

(ii) If GET = '1', then

W_1(R*8) = Rotation angle about the z-axis (Deg)

W_2(R*8) = Rotation angle about the y-axis (Deg)

W_3(R*8) = Rotation angle about the x-axis (Deg)

The direction cosines are calculated as follows.

DATA(1,1,2) = $\cos w_1 \cdot \cos w_2$

DATA(2,1,2) = $\sin w_1 \cdot \cos w_2$

DATA(3,1,2) = $-\sin w_2$

DATA(1,2,2) = $\cos w_1 \cdot \sin w_2 \cdot \sin w_3 - \sin w_1 \cdot \cos w_3$

DATA(2,2,2) = $\sin w_1 \cdot \sin w_2 \cdot \sin w_3 + \cos w_1 \cdot \cos w_3$

DATA(3,2,2) = $\cos w_2 \cdot \sin w_3$

(ii) If GET = Euler option,

W_1(R*8) = Rotation angle about the z-axis (Deg)

W_2(R*8) = Rotation angle about the y-axis (Deg)

W_3(R*8) = Rotation angle about the z-axis (Deg)

The direction cosines are:

DATA(1,1,2) = $\cos w_1 \cdot \cos w_2 \cdot \cos w_3 - \sin w_1 \cdot \sin w_3$

DATA(2,1,2) = $\sin w_1 \cdot \cos w_2 \cdot \cos w_3 + \cos w_1 \cdot \sin w_3$

DATA(3,1,2) = $-\sin w_2 \cdot \cos w_3$

DATA(1,2,2) = $-\cos w_1 \cdot \cos w_2 \cdot \sin w_3 - \sin w_1 \cdot \cos w_3$

DATA(2,2,2) = $-\sin w_1 \cdot \cos w_2 \cdot \sin w_3 + \cos w_1 \cdot \cos w_3$

DATA(3,2,2) = $\sin w_2 \cdot \sin w_3$

Record 11 If GET = '0', then

DATA(1,2,2) = OX(R*8) = x-component of sliding vector O

DATA(2,2,2) = OY(R*8) = y-component of sliding vector O

DATA(3,2,2) = OZ(R*8) = z-component of sliding vector O

Appendix 3 Computation of coefficients

(1)
$$b_0 = a_5(n_y y_n - n_x x_n)$$
$$b_1 = 2a_5\{(o_x x_n - o_y y_n) - a_5(n_x o_x + n_y o_y)\}$$
$$b_2 = a_5\{(n_x x_n - n_y y_n) + (n_y y_p - n_x x_p) + 4a\ (o_x{}^2 + o_y{}^2)\}$$
$$b_3 = 2a_5\{(o_x x_p - o_y y_p) + a_5(n_x o_x + n_y o_y)\}$$
$$b_4 = a_5(n_x x_p - n_y y_p)$$
$$b_5 = XX \cdot x_n - YY \cdot y_n$$
$$b_6 = 2a_5(XX \cdot o_x + YY \cdot o_y)$$
$$b_7 = XX \cdot x_n - YY \cdot y_n + XX \cdot x_p - YY \cdot y_p$$
$$b_8 = 2a_5(XX \cdot o_x + YY \cdot o_y)$$
$$b_9 = XX \cdot x_p - YY \cdot y_p$$

(2)
$$\overline{cc}_i = b_i + b_{i+5} \qquad (i = 0, \cdots, 4)$$

(3)
$$do_1 = a_5(n_z - ZZ)$$
$$do_2 = -2a_5 o_z$$
$$do_3 = -(a_5 n_z + ZZ)$$

(4)
$$\sum_{i=0}^{4} d_i\ t_i = \left(\sum_{i=0}^{3} do_i\ t^{i-1} \right)^2$$

d_i is computed from Subroutine CFSET. $(i = 0, \cdots, 4)$

(5)
$$\sum_{i=0}^{8} e_i\ t^i = \left(\sum_{i=0}^{4} \overline{cc}_i\ t^i \right)^2$$

Coefficient e_i is computed from Subroutine CFSET.

(6) $f_0 = x_n^2 + y_n^2$

$f_1 = 4a_5(o_x x_y - o_y y_n)$

$f_2 = 2(x_n x_p + y_n y_p) + 4a_5^2(o_x^2 + o_y^2)$

$f_3 = 4a_5(o_x x_p - o_y y_p)$

$f_4 = x_p^2 + y_p^2$

(7) $\sum_{i=0}^{8} g_i t^i = \left(\sum_{i=0}^{4} d_i t^i\right)\left(\sum_{i=0}^{4} f_i t^i\right)$

Coefficient q_i is computed from Subroutine CFSET. $(i=0,\cdots,8)$

(8) $h_0 = f_0$

$h_1 = f_1$

$h_2 = f_2 + 2f_0$

$h_3 = f_3 + 2f_1$

$h_4 = f_0 + 2f_2 + f_4$

$h_5 = f_1 + 2f_3$

$h_6 = f_2 + 2f_4$

$h_7 = f_3$

$h_8 = f_4$

(9) $J_i = e_i + g_i - ah_i \qquad (i = 0, \cdots, 8)$

(10) $I_0 = o_y y_n - o_x x_n$

$I_1 = -2a_5(o_x^2 + o_y^2) + 2(n_y y_n - n_x x_n)$

$I_2 = o_y y_p - o_x x_p + o_x x_n - o_y y_n - 4a_5(n_x o_x + n_y o_y)$

$I_3 = 2a_5(o_x^2 + o_y^2) + 2(n_y y_p - n_x x_p)$

$I_4 = o_x x_p - o_y y_p$

(11) $\left(\sum_{i=0}^{8} I_{2i} t^i\right) = \left(\sum_{i=0}^{4} I_i t^i\right)^2$

I_{2i} is computed from Subroutine CFSET. $(i = 0, \cdots, 8)$

(12) $$\left(\sum_{i=0}^{6} II_i\ t^i\right) = \left(\sum_{i=0}^{4} I_i\ t^i\right)\left(\sum_{i=1}^{3} do_i\ t^{i-1}\right)$$

II_i is computed from Subroutine CFSET. $(i = 0, \cdots, 6)$

(13) $$K_0 = -o_z \cdot \overline{cc}_0$$

$$K_1 = -(o_z\ \overline{cc}_1 + 2n_z\ \overline{cc}_0)$$

$$K_2 = o_z \cdot \overline{cc}_0 - 2n_z \cdot \overline{cc}_1 - o_z \cdot \overline{cc}_2$$

$$K_3 = o_z \cdot \overline{cc}_1 - 2n_z \cdot \overline{cc}_2 - o_z \cdot \overline{cc}_3$$

$$K_4 = o_z \cdot \overline{cc}_2 - 2n_z \cdot \overline{cc}_3 - o_z \cdot \overline{cc}_4$$

$$K_5 = o_z \cdot \overline{cc}_3 - 2n_z \cdot \overline{cc}_4$$

$$K_6 = o_z \cdot \overline{cc}_4$$

(14) $$PI_i = K_i + II_i \qquad (i = 0, \cdots, 6)$$

(15) $$LL_0 = o_z^{\,2}$$

$$LL_1 = 4n_z\ o_z$$

$$LL_2 = 4n_z^{\,2} - 2o_z^{\,2}$$

$$LL_3 = -4n_z\ o_z$$

$$LL_4 = o_z^{\,2}$$

(16) $$\sum_{i=0}^{8} PL_i\ t^i = \left(\sum_{i=0}^{4} LL_i\ t^i\right)\left(\sum_{i=0}^{4} f_i\ t^i\right)$$

Coefficient PL_i is computed from Subroutine CFSET. $(i=0,\cdots,8)$

(17) $$L_i = I_{2i} + PL_i \qquad (i = 0, \cdots, 8)$$

(18) $$\left(\sum_{i=0}^{16} N_i\ t^i\right) = \left(\sum_{i=0}^{8} J_i\ t^i\right)^2$$

N_i is computed from Subroutine CFSET. $(i = 0, \cdots, 16)$

(19) $$\left(\sum_{i=0}^{12} m_i\ t^i\right) = \left(\sum_{i=0}^{6} PI_i\ t^i\right)^2 \times 4a_{34}{}^2$$

m_i is computed from Subroutine CFSET. $(i = 0,\cdots, 12)$

(20) $$\sum_{i=0}^{12} f_i\ t^i = (1+t^2)^2\left(\sum_{i=0}^{4} f_i\ t^i\right)^2$$

f_i $(i = 0,\cdots, 12)$ is calculated from CFSET routine.

(21) $$\sum_{i=0}^{24} P_i\ t^i = \left(\sum_{i=0}^{12} m_i\ t^i\right)\left(\sum_{i=0}^{12} f_i\ t^i\right)$$

$$\sum_{i=0}^{24} q_i\ t^i = \left(\sum_{i=0}^{8} L_i\ t^i\right)\left(\sum_{i=0}^{16} N_i\ t^i\right)$$

P_i and q_i are computed from CFSET routine. $(i = 0,\cdots, 24)$

(22) $r_i = p_i - q_i$ $\quad (i = 0,\cdots, 24)$

Appendix 3. List of Input Data

(1) Sample Problem 1

```
 INPUT --- TITLE
++++++++++++++++++ T  I  T  L  E ++++++++++++++++++
+                                                  +
+ +++  BENCH  MARK  1  +++                         +
+                                                  +
++++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>  100

 INPUT --- EPS , EPS1

 EPS    ===>  1.00000D-04    EPS1    ===>  1.00000D-03

---------- INITIAL   POINT ----------

 INPUT --- PX : PY :PZ

 PX ==> -2.50000D-01   PY ==>  7.80000D-01   PZ ==>  1.20100D+00 ( M )

 KEYIN        0 : USER SPECIFIED
              1 : RPY
              Z : EARLY STAGES
       DEFAULTS : EULER

 INPUT --- NX , NY , NZ

 NX ==>  0.0              NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0              OZ ==>  0.0

---------- TERMINAL  POINT ----------

 INPUT --- PX : PY :PZ

 PX ==>  2.50000D-01   PY ==>  7.80000D-01   PZ ==>  1.20100D+00 ( M )

 KEYIN        0 : USER SPECIFIED
              1 : RPY
              Z : EARLY STAGES
       DEFAULTS : EULER

 INPUT --- NX , NY , NZ

 NX ==>  0.0              NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0              OZ ==>  0.0
```

(2) Sample Problem 2

```
 INPUT --- TITLE
++++++++++++++++++ T  I  T  L  E ++++++++++++++++++
+                                                 +
+ +++  BENCH  MARK  2  +++                        +
+                                                 +
+++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>   41

 INPUT --- EPS , EPS1

 EPS    ===>  1.00000D-04    EPS1    ===>  1.00000D-03

---------- INITIAL   POINT ----------

 INPUT --- PX : PY :PZ

 PX ==> -1.00000D-01   PY ==>  3.50000D-01   PZ ==>  1.63100D+00 ( M )

 KEYIN       0 : USER SPECIFIED
             1 : RPY
             Z : EARLY STAGES
     DEFAULTS : EULER

 INPUT --- NX , NY , NZ

 NX ==>  0.0            NY ==>  0.0            NZ ==>  1.00000D+00

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0            OZ ==>  0.0

---------- TERMINAL  POINT ----------

 INPUT --- PX : PY :PZ

 PX ==>  1.00000D-01   PY ==>  3.50000D-01   PZ ==>  1.63100D+00 ( M )

 KEYIN       0 : USER SPECIFIED
             1 : RPY
             Z : EARLY STAGES
     DEFAULTS : EULER

 INPUT --- NX , NY , NZ

 NX ==>  0.0            NY ==>  0.0            NZ ==>  1.00000D+00

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0            OZ ==>  0.0
```

(3) Sample Problem 3

```
 INPUT --- TITLE
+++++++++++++++++ T  I  T  L  E +++++++++++++++++++
+                                                 +
+ +++  BENCH  MARK  3  +++                        +
+                                                 +
+++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>   40

 INPUT --- EPS , EPS1

 EPS   ===>  1.00000D-04    EPS1   ===>  1.00000D-03

---------- INITIAL  POINT ----------

 INPUT --- PX : PY :PZ

 PX ==>  0.0           PY ==>  4.30000D-01   PZ ==>  1.35100D+00 ( M )

 KEYIN       0 : USER SPECIFIED
             1 : RPY
             2 : EARLY STAGES
     DEFAULTS : EULER

 INPUT --- NX , NY , NZ

 NX ==>  0.0           NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0           OZ ==>  0.0

---------- TERMINAL  POINT ----------

 INPUT --- PX : PY :PZ

 PX ==>  0.0           PY ==>  4.30000D-01   PZ ==>  1.55100D+00 ( M )

 KEYIN       0 : USER SPECIFIED
             1 : RPY
             2 : EARLY STAGES
     DEFAULTS : EULER

 INPUT --- NX , NY , NZ

 NX ==>  0.0           NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0           OZ ==>  0.0
```