JAERI - M
86-096

FINITE ELEMENT CIRCUIT THEORY OF THE NUMERICAL CODE EDDYMULT FOR SOLVING EDDY CURRENT PROBLEMS IN A MULTI-TORUS SYSTEM

July 1986

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Japan Atomic Energy Research Institute

編集兼発行　日本原子力研究所
印　　刷　㈱原子力資料サービス

Finite Element Circuit Theory of the Numerical Code EDDYMULT for Solving Eddy Current Problems in a Multi-Torus System

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(Received June 9, 1986)

The finite element circuit theory is extended to the general eddy current problem in a multi-torus system, which consists of various torus conductors and axisymmetric coil systems. The numerical procedures are devised to avoid practical restrictions of computer storage and computing time, that is, the reduction technique of eddy current eigen modes to save storage and the introduction of shape function into the double area integral of mode coupling to save time. The numerical code EDDYMULT based on the theory is developed to use in designing tokamak device from the viewpoints of the evaluation of electromagnetic loading on the device components and the control analysis of tokamak equilibrium.

Keywords: Eddy Current, Finite Element Circuit Method, Tokamak, Eigen Mode, EDDYMULT, Multi-Torus System

複合トーラス導体系の渦電流問題を解くための計算コードEDDYMULTの有限要素回路理論

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（1986年6月9日受理）

多数のトーラス状導体や軸対称ポロイダルコイル群から成る複合トーラス導体系における一般的な渦電流問題を解くために有限要素回路理論を拡張・発展させた。この種の問題は，トカマク装置を構成する主たる構造物を全て含む大規模な数値計算となるため，計算機容量，計算時間の制約を受けない計算手法をくふうした。すなわち，渦電流固有モードの縮少およびエネルギー積分に際しての形状関数の導入である。本理論に基づき計算コードEDDYMULTを開発したが，トカマク装置の電磁力・誤差磁場設計さらにプラズマ制御解析に有用である。

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1. Introduction

As is well recognized in a tokamak fusion research and development, eddy current problem is one of the most important and essential subjects from the standpoints of mechanical stress design against an intensive electromagnetic loading on devices components and control analysis of plasma current, plasma position and its shape. The intensive electromagnetic loading on device components due to a rapid plasma disruption causes a severe problem in assessing the structural integrity of tokamak device. In addition, the transient stray field produced by eddy currents causes unfavorable displacement and distortion of the tokamak equilibrium from its ideal state. Therefore, the accurate evaluation of transient eddy current is absolutely necessary. These demands on the eddy current study in a tokamak fusion research would go on increasing with a scale-up of tokamak device.

In spite of these urgent requirement, the rational evaluation of eddy current in a tokamak system is quite difficult because of the complexities of machine geometry and electrical characteristic. In the early years of a tokamak fusion research, a stabilized confinement of tokamak plasma had been attained only by the so-called shell effect of conducting walls such as a vacuum vessel or copper shell. Namely, tokamak equilibria during a short discharge duration had been established by the passive feedback control action due to an image current induced on ideally conducting walls. Where, the study of eddy current itself was not necessary since the ideal shell effect of conducting walls automatically guarantees the equilibrium of tokamak plasma column[1], that is, there was no need for precise considerations taking account of the wall geometry or its electrical characteristics. Here, one must notice the following remarkable features of actual tokamak system from a viewpoint of a transient analysis of eddy current. In general, a tokamak device consists of a composite multi-torus system, i.e., a vacuum vessel, toroidal field coils, support structures and various kinds of poloidal field coil systems. Each torus component has individually a complicated geometry with regard to the cross-sectional shape. There is a great deal of complexity due to port holes and electrical insulations. Moreover, one must take account of non-uniform and non-isotropic electric resistivities due to the bellows section of a vacuum vessel commonly used in the actual tokamak.

From the mentioned above, the eddy current problem must be solved for the multi-torus system composed of individually complicated torus conductors including various kinds of poloidal field coil systems, since each torus component magnetically couples with the others.

In the recent years, numerical and analytical procedures have been considerably advanced on the increasing demands for design studies of the present generation large tokamaks such as JT-60, JET and TFTR[2-13]. On the assumption of axisymmetry, the eddy current problems had been investigated considering a vacuum vessel magnetically coupled with a plasma loop or poloidal field coil system[3,4]. The finite element method was applied to three dimensional eddy current problem, in which the penetration of electromagnetic field into a conductor with the finite thickness was taken into account[5,6]. On the other hand, the eddy current in an infinitely thin surface had been investigated to use in a practical design of tokamak device[8-12]. In these studies, a torus with an arbitrary cross-section was discretized into a set of finite element circuits, which provide the eddy current induced on a vacuum vessel. Among them, the finite element circuit method developed by Kameari et al.[11,12] is considered to be most promising way from a viewpoint of the designing of tokamak device and the control analysis of plasma equilibrium. Since the finite element circuit method is based on the eigenvalue expansion, the method has the advantage that the obtained eigen modes of eddy current characterize the electrical and geometrical features of a considered torus. The method was extended to solve a general eddy current problem in a composite multi-torus system because that the actual tokamak system is composed of many torus conductors such as a vacuum vessel, toroidal field coils and the support structures which are magnetically coupled with each other[13]. Using the extended finite element circuit method, the computer program EDDYMULT has been developed in JAERI and used to evaluate the transient eddy current on device components of JT-60 multi-torus system[14,15]. Moreover, it was shown that the obtained eigen modes of eddy current by the program describe the actual eddy current effects on the tokamak equilibrium[16]. Therefore, it is anticipated to use the numerical results in the advanced control analysis of fusion plasma[17-19].

The paper is arranged as follows. The following section describes outline of the finite element circuit method. The general formulation of circuit constants and the introduction of boundary conditions will be mentioned, respectively, in sections 3 and 4 without the assumption of toroidal geometry. In section 5, the concrete representation in a multi-torus system is discussed, where the formulation to evaluate the three dimensional magnetic field produced by eddy current is also described. Summary of the paper is presented in section 6. The description of the program EDDYMULT and the representation of the numerical results are outside of the scope of this paper.

2. Basic Concept of Finite Element Circuit Method

This section is devoted to outline a basic concept of finite element circuit method. Throughout the paper, all of the conductors is approximated to be infinitesimally thin neglecting the thickness and the conductor material is considered non-ferromagnetic. In the application of the thin conductor approximation, the skin time of the conductor $\tau_{skin} \approx \mu_o d^2/\eta$ (μ_o is the vacuum permeability, and η and d are the electric resistivity and the thickness of the conductor, respectively)[20] must be sufficiently small compared with the characteristic time of the external magnetic field variations.

Since a surface current density J on a thin conductor is divergence free, so the surface current density is described by a current vector potential V which is a function of time and position on the conductor and given by

$$J = \nabla \times V \ . \tag{1}$$

Apparently, the current vector potential V always refers to normal direction of the conductor surface. Let V be the normal component of the current vector potential V. We call V a current function. The current flows along the lines of V = constant. An arbitrary constant can be added to V, and V can and must be fixed to zero at a certain point on a conductor to eliminate the arbitrariness. These are illustrated in Fig. 1.

If a current function V is represented by a linear combination of linear-independent functional series $a_n (n = 1, 2, \ldots, \infty)$, then V can be described using each coefficient $V_n (n = 1, 2, \ldots, \infty)$:

formulation of circuit constants and the introduction of boundary conditions will be mentioned, respectively, in sections 3 and 4 without the assumption of toroidal geometry. In section 5, the concrete representation in a multi-torus system is discussed, where the formulation to evaluate the three dimensional magnetic field produced by eddy current is also described. Summary of the paper is presented in section 6. The description of the program EDDYMULT and the representation of the numerical results are outside of the scope of this paper.

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If a current function V is represented by a linear combination of linear-independent functional series a_n ($n = 1, 2, \ldots, \infty$), then V can be described using each coefficient V_n ($n = 1, 2, \ldots, \infty$):

$$V = \sum_{n=1}^{\infty} a_n V_n . \tag{2}$$

Although a distributed eddy current problem is infinite dimensional, we must consider a finite dimensional problem in practice. Here, we divide the conductor surface into finite elements as shown in Fig. 1. Furthermore, it is assumed that the unknown function of distributed eddy current in the individual finite element can be approximated by using an interpolation polynominal $a_n(u,v)$. Then, the current function given by Eq. (1) is rewritten as:

$$V = \sum_{n=1}^{N} a_n(u,v)\ V_n , \tag{3}$$

here, N denotes the number of independent nodal points on the conductor. If the nodal points lie on a same boundary of the considered conductor, then these are linear-dependent with each other. The detailed discussion to introduce the boundary condition and symmetry of the current function will be carried out in section 4.

From the above mentioned, a finite-dimensional linear circuit equation of eddy current is given as:

$$M\dot{X} + RX = E , \tag{4}$$

where X is the current vector corresponding to the nodal value V_n ($n = 1, 2, \ldots, N$). Inductance matrix $M \varepsilon R^{N\times N}$ and resistance matrix $R \varepsilon R^{N\times N}$ are both real symmetric and positive-definite since the physical nature. $E \varepsilon R^N$ means the externally applied electromotive force to the individual finite element circuit. In the following section, it is shown that the inductance and resistance matrices and the voltage vector can be described by means of the corresponding energy integrals.

Consider the following generalized eigenvalue problem as:

$$MX = \mathrm{diag}(\lambda)RX , \tag{5}$$

here $\mathrm{diag}(\lambda)$ denotes the diagonal matrix whose diagonal element is λ. The obtained eigen modes are magnetically and resistively decoupled with each other. The physical meaning of eigenvalue λ is decay time of the eigen mode. After solving the generalized eigenvalue problem, we can represent the circuit equation of the ℓ-th eigen mode of eddy current as follows:

$$\lambda_\ell \dot{\xi}_\ell + \xi_\ell = \varepsilon_\ell , \tag{6}$$

where ε_ℓ means the externally applied electromotive force to the ℓ-th eigen mode, which can be expressed as:

$$\varepsilon = \Phi^T E \tag{7}$$

here, Φ is the modal matrix of the generalized eigenvalue problem given by Eq. (5) and $(\ )^T$ denotes a transpose of matrix or vector. The time evolutional solution of Eq. (6) is easily obtained as:

$$\xi_\ell(t) = \xi_\ell(0) \exp(-\frac{t}{\lambda_\ell}) + \frac{1}{\lambda_\ell}\int_0^t \varepsilon_\ell \exp(\frac{t'-t}{\lambda_\ell})dt' , \tag{8}$$

where, $\xi_\ell(0)$ means an initial current of the ℓ-th eigen mode at $t = 0$.

3. Circuit Constants

In this section, we firstly present the general formulation of respective circuit constants such as an inductance and resistance matrices. These matrices can be obtained from the integral calculations of the magnetic energy and Joule loss. Formulations in a discrecte form for the finite element system is described in the later part of this section.

Let us consider the multi-conductor system composed of N_{cond} conductor surfaces as is shown schematically in Fig. 2. The curvi-linear coordinate system (u^i,v^i,w^i) is defined separetely corresponding to the individual conductor. Therefore, a point P on the conductor S_i is designated by the two dimensional orthogonal coordinate system (u^i,v^i). Using this coordinate system, Cartesian coordinate (x,y,z) of the point P on the conductor S_i is represented as:

$$x = x(u^i,v^i), \quad y = y(u^i,v^i), \quad z = z(u^i,v^i) . \tag{9}$$

A line element ds_i is

$$ds_i^2 = f_i^2 \, du^{i^2} + g_i^2 \, dv^{i^2} , \tag{10}$$

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$$ds_i^2 = f_i^2 \, du^{i^2} + g_i^2 \, dv^{i^2} , \tag{10}$$

where the scale factors f_i and g_i are respectively defined as:

$$f_i^2(u^i,v^i) = \left(\frac{\partial x}{\partial u^i}\right)^2 + \left(\frac{\partial y}{\partial u^i}\right)^2 + \left(\frac{\partial z}{\partial u^i}\right)^2 ,$$
$$g_i^2(u^i,v^i) = \left(\frac{\partial x}{\partial v^i}\right)^2 + \left(\frac{\partial y}{\partial v^i}\right)^2 + \left(\frac{\partial z}{\partial v^i}\right)^2 . \quad (11)$$

Because of the orthogonality of (u^i,v^i) coordinate, we obtain the equation:

$$\frac{\partial x}{\partial u^i}\frac{\partial x}{\partial v^i} + \frac{\partial y}{\partial u^i}\frac{\partial y}{\partial v^i} + \frac{\partial z}{\partial u^i}\frac{\partial z}{\partial v^i} = 0 . \quad (12)$$

Let $\mathbf{e}_{u^i}$ and $\mathbf{e}_{v^i}$ be unit tangent vectors along v^i = constant and u^i = constant, respectively. $\mathbf{e}_{u^i}$ and $\mathbf{e}_{v^i}$ are mutually orthogonal and the components are given by

$$\mathbf{e}_{u^i} = \begin{pmatrix} \dfrac{\partial x}{f_i\,\partial u^i} \\ \dfrac{\partial y}{f_i\,\partial u^i} \\ \dfrac{\partial z}{f_i\,\partial u^i} \end{pmatrix} , \quad \mathbf{e}_{v^i} = \begin{pmatrix} \dfrac{\partial x}{g_i\,\partial v^i} \\ \dfrac{\partial y}{g_i\,\partial v^i} \\ \dfrac{\partial z}{g_i\,\partial v^i} \end{pmatrix} . \quad (13)$$

From the definition of Eq. (1), a surface current density $J^i(u^i,v^i)$ is expressed by a current function which is a function of time and position on the conductor S_i and given by

$$J^i(u^i,v^i) = j_{u^i}(u^i,v^i)\,\mathbf{e}_{u^i}(u^i,v^i) + j_{v^i}(u^i,v^i)\,\mathbf{e}_{v^i}(u^i,v^i), \quad (14)$$

where, the components of current density can be expressed as:

$$j_{u^i} = \frac{1}{g^i}\frac{\partial V^i}{\partial v^i} , \quad j_{v^i} = -\frac{1}{f^i}\frac{\partial V^i}{\partial u^i} . \quad (15)$$

As was discussed in the previous section, the usual current function V^i can be expanded by the infinite sets of linear-independent functional series. By substituting Eq. (2) into Eq. (15), we can obtain each component of current density expanded by the linear-independent function a_n^i as follows:

$$j_{ui} = \sum_{n=1}^{\infty} \left(\frac{1}{g_i} \frac{\partial a_n^i}{\partial v^i} V_n^i \right) ,$$
$$j_{vi} = \sum_{n=1}^{\infty} \left(- \frac{1}{f_i} \frac{\partial a_n^i}{\partial u^i} V_n^i \right) . \tag{16}$$

Here-in-after, the functional series $a_n^i(u^i,v^i)$ $(n = 1,2,....)$ is abbreviated as a_n^i for convenience. Consider the following energy integrals to obtain the respective element of inductance and resistance matrices

$$U_{nm}^{ij} = \frac{\mu_0}{8\pi} \int_{S_i} \int_{S_j} \frac{J_n^i \cdot J_m^j}{\rho_{12}} dS_i \, dS_j , \quad (i,j = 1,...,N_{cond}) \tag{17}$$

$$W_{nm}^i = \int_{S_i} J_n^{i^T} \eta^i J_m^i \, dS_i , \qquad (i = 1,...,N_{cond}) \tag{18}$$

here ρ_{12} denotes the distance between a point $P_1(x_1,y_1,z_1)$ on the conductor S_i and a point $P_2(x_2,y_2,z_2)$ on the conductor S_j; $\rho_{12}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$. The electrical resistivity matrix $\eta^i \varepsilon R^{2\times2}$ is real symmetric and positive-definite

$$\eta^i = \begin{pmatrix} \eta_u^i & \eta_{uv}^i \\ \eta_{vu}^i & \eta_v^i \end{pmatrix} . \tag{19}$$

If the electrical resistivity is uniform, then $\eta_u^i = \eta_v^i$ and $\eta_{uv}^i = 0$. By substituting Eq. (16) into Eqs. (17) and (18), we obtain the element M_{nm}^{ij} of inductance matrix M^{ij} between the current components V_n^i and V_m^j, and the element R_{nm}^i of resistance matrix R^i between the current components V_n^i and V_m^i

$$M_{nm}^{ij} = \frac{\mu_0}{4\pi} \iiiint \frac{1}{\rho} \Big(f_i f_j \frac{\partial a_n^i}{\partial v^i} \frac{\partial a_m^j}{\partial v^j} e_{u^i} \cdot e_{u^j} - f_i g_j \frac{\partial a_n^i}{\partial v^i} \frac{\partial a_m^j}{\partial u^j} e_{u^i} \cdot e_{v^j} - g_i f_j \frac{\partial a_n^i}{\partial u^i} \frac{\partial a_m^j}{\partial v^j} e_{v^i} \cdot e_{u^j} + g_i g_j \frac{\partial a_n^i}{\partial u^i} \frac{\partial a_m^j}{\partial u^j} e_{v^i} \cdot e_{v^j} \Big) du^i dv^i du^j dv^j \quad , \tag{20}$$

$$R_{nm}^{i} = \iint \Big(\eta_u^i \frac{f_i}{g_i} \frac{\partial a_n^i}{\partial v^i} \frac{\partial a_m^i}{\partial v^i} - 2\eta_{uv}^i \frac{\partial a_n^i}{\partial v^i} \frac{\partial a_m^i}{\partial u^i} + \eta_v^i \frac{g_i}{f_i} \frac{\partial a_n^i}{\partial u^i} \frac{\partial a_m^i}{\partial u^i} \Big) du^i dv^i \quad . \tag{21}$$

It is clear that the inductance matrix M^{ij} and the resistance matrix R^i are both real symmetric and positive-definite.

If an externally applied magnetic field on the conductor S_i is described by the vector potential $A(u^i,v^i)$, then the mutual magnetic energy U_{en}^i between the current component V_n^i and $A(u^i,v^i)$ is represented by

$$U_{en}^i = \int_{S_i} A \cdot J_n^i \; dS_i \quad . \tag{22}$$

Introduction of Eq. (16) into Eq. (22) leads to the following equation:

$$E_n^i = \iint \Big(f_i \frac{\partial a_n^i}{\partial v^i} A_u^i - g_i \frac{\partial a_n^i}{\partial u^i} A_v^i \Big) du^i dv^i \quad , \tag{23}$$

in which, E_n^i denotes the extermally applied electromotive force on the current component V_n^i. A_u^i and A_v^i are u^i and v^i components of $A(u^i,v^i)$, respectively.

For more explanation, we divide the conductor into finite elements as schematically shown in Fig. 3, so that, the energy integrals (17), (18), (22) are reduced to discrete forms. The finite element circuit $\bar{\Omega}_i(m,n)$ is assumed to be constructed of the corresponding four finite elements $\Omega_i(\nu)$ $(\nu = 1,2,3,4)$ to each node (m,n)

$$\bar{\Omega}_i(m,n) = \Omega_i(1) \cup \Omega_i(2) \cup \Omega_i(3) \cup \Omega_i(4) \quad . \tag{24}$$

According to the discussion carried out in the previous section, let us assume that the function a_n^i is represented by coordinates u^i and v^i in the finite element $\Omega_i(\nu)$

$$a_n^i = P(\nu)\ Q(\nu) \qquad (\nu = 1,2,3,4) \ , \tag{25}$$

in which

$$\begin{aligned} P(\nu) &= X_\nu && (\nu = 1,\ 4) \ , \\ P(\nu) &= 1 - X_\nu && (\nu = 2,\ 3) \ , \\ Q(\nu) &= Y_\nu && (\nu = 1,\ 2) \ , \\ Q(\nu) &= 1 - Y_\nu && (\nu = 3,\ 4) \ . \end{aligned} \tag{26}$$

Here X_ν and Y_ν are the following local coordinates given in the finite element $\Omega_i(\nu)$, respectively

$$\begin{aligned} X_\nu &= \frac{1}{\Delta u^i(\nu)} (u^i - u_L^i(\nu)) \ , \\ Y_\nu &= \frac{1}{\Delta v^i(\nu)} (v^i - v_L^i(\nu)) \ . \end{aligned} \tag{27}$$

where, $\Delta u^i(\nu)$ and $\Delta v^i(\nu)$ are widths of the finite element $\Omega_i(\nu)$ along u^i and v^i directions, respectively. It is easily found

$$u_m^i = u_U^i(1) = u_U^i(4) = u_L^i(2) = u_L^i(3) \ ,$$

$$u_{m-1}^i = U_L^i(1) = u_L^i(4) \ , \quad u_{m+1}^i = u_U^i(2) = u_U^i(3) \ ,$$

$$v^i_n = v^i_U(1) = v^i_U(2) = v^i_L(3) = v^i_L(4)$$
$$v^i_{n-1} = v^i_L(1) = v^i_L(2) \quad , \quad v^i_{n+1} = v^i_U(3) = v^i_U(4) \quad , \tag{28-a}$$

and

$$\Delta u^i(1) = \Delta u^i(4) = u^i_m - u^i_{m-1} \quad ,$$
$$\Delta u^i(2) = \Delta u^i(3) = u^i_{m+1} - u^i_m \quad ,$$
$$\Delta v^i(1) = \Delta v^i(2) = v^i_n - v_{n-1} \quad , \tag{28-b}$$
$$\Delta v^i(3) = \Delta v^i(4) = v^i_{n+1} - v^i_n \quad .$$

Using the approximation given by Eq. (25), the partial differential terms of the integland in Eqs. (20), (21) and (23) are represented as:

$$\left(\frac{\partial a^i_n}{\partial v^i} \right)_{\Omega_i(\nu)} = \varepsilon^Y(\nu)\ P(\nu)/\Delta v^i(\nu) \quad ,$$
$$\left(\frac{\partial a^i_n}{\partial u^i} \right)_{\Omega_i(\nu)} = \varepsilon^X(\nu)\ Q(\nu)/\Delta u^i(\nu) \quad . \tag{29}$$

Where, $\varepsilon^X(1) = \varepsilon^X(4) = 1$, $\varepsilon^X(2) = \varepsilon^X(3) = -1$, $\varepsilon^Y(1) = \varepsilon^Y(2) = 1$ and $\varepsilon^Y(3) = \varepsilon^Y(4) = -1$.

Now, the integral of Eq. (20) must be carried out for $\bar{\Omega}_i(m,n)$ and $\bar{\Omega}_j(m',n')$. By putting Eq. (29) into Eq. (20), we can obtain the concrete form of the mutual inductance $M_{\bar{\Omega}_i(m,n);\bar{\Omega}_j(m',n')}$ between finite element circuits $\bar{\Omega}_i(m,n)$ and $\bar{\Omega}_j(m',n')$, which is

$$M_{\bar{\Omega}_i(m,n)\ ;\ \bar{\Omega}_j(m',n')} = \frac{\mu_0}{4\pi} \sum_{\nu=1}^{4} \sum_{\mu=1}^{4}$$
$$\begin{aligned} &(\Delta u^i(\nu)\ \Delta u^j(\mu)\ \varepsilon^Y(\nu)\ \varepsilon^Y(\mu)\ A^1_{\nu\mu} \\ &- \Delta u^i(\nu)\ \Delta v^j(\mu)\ \varepsilon^Y(\nu)\ \varepsilon^X(\mu)\ A^2_{\nu\mu} \\ &- \Delta v^i(\nu)\ \Delta u^j(\mu)\ \varepsilon^X(\nu)\ \varepsilon^Y(\mu)\ A^3_{\nu\mu} \\ &+ \Delta v^i(\nu)\ \Delta v^j(\mu)\ \varepsilon^X(\nu)\ \varepsilon^X(\mu)\ A^4_{\nu\mu}) \quad , \end{aligned} \tag{30}$$

where

$$A^1_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{f_i\ f_j}{\rho_{12}}\ e_{ui}\cdot e_{uj}\ P(\nu)\ P(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu\ ,$$

$$A^2_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{f_i\ g_j}{\rho_{12}}\ e_{ui}\cdot e_{vj}\ P(\nu)\ Q(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu\ ,$$

$$A^3_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{g_i\ f_j}{\rho_{12}}\ e_{vi}\cdot e_{uj}\ Q(\nu)\ P(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu\ , \tag{31}$$

$$A^4_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{g_i\ g_j}{\rho_{12}}\ e_{vi}\cdot e_{vj}\ Q(\nu)\ Q(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu\ .$$

In the case $\Omega_i(\nu) = \Omega_j(\mu)$, the integral of Eq. (31) has a singularity since the denominator ρ_{12} becomes zero. The analytical way to avoid the singularity will be described in Appendix A.

By substituting Eq. (29) into Eq. (21), we obtain the concrete form of mutual resistnace $R_{\bar{\Omega}_i(m,n);\bar{\Omega}_i(m',n')}$ between the finite element circuits $\bar{\Omega}_i(m,n)$ and $\bar{\Omega}_i(m',n')$

$$R_{\bar{\Omega}_i(m,n);\bar{\Omega}_i(m',n')} = \sum_{\substack{\nu=1 \\ (\Omega_i(\nu)\,=\,\Omega_i(\nu'))}}^{4} \left(\frac{\Delta u^i(\nu)}{\Delta v^i(\nu)}\ \varepsilon^Y(\nu)\ \varepsilon^Y(\nu')\ B^1_{\nu\mu'} \right. \tag{32}$$

$$\left. -\ 2\ \varepsilon^Y(\nu)\ \varepsilon^X(\nu')\ B^2_{\nu\nu'} + \frac{\Delta v^i(\nu)}{\Delta u^i(\nu)}\ \varepsilon^X(\nu)\ \varepsilon^X(\nu')\ B^3_{\nu\nu'} \right)\ ,$$

where

$$B^1_{\nu\nu'} = \int_0^1\int_0^1 \eta^i_u\ \frac{f_i}{g_i}\ P(\nu)\ P(\nu')\ dX_\nu\ dY_\nu\ ,$$

$$B^2_{\nu\nu'} = \int_0^1\int_0^1 \eta^i_{uv}\ P(\nu)\ Q(\nu')\ dX_\nu\ dY_\nu\ , \tag{33}$$

$$B^3_{\nu\nu'} = \int_0^1\int_0^1 \eta^i_v\ \frac{g_i}{f_i}\ Q(\nu)\ Q(\nu')\ dX_\nu\ dY_\nu\ .$$

The area integral of Eq. (21) must be carried out for the finite element which coincides as $\Omega_i(\nu) = \Omega_i(\nu')$.

By substituting Eq. (29) into Eq. (23), we obtain the concrete form of the electromotive force due to an externally applied field $A(u^i, v^i)$

$$e^i_n = \sum_{\nu=1}^{4} (\Delta u^i(\nu)\ \varepsilon^Y(\nu)\ C^1_\nu - \Delta v^i(\nu)\ \varepsilon^X(\nu)\ C^2_\nu) \ , \tag{34}$$

where

$$C^2_\nu = \int_0^1\int_0^1 f_i\ P(\nu)\ A^i_u\ dX_\nu\ dY_\nu \ ,$$

$$C^2_\nu = \int_0^1\int_0^1 g_i\ Q(\nu)\ A^i_v\ dX_\nu\ dY_\nu \ . \tag{35}$$

Using the circuit constants given by Eqs. (30) and (32) and the externally applied electromotive force (34), we can represent the circuit equations governing the eddy current, which can be solved directly with the aid of Runge-Kutta method or analytically with the help of eigen mode expansion.

4. Boundary Condition and Symmetry

In the foregoing section, the circuit equations of finite element circuits in a multi-conductor system with arbitrary shape and electrical resistivity distribution are formulated to solve a general eddy current problem. In this section, boundary condition of the individual conductor is introduced by adopting a linear map of linear vector space. The symmetry of system geometry is also discussed making use of the same operation on the nodal variables of current function.

4.1 Boundary condition

Let us consider a bounded surface S with an outer boundary ∂C^{out} as shown in Fig. 4. In the figure, an inner boundary C^{in} denotes the hole and an inner boundary ∂C^{in} means the electrical insulation. From the definition given by Eq. (1), it is easily found that the difference of current functions between the certain two points denotes the total

The area integral of Eq. (21) must be carried out for the finite element which coincides as $\Omega_i(\nu) = \Omega_i(\nu')$.

By substituting Eq. (29) into Eq. (23), we obtain the concrete form of the electromotive force due to an externally applied field $A(u^i, v^i)$

$$e_n^i = \sum_{\nu=1}^{4} (\Delta u^i(\nu)\ \varepsilon^Y(\nu)\ C_\nu^1 - \Delta v^i(\nu)\ \varepsilon^X(\nu)\ C_\nu^2) \ , \tag{34}$$

where

$$\begin{aligned} C_\nu^2 &= \int_0^1\int_0^1 f_i\ P(\nu)\ A_u^i\ dX_\nu\ dY_\nu \ , \\ C_\nu^2 &= \int_0^1\int_0^1 g_i\ Q(\nu)\ A_v^i\ dX_\nu\ dY_\nu \ . \end{aligned} \tag{35}$$

Using the circuit constants given by Eqs. (30) and (32) and the externally applied electromotive force (34), we can represent the circuit equations governing the eddy current, which can be solved directly with the aid of Runge-Kutta method or analytically with the help of eigen mode expansion.

4. Boundary Condition and Symmetry

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4.1 Boundary condition

Let us consider a bounded surface S with an outer boundary ∂C^{out} as shown in Fig. 4. In the figure, an inner boundary C^{in} denotes the hole and an inner boundary ∂C^{in} means the electrical insulation. From the definition given by Eq. (1), it is easily found that the difference of current functions between the certain two points denotes the total current across a line connecting these points. Therefore, we can regard the nodal current functions on the outer boundary ∂C^{out}, the inner boundary ∂C^{in} and within the inner boundary C^{in} to be constant, respectively. Moreover, we can also regard all of the nodal current functions on the outer boundary ∂C^{out} to be zero since the current function has an arbitrariness of constant. From the mentioned above, it is evident that the current function of a point on the conductor surface S denotes the total current flowing between the point and the outer boundary ∂C^{out}.

Let N be the total number of finite element circuits on the conductor S. And, let $N_{C^{in}}$, $N_{\partial C^{in}}$, $N_{\partial C^{out}}$ and N_0 be the numbers of all nodal points within the hole, on the electrical cut, on the outer boundary and the numbers of all nodal points except those within the inner boundaries and on the outer boundary, respectively. Therefore, the number of independent nodal value of current function is $N' = N_0 + 2 = N - (N_{C^{in}} - 1) - (N_{\partial C^{in}} - 1) - N_{\partial C^{out}}$ in total. Letting V be an ordered set whose member denotes the current function of each node and, then V can be written

$$V = A\ V' \ , \tag{36-a}$$

$$A = A_{\partial C^{out}}\ A_{\partial C^{in}}\ A_{C^{in}} \ . \tag{36-b}$$

Where, V' means an ordered set whose members denote the $N_0 + 2$ independent current function of nodes, which is reprecented:

$$V' = (V_1 \cdots V_{N_0}\ V_{C^{in}}\ V_{\partial C^{in}})^T \ , \tag{37}$$

V is also represented

$$V = (\overbrace{V_1 \cdots V_{N_0}}^{N_0}\ \overbrace{V_{C^{in}} \cdots V_{C^{in}}}^{N_{C^{in}}}\ \overbrace{V_{\partial C^{in}} \cdots V_{\partial C^{in}}}^{N_{\partial C^{in}}}$$

$$\overbrace{V_{\partial C^{out}} \cdots V_{\partial C^{out}}}^{N_{\partial C^{out}}})^T \ , \tag{38-a}$$

where,

$$V_{\partial C^{out}} = 0 \ . \tag{38-b}$$

$A_{\partial C^{out}} \in I^{N\times(N_0+N_{C^{in}}+N_{\partial C^{in}})}$, $A_{\partial C^{in}} \in I^{(N_0+N_{C^{in}}+N_{\partial C^{in}})\times(N_0+N_{C^{in}}+1)}$ and $A_{C^{in}} \in I^{(N_0+N_{C^{in}}+1)\times(N_0+2)}$ denote the respective linear maps in linear vector space corresponding to the outer boundary ∂C^{out}, the inner boundaries ∂C^{in} and C^{in} as follows:

$$A_{\partial C^{out}} = \begin{array}{c} \begin{matrix} \overbrace{}^{N_0} & \overbrace{}^{N_{C^{in}}} & \overbrace{}^{N_{\partial C^{in}}} \end{matrix} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{matrix} \} N_0 \\ \} N_{C^{in}} \\ \} N_{\partial C^{in}} \\ \} N_{\partial C^{out}} \end{matrix} , \qquad (39\text{-a})$$

$$A_{\partial C^{in}} = \begin{array}{c} \begin{matrix} \overbrace{}^{N_0} & \overbrace{}^{N_{C^{in}}} & \overbrace{}^{1} \end{matrix} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \end{array} \begin{matrix} \} N_0 \\ \} N_{C^{in}} \\ \} N_{\partial C^{in}} \end{matrix} , \qquad (39\text{-b})$$

$$A_{C^{in}} = \begin{array}{c} \begin{matrix} \overbrace{}^{N_0} & \overbrace{}^{1} & \overbrace{}^{1} \end{matrix} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{matrix} \} N_0 \\ \} N_{C^{in}} \\ \} 1 \end{matrix} , \qquad (39\text{-c})$$

where, I is unit matrix. If a conductor has many holes or electrical cuts, then the corresponding linear mapping to these boundary conditions successively done by means of the similar procedure.

From the mentioned above, the similar linear map $\bar{A}$ in a multi-conductor system is obtainable

$$\bar{A} = \text{quasi-diag.}\ (A_i) \quad (i = 1, \ldots, N_{cond}) \ . \qquad (40)$$

Here, $\bar{A} \in I^{\Sigma N_i \times \Sigma N_i{}'}$ has a structure of so-called quasi-diagonal matrix composed of the submatrices A_i of the i-th conductor, which is already given by Eq. (36-b). The summation Σ must be carried out over the N_{cond} conductors of a considered system. Let $M \in R^{\Sigma N_i \times \Sigma N_i}$, $R \in R^{\Sigma N_i \times \Sigma N_i}$

and $\varepsilon \in R^{\Sigma N_i}$ be the inductance, resistance matrices and the voltage vector for the variable sets of current function in a multi-conductor system without the boundary conditions in each conductor. Then, the reduced inductance, resistance matrices $M' \in R^{\Sigma N_i' \times \Sigma N_i'}$, $R' \in R^{\Sigma N_i' \times \Sigma N_i'}$ and the reduced voltage vector $\varepsilon' \in R^{\Sigma N_i'}$ are represented by use of the boundary condition (40)

$$\begin{aligned} M' &= \bar{A}^T M \bar{A} \quad , \\ R' &= \bar{A}^T R \bar{A} \quad , \\ \varepsilon' &= \bar{A} \varepsilon \quad . \end{aligned} \tag{41}$$

It is clear that the reduced matrices M' and R' are also real symmetric because $M = M^T$ and $R = R^T$.

4.2 Symmetry

When the geometry of a multi-conductor system has a symmetry with respect to a common symmetric plane Π, then the usual current function on the conductor S can be separated into the following odd parity part and even parity part as:

$$\begin{aligned} V^{odd}(u,v) &= \frac{1}{2}(V(u,v) - V(-u,v)) \quad , \\ V^{even}(u,v) &= \frac{1}{2}(V(u,v) + V(-u,v)) \quad . \end{aligned} \tag{42}$$

Here, the common symmetric plane Π is given by $u^i = 0$ $(i = 1, \ldots, N_{cond})$. The odd parity part of current function along the symmetric line $u = 0$ is always zero, and the even parity part $V^{even}(0,v)$ along the symmetric line $u = 0$ is given by the usual current function $V(0,v)$. Fig. 5 shows the symmetry of a multi-conductor system, where conductors are discretized into finite elements. Using the representation given by Eq. (42), the ordered set of nodal current function V on the conductor S can be described as follows:

$$V = P\, V' \quad , \tag{43}$$

in which

$$V = (V(m>0,n)^{\mathrm{T}}\ V(m=0,n)^{\mathrm{T}}\ \ (m<0,n)^{\mathrm{T}})^{\mathrm{T}}\ , \tag{44-a}$$

$$V' = (V^{\mathrm{odd}}(m,n)^{\mathrm{T}}\ V^{\mathrm{even}}(m=0,n)^{\mathrm{T}}\ V^{\mathrm{even}}(m,n)^{\mathrm{T}})^{\mathrm{T}}\ , \tag{44-b}$$

$$(1 \le m \le M,\ 1 \le n \le N)\ .$$

Here the both ordered sets $V^{\mathrm{odd}}(m,n)$ and $V^{\mathrm{even}}(m,n)$ are arranged according to the same order of the set $V(m>0,n)$, and the order of $V(m>0,n)$ is arranged according to the reverse order of $V(m<0,n)$. Therefore, the matrix $\mathrm{P} \in I^{N(2M+1)\times N(2M+1)}$ is given by

$$\mathrm{P} = \begin{pmatrix} \overbrace{\mathrm{I}}^{MN} & \overbrace{0}^{N} & \overbrace{\mathrm{I}}^{MN} \\ 0 & \mathrm{I} & 0 \\ -\mathrm{I}^* & 0 & \mathrm{I}^* \end{pmatrix} \begin{matrix} \}\ MN \\ \}\ N \\ \}\ MN \end{matrix}\ , \tag{45}$$

where, $\mathrm{I}^* \in I^{MN\times MN}$ is a matrix to rearrange the members of ordered set in reverse. The entire matrix $\bar{\mathrm{P}} \in I^{N_i(2M_i+1)\times\Sigma N_i(2M_i+1)}$ in a multi-conductor system is described

$$\bar{\mathrm{P}} = \text{block diag.}\ (\mathrm{P}_{\mathrm{i}}) \qquad (\mathrm{i} = 1, \ldots, \mathrm{N}_{\mathrm{cond}})\ , \tag{46}$$

here P_{i} denotes the linear map of nodal current function for the i-th conductor, which is given by Eq. (45).

Let $\mathrm{M} \in \mathrm{R}^{\Sigma N_i(2M_i+1)\times\Sigma N_i(2M_i+1)}$ be the inductance matrix corresponding to the ordered set $\bar{V} = (\bar{V}_1^{\mathrm{T}}\bar{V}_2^{\mathrm{T}}\ \ldots\ \bar{V}_{\mathrm{N_{cond}}}^{\mathrm{T}})^{\mathrm{T}}$ in a multi-conductor system, which is given by

$$\mathrm{M} = \left(\begin{pmatrix} \mathrm{M}^{\mathrm{ij}}_{m>0,\ m'>0} & \mathrm{M}^{\mathrm{ij}}_{m>0,\ m'=0} & \mathrm{M}^{\mathrm{ij}}_{m>0,\ m'<0} \\ \mathrm{M}^{\mathrm{ij}}_{m=0,\ m'>0} & \mathrm{M}^{\mathrm{ij}}_{m=0,\ m'=0} & \mathrm{M}^{\mathrm{ij}}_{m=0,\ m'<0} \\ \mathrm{M}^{\mathrm{ij}}_{m<0,\ m'>0} & \mathrm{M}^{\mathrm{ij}}_{m<0,\ m'=0} & \mathrm{M}^{\mathrm{ij}}_{m<0,\ m'<0} \end{pmatrix}\right)\ . \tag{47}$$

Then, the inductance matrix $\bar{\mathrm{M}} \in R^{N_i(2M_i+1)\times N_i(2M_i+1)}$ corresponding to the ordered sets $\overline{V'} = (\bar{V}_1'^{\mathrm{T}}\ \bar{V}_2'^{\mathrm{T}}\ \ldots\ \bar{V}'^{\mathrm{T}}_{\mathrm{N_{cond}}})^{\mathrm{T}}$ can be represented

$$\bar{M} = \bar{P}^T \; M \; \bar{P} \quad . \tag{48}$$

That is,

$$\bar{M} = \left(\left(\begin{array}{ccc} \bar{M}^{ij}_{odd,odd} & \bar{M}^{ij}_{odd,even(m'=0)} & \bar{M}^{ij}_{odd,even} \\ \bar{M}^{ij}_{even(m=0),odd} & \bar{M}^{ij}_{even(m=0),even(m'=0)} & \bar{M}^{ij}_{even(m=0),even} \\ \bar{M}^{ij}_{even,odd} & \bar{M}^{ij}_{even,even(m'=0)} & \bar{M}^{ij}_{even,even} \end{array}\right)\right), \tag{49-a}$$

where

$$\bar{M}^{ij}_{odd,odd} = M^{ij}_{m>0,\ m'>0} - M^{ij}_{m>0,\ m'<0}\ I^*_j - I^{*T}_i\ M^{ij}_{m<0,\ m'>0} + I^{*T}_i\ M^{ij}_{m<0,\ m'<0}\ I^*_j \quad , \tag{49-b}$$

$$\bar{M}^{ij}_{odd,even(m'=0)} = M^{ij}_{m>0,\ m'=0} - I^{*T}_i\ M^{ij}_{m<0,\ m'=0} \quad , \tag{49-c}$$

$$\bar{M}^{ij}_{odd,even} = M^{ij}_{m>0,\ m'>0} + M^{ij}_{m>0,\ m'<0} I^*_j - I^{*T}_i\ M^{ij}_{m<0,\ m'>0} - I^{*T}_i\ M^{ij}_{m<0,\ m'<0}\ I^*_j \tag{49-d}$$

$$\bar{M}^{ij}_{even(m=0),odd} = M^{ij}_{m=0,\ m'>0} - M^{ij}_{m=0,\ m'<0}\ I^*_j \tag{49-e}$$

$$\bar{M}^{ij}_{even(m=0),even(m'=0)} = M^{ij}_{m=0,\ m'=0} \tag{49-f}$$

$$\bar{M}^{ij}_{even(m=0),even} = M^{ij}_{m=0,\ m'>0} + M^{ij}_{m=0,\ m'<0}\ I^*_j \tag{49-g}$$

$$\bar{M}^{ij}_{even,odd} = M^{ij}_{m>0,\ m'>0} - M^{ij}_{m<0,\ m'<0}\ I^*_j + I^{*T}_i\ M^{ij}_{m<0,\ m'>0} - I^{*T}_i\ M^{ij}_{m<0,\ m'<0}\ I^*_j \tag{49-h}$$

$$\bar{M}^{ij}_{even,even(m'=0)} = M^{ij}_{m>0,\ m'=0} + I^{*T}_i\ M^{ij}_{m>0,\ m'=0} \tag{49-i}$$

$$\bar{M}^{ij}_{\mathrm{even,even}} = M^{ij}_{m>0,\ m'>0} + M^{ij}_{m>0,\ m'<0}\ I^{*}_{j} + I^{*T}_{i}\ M^{ij}_{m<0,\ m'>0} + I^{*T}_{i}\ M^{ij}_{m<0,\ m'<0}\ I^{*}_{j} \tag{49-j}$$

From the symmetry of the multi-conductor system with respect to the plane Π, we obtain the following equations for the inductance submatrices

$$M^{ij}_{m>0,\ m'<0}\ I^{*}_{j} = I^{*T}_{i}\ M^{ij}_{m<0,\ m'>0}\ , \tag{50-a}$$

$$M^{ij}_{m>0,\ m'>0} = I^{*T}_{i}\ M^{ij}_{m<0,\ m'<0}\ I^{*}_{j}\ , \tag{50-b}$$

$$M^{ij}_{m>0,\ m'=0} = I^{*T}_{i}\ M^{ij}_{m<0,\ m'=0}\ , \tag{50-c}$$

$$M^{ij}_{m=0,\ m'>0} = M^{ij}_{m=0,\ m'<0}\ I^{*}_{j}\ . \tag{50-d}$$

Consequently, all of the inductance submatrices between the odd and even parity parts of current function are always zero in the multi-conductor system which is symmetric with respect to the common symmetric plane Π. Finally, we obtain the inductance matrix $\bar{M}$

$$\bar{M} = \left(\left(\begin{array}{ccc} 2(M^{ij}_{m>0,\ m'>0} - M^{ij}_{m>0,\ m'<0}\ I^{*}_{j}) & 0 & 0 \\ 0 & M^{ij}_{m=0,\ m'=0} & 2M^{ij}_{m=0,\ m'>0} \\ 0 & 2M^{ij}_{m>0,\ m'=0} & 2(M^{ij}_{m>0,\ m'>0} + M^{ij}_{m>0,\ m'>0}\ I^{*}_{j} \end{array}\right)\right)\ . \tag{51}$$

In a similar way, we can verify the orthogonality of resistance, that is, the odd parity and even parity parts of current function are decoupled with each other.

5. Formulation in a Multi-Torus System

This section describes the detailed investigation on an application of the finite element circuit theory discussed in the previous sections to the eddy current problem in a composite multi-torus system. It is supposed that the considered multi-torus system, in which each torus is disconnected with others, is symmetric with respect to an equatorial plane of the system geometry. Therefore, the odd and even parity parts of current function with respect to the equatorial plane are separately treated in what follows. As is described in subsection 5.4, the odd parity part of current function with respect to a toroidal symmetry couples only the magnetic field provided by a toroidal field coils. For this reason, parity of current function is assumed to be even with respect to the toroidal symmetry.

In order to analyze the eddy current in a multi-torus system composed of many tori, one must customarily carry out multiple integrals to obtain the circuit constants and solve a large-scale generalized eigenvalue problem to get the eigen mode of eddy current paying for a great deal of computational cost. Although a reckless practice with the aid of the theory discussed in the foregoning sections is basically possible, however, this will not be practical because of the existing restrictions of computer storage and computing time. In order to avoid these restrictions of computer, the improved method is described in subsections 5.1 and 5.2. The improvement is provided by the mode reduction of eigen eddy current and the introduction of shape function into the double area integral for the coupling between eigen modes. Firstly, the eddy current in a multi-torus system is obtained for the bases of eigen function, which indicates the eigen mode of eddy current on a corresponding torus. In subsection 5.1, formulation of the eigenvalue problem is given for the system composed of a torus. In this step, the inductance matrix of the multi-torus system has yet off-diagonal block submatrices (see Fig. 8), that is, the eigen modes defined on different tori couples with each other. Next, we individually select a set of the dominant modes from the entire modes of eddy current on the respective tori by eliminating the uncontrollable and higher modes. The discussion on this mode elimination is carried out in Appendix B by means of "controllability" familiar in the theory of linear multivariable control system. Finally, after evaluating the mutual inductance between the reduced eigen modes, we solve the reduced eigenvalue problem in an entire multi-torus system. Those are described in subsection 5.2. In subsection 5.3, the external coil systems controlled by the given voltage are introduced to simulate the poloidal field coils in an actual tokamak. Formulation to evaluate the three-dimensional magnetic field due to an eddy current is represented in subsection 5.4.

5.1 Eigen mode of eddy currents localized in a torus

Consider a multi-torus system composed of N_{cond} tori as is illustrated in Fig. 6. The system geometry is supposed to be symmetric with respect to the equatorial plane z = 0. Let (ϕ,ℓ) be the orthogonal coordinate system defined on the torus S. In a cylindrical coordinate system (r,ϕ,z),

$$r = r(\ell) \quad , \qquad z = z(\ell) \quad . \tag{52}$$

Taking x-axis as the direction $\phi = 0$, Eq. (9) is now rewritten as:

$$x = r(\ell)\cos\phi, \quad y = r(\ell)\sin\phi, \quad z = z(\ell) \quad . \tag{53}$$

Here, the scale factors f and g given by Eq. (11) are also rewritten as:

$$f = r(\ell) \quad , \qquad g = 1 \quad . \tag{54}$$

Coordinate line ℓ and ϕ are defined between the interval $[\ell^-,\ell^+]$ and between the interval $[-\pi,\pi]$, respectively. In the case that a torus cross-section is closed, two points (ϕ,ℓ^-) and (ϕ,ℓ^+) conincide with each other. Unit tangent vectors along ℓ = constant and ϕ = constant are respectively expressed as

$$\begin{aligned} e_\phi &= (-\sin\phi,\ \cos\phi,\ 0)^T \ , \\ e_\ell &= \left(\frac{dr}{d\ell}\cos\phi,\ \frac{dr}{d\ell}\sin\phi,\ \frac{dz}{d\ell} \right)^T \end{aligned} \tag{55}$$

Consequently, the double area integrals for the mutual inductance $M_{\bar{\Omega}(m,n);\bar{\Omega}(m',n')}$ between the finite element circuits $\bar{\Omega}(m,n)$ and $\bar{\Omega}(m',n')$, which is given by Eq. (31), are represented in the following concrete forms as:

$$A^1_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{r_1 r_2}{\rho_{12}} \cos(\phi_2 - \phi_1)\ P(\nu)\ P(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu \ ,$$

$$A^2_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{r_1}{\rho_{12}} \frac{dr_2}{d\ell_2} \sin(\phi_2 - \phi_1)\ P(\nu)\ Q(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu \ ,$$

$$A^3_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{r_2}{\rho_{12}} \frac{dr_1}{d\ell_1} \sin(\phi_1 - \phi_2)\ Q(\nu)\ P(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu \ , \tag{56}$$

$$A^4_{\nu\mu} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{1}{\rho_{12}} \left\{ \frac{dr_1}{d\ell_1}\frac{dr_2}{d\ell_2} \cos(\phi_2 - \phi_1) + \frac{dz_1}{d\ell_1}\frac{dz_2}{d\ell_2} \right\} Q(\nu)\ Q(\mu)\ dX_\nu\ dY_\nu\ dX_\mu\ dY_\mu \ .$$

And, the area integrals for the resistance $R_{\bar{\Omega}(m,n);\bar{\Omega}(m',n')}$, which is given by Eq. (33), are also represented in the following concrete forms as:

$$B^1_{\nu\nu'} = \int_0^1\int_0^1 \eta_\phi(\nu)\ r_1\ P(\nu)\ P(\nu')\ dX_\nu\ dY_\nu \ ,$$

$$B^2_{\nu\nu'} = \int_0^1\int_0^1 \eta_{\phi\ell}(\nu)\ P(\nu)\ Q(\nu')\ dX_\nu\ dY_\nu \ , \tag{57}$$

$$B^3_{\nu\nu'} = \int_0^1\int_0^1 \eta_\ell(\nu)\ \frac{1}{r_1}\ Q(\nu)\ Q(\nu')\ dX_\nu\ dY_\nu \ .$$

If the finite element is sufficiently small comparing with the size and radius of curvature of the torus, then f, g, e_ϕ and e_ℓ can be approximated to be constant as $\bar{f}$, $\bar{g}$, $\bar{e}_\phi$ and $\bar{e}_\ell$ within the respective rectangular finite element. In the case the finite elements $\Omega(\nu)$ and $\Omega(\mu)$ do not overlap each other, ρ_{12} can be well approximated by $\bar{\rho}_{12}$, which is the distance between centroids. Therefore, one can rewrite Eqs. (56) and (57) into the following simple forms, respectively

$$A^1_{\nu\mu} = \frac{\bar{r}_1\ \bar{r}_2}{4\ \bar{\rho}_{12}}\ \bar{e}_{\phi_1} \cdot \bar{e}_{\phi_2} \quad ,$$

$$A^2_{\nu\mu} = \frac{\bar{r}_1}{4\ \bar{\rho}_{12}}\ \bar{e}_{\phi_1} \cdot \bar{e}_{\phi_2} \quad ,$$

$$A^3_{\nu\mu} = \frac{\bar{r}_2}{4\ \bar{\rho}_{12}}\ \bar{e}_{\ell_1} \cdot \bar{e}_{\phi_2} \quad , \qquad (56')$$

$$A^4_{\nu\mu} = \frac{1}{4\ \bar{\rho}_{12}}\ \bar{e}_{\ell_1} \cdot \bar{e}_{\ell_2} \quad ,$$

and

$$B^1_{\nu\nu'} = \frac{\eta_\phi(\nu)}{4}\ \bar{r}_1 \quad ,$$

$$B^2_{\nu\nu'} = 0 \quad , \qquad (57')$$

$$B^3_{\nu\nu'} = \frac{\eta_\ell(\nu)}{4\ \bar{r}_1} \quad .$$

Here, $\eta_\phi(\nu)$ and $\eta_\ell(\nu)$, which mean the electrical resistivities in ϕ and ℓ directions, are supposed to be respectively constant in the finite element $\Omega(\nu)$. On the other hand, in the case that the finite elements $\Omega(\nu)$ and $\Omega(\mu)$ overlap each other, it is difficult to carry out the integral (56) since the integrands are singular at $\rho_{12} = 0$. The analytical way to aovid the singularity will be described in Appendix A.

For more explanation, let us consider a finite element mesh of the torus as is illustrated in Fig. 7, where the three-dimensional torus conductor is extended to a plane. N_ϕ denotes the number of toroidal periodicity of the considered multi-torus system. $\ell = 0$ line on the torus coincides with $r = r(0)$, $z = 0$ line. The torus is divided into $4N_\phi MN$ rectangular finite elements with a same toroidal width but a different width along the torus cross-section. Following features of the torus geometry are obvious with regard to the inductance M.

(a) The symmetry of torus geometry with respect to the equatorial plane gives the relation

$$M_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = M_{\bar{\Omega}(m,-n)\,;\,\bar{\Omega}(m',-n')} \quad . \tag{58-a}$$

(b) The axisymmetry of a torus conductor gives the relation

$$M_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = M_{\bar{\Omega}(m+k,n)\,;\,\bar{\Omega}(m'+k,n')} \quad . \tag{58-b}$$

(c) The symmetry of torus gemometry with respect to the $\phi = 0$ plane gives the relation

$$M_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = M_{\bar{\Omega}(-m,n)\,;\,\bar{\Omega}(-m',n')} \quad . \tag{59-c}$$

Here, $|m|,|m'| \leqq MN_{\phi}$ and $|n|,|n'| \leqq N$. It is also obvious $M_{\bar{\Omega}(m,n);\bar{\Omega}(m',n')} = M_{\bar{\Omega}(m',n');\bar{\Omega}(m,n)}$. Consequently, the numerical calculations only of $M_{\bar{\Omega}(0,n);\bar{\Omega}(m',n')}$ $(n = 0,\ldots,N; m' = 0,\ldots,MN_{\phi}; n' = -N, \ldots, 0, \ldots N)$ are sufficient in order to obtain the objective inductance matrix. In Fig. 7, it must be noted that the boundary $\partial C(\pi)$ always coincides with $\partial C(-\pi)$, and the finite element circuits $\partial\bar{\Omega}(m,N)$ along the boundary $\partial C(\ell^{+})$ or $\partial\bar{\Omega}(m,-N)$ along the boundary $\partial C(\ell^{-})$ are composed only of $\Omega(1)$ and $\Omega(2)$ or $\Omega(4)$ and $\Omega(3)$ rectangular finite elements, respectively. The finite element circuit $\bar{\Omega}(MN_{\phi},n)$ overlaps with $\bar{\Omega}(-MN_{\phi},n)$.

From the periodicity of current function whose interval is $[\frac{2\pi}{N_\phi} i_\phi, \frac{2\pi}{N_\phi}(i_\phi + 1)]$ $(i_\phi = 0,\ldots, N_\phi - 1)$ and on the assumption of even parity of current function with respect to $\frac{2\pi}{N_\phi} i_\phi$, we can reduce a dimension of the inductance matrix $M \in R^{2MN_\phi(2N+1)\times 2MN_\phi(2N+1)}$ to $M' \in R^{(M+1)(2N+1)\times(M+1)(2N+1)}$ making use of a similar manner to the discussion on symmetry described in subsection 4.2. Namely, because that the current function $V(m,n)$ on the node (m,n) $(0 \leqq m \leqq 2M-1; -N \leqq n \leqq N)$ is same with $V(2Mi_\phi+m,n)$ due to the toroidal periodicity and $V(2Mi_\phi-m,n)$ due to the even parity, we can obtain the following equation

$$M'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = \alpha_m N_\phi \sum_{i_\phi=0}^{N_\phi-1} \{ M_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(2Mi_\phi+m',n')} + \beta_{m'} M_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(2Mi_\phi-m',n')} \} . \quad (60)$$

Where, $0 \leqq m,\ m' \leqq M$ and $|n|,|n'| \leqq N$. If $m = 0,\ M$, then $\alpha_m = 1$, otherwise $\alpha_m = 2$. If $m' = 0,\ M$, then $\beta_{m'} = 0$, otherwise $\beta_{m'} = 1$.

Furthermore, the parity condition with respect to the equatorial plane divides the inductance matrix M' into an even and odd matrices with reduced dimensions. As is emphasized in subsection 4.2, these parity parts decouple with each other. As for the even parity part, the mutual inductance $M'^{\,even}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')}$ is represented by

$$M'^{\,even}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = \gamma_n (M'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} + \delta_{n'} M'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',-n')}) , \quad (61)$$

here, $0 \leqq m,\ m' \leqq M$ and $0 \leqq n,\ n' \leqq N$. If $n = 0$, then $\gamma_n = 1$, otherwise $\gamma_n = 2$. If $n' = 0$, then $\delta_{n'} = 0$, otherwise $\delta_{n'} = 1$. Consequently, the dimension of inductance matrix $M'^{\,even}$ is $(M+1)(N+1)\times(M+1)(N+1)$. As for the odd parity part, it is evident that the mutual inductance $M'^{\,odd}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')}$ can be represented by

$$M'^{\,odd}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = 2\,(M'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} - M'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',-n')}) , \quad (62)$$

where, $0 \leqq m,\ m' \leqq M$ and $1 \leqq n,\ n' \leqq N$. Now, the odd parity part of current function with respect to the equatorial plane has no current component across the boundary $\partial C(\ell^+)$ or $\partial C(\ell^-)$, that is, the current function of odd parity part is constant along the boundary $\partial C(\ell^+)$, even if the torus cross-section closes its circumference. Therefore, these boundary conditions given the reduction of matrix dimension in terms of the following representation as

$$M'^{odd}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(0,N)} = \sum_{m'=0}^{M} M'^{odd}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',N)} \quad (n \neq N) \;, \tag{63-a}$$

and

$$M'^{odd}_{\bar{\Omega}(0,N)\,;\,\bar{\Omega}(0,N)} = \sum_{m=0}^{M} \sum_{m'=0}^{M} M'^{odd}_{\bar{\Omega}(m,N)\,;\,\bar{\Omega}(m',N)} \;. \tag{63-b}$$

Consequently, the dimension of inductance matrix M'^{odd} for the odd parity part of current function is $(MN-M+N)\times(MN-M+N)$.

Now, consider the resistance matrix. It is sufficient to calculate only $R_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')}$ $(m = 0,\ldots, M,\ m' = m,\ m+1\,;\ n = 0,\ldots, N,\ n' = n,\ n+1)$ in order to obtain the objective resistance matrix. Here, the following features of the torus geometry are used.

(a) The symmetry of torus geometry and electrical resistivity with respect to the equatorial plane gives the relation

$$R_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = R_{\bar{\Omega}(m,-n)\,;\,\bar{\Omega}(m',-n')} \;. \tag{64-a}$$

(b) The periodicity of electrical resistivity gives the relation

$$R_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = R_{\bar{\Omega}(2Mi_\phi + m,n)\,;\,\bar{\Omega}(2Mi_\phi + m',n')} \tag{64-b}$$

$$(i_\phi = 0,\ldots, N_\phi - 1) \;.$$

(c) The symmetry of torus geometry and electrical resistivity with respect to $\phi = 0$ plane gives the relation

$$R_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = R_{\bar{\Omega}(-m,n)\,;\,\bar{\Omega}(-m',n')} \;. \tag{64-c}$$

It is evident from the definition of electrical resistivity that if $|m - m'| \geq 2$ or $|n - n'| \geq 2$, then $R_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = 0$, and $R_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = R_{\bar{\Omega}(m',n')\,;\,\bar{\Omega}(m,n)}$.

According to the procedure described in the calculation of the inductance matrix, from the periodicity of current function with interval $[\frac{2\pi}{N_\phi} i\ ,\ \frac{2\pi}{N_\phi}(i_\phi+1)]$ $(i_\phi = 0,\ldots, N_\phi-1)$ and on the assumption of even parity with respect to $\frac{2\pi}{N_\phi} i_\phi$, we can reduce the order of the resistance matrix $R \in \mathcal{R}^{2MN_\phi(2N+1)\times 2MN_\phi(2N+1)}$ to $R' \in \mathcal{R}^{(M+1)(2N+1)\times(M+1)(2M+1)}$. The element of R' is denoted by

$$R'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = \alpha_{mm'} N_\phi R_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} \quad . \tag{65}$$

Where, $|m|,|m'| \leqq M$ and $|n|,|n'| \leqq N$. If $m, m' = 0$ or M, then $\alpha_{mm'} = 1$, otherwise $\alpha_{mm'} = 2$. Furthermore, the parity condition reduces the dimension of resistance matrix. Namely, the resistance matrix $R'^{even} \varepsilon R^{(M+1)(N+1)\times(M+1)(N+1)}$ of the even parity part can be described such that

$$R'^{even}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = \gamma_{nn'} R'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} \quad , \tag{66}$$

where, $0 \leqq m, m' \leqq M$ and $0 \leqq n, n' \leqq N$. If $n = n' = 0$, then $\gamma_{nn'} = 1$, otherwise $\gamma_{nn'} = 2$. On the other, the resistance matrix R'^{odd} of the parity part is described by

$$R'^{odd}_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} = 2R'_{\bar{\Omega}(m,n)\,;\,\bar{\Omega}(m',n')} \quad , \tag{67}$$

where, $0 \leqq m, m' \leqq M$ and $1 \leqq n, n' \leqq N$. Now the current function of odd parity part along the boundary $\partial C(\ell^+)$ is constant. Therefore, this gives the reduction of matrix dimension in terms of the following representation as

$$R'^{odd}_{\bar{\Omega}(m,N-1)\,;\,\bar{\Omega}(m',N)} = \sum_{m'=m-1}^{m+1} R'^{odd}_{\bar{\Omega}(m,N-1)\,;\,\bar{\Omega}(m',N)} \qquad (0 \leqq m \leqq M) \quad , \tag{68-a}$$

and

$$R'^{odd}_{\bar{\Omega}(0,N)\,;\,\bar{\Omega}(0,N)} = \sum_{m=0}^{M} \sum_{m'=m-1}^{m+1} R'^{odd}_{\bar{\Omega}(m,N)\,;\,\bar{\Omega}(m',N)} \quad . \tag{68-b}$$

Consequently, the dimension of resistance matrix R'^{odd} for the odd parity part of current function is $(MN-M+N)\times(MN-M+N)$.

When the torus has port holes and electrical insulations, the boundary conditoins can be introduced making use of a corresponding linear map in linear vector space, which is given by Eq. (41). In addition, the constant arbitrariness of current function can be removed by putting the current function of even parity to be zero at a given node. Now, the preparatory formulations are completed for the generalized eigenvalue problem (5), which should be individually solved subject to the considered N_{cond} tori. Here, the eigenfunction is normalized in its joule loss per unit time (sec) on the corresponding torus conductor. It is clear that eigen modes of eddy current on a torus are decoupled with each other, however, these are coupled with the eddy current modes on the other tori through magnetic interaction.

5.2 Eddy current problem in a multi-torus system

In this subsection, formulations to solve an eddy current problem in the multi-torus system are described. firstly, basis vectors of eddy current in the multi-torus system are defined by selecting the dominant eigen modes among the entire sets previously obtained for the individual torus. After that, the mutual inductance $M_{k_i k_j}$ between the k_i-th mode on the i-th torus and the k_j-th mode on the i-th torus is expressed in terms of these basis vectors. Here, one must evaluate a great number of the mutual inductance given by Eq. (20). It is shown that the double area integral can be replaced by a linear combination of the nodal value of eigenfunction and the so-called shape function which depends only on the torus geometry but current function. Therefore, it is sufficient to carry out once the double are integral of the i-th and the j-th (j≠i) tori. The structure of inductance matrix in the multi-torus system is shown in Fig. 8.

Because that the torus S_i are divided into finite elements $\Omega_i(m,n)$ ($|m| \leq MN_\phi$, $|n| \leq N$), Eq. (20) is rewritten in the discrete form

$$M_{k_i k_j} = \frac{\mu_0}{4\pi} \sum_{\Omega_i(m,n)} \sum_{\Omega_j(m',n')} \Delta M_{k_i k_j}(\Omega_i,\Omega_j) \quad , \tag{69}$$

here, $\Delta M_{k_i k_j}(\Omega_i,\Omega_j)$ denotes the partial inductance between the k_i-th eigen mode on the finite element $\Omega_i(m,n)$ of the i-th torus and the k_j-th eigen mode on the finite element $\Omega_j(m',n')$ of the j-th torus. The finite element $\Omega(m,n)$ is redefined in Fig. 9. Double summation must be carried out for all of the finite elements on the respective tori. Now, the current function in the finite element $\Omega(m,n)$ is represented in terms of four vertices $V(\sigma,\mu)$ (σ,μ = L,U) which are shown in Fig. 9

$$V(\phi,\ell) = \sum_{\sigma=L,U} \sum_{\mu=L,U} P(\sigma)\, Q(\mu)\, V(\sigma,\mu) \quad . \tag{70}$$

Where

$$P(U) = X, \qquad P(L) = 1 - X \quad ,$$
$$Q(U) = Y, \qquad Q(L) = 1 - Y \quad . \tag{71}$$

here, X and Y are the following local coordinates in the finite element $\Omega(m,n)$, respectively

$$X = \frac{1}{\Delta\phi}(\phi - \phi_L) \quad ,$$
$$Y = \frac{1}{\Delta\ell}(\ell - \ell_L) \quad . \tag{72}$$

Therefore, on the assumption that the unknown function a_n is eigenfunction $V(\phi,\ell)$, the differential terms of the integrand in Eq. (20) are expressed as

$$\left(\frac{\partial V}{\partial \ell}\right)_{\Omega(m,n)} = \frac{1}{\Delta\ell}\sum_{\sigma=L,U}\sum_{\mu=L,U} \varepsilon(\mu)\ P(\sigma)\ V(\sigma,\mu) \quad ,$$
$$\left(\frac{\partial V}{\partial \phi}\right)_{\Omega(m,n)} = \frac{1}{\Delta\phi}\sum_{\sigma=L,U}\sum_{\mu=L,U} \varepsilon(\sigma)\ Q(\mu)\ V(\sigma,\mu) \quad , \tag{73}$$

here, $\varepsilon(L) = -1$ and $\varepsilon(U) = 1$. Using Eq. (73), the partial inductance $\Delta M_{k_i k_j}(\Omega_i,\Omega_j)$ is given by

$$\Delta M_{k_i k_j}(\Omega_i,\Omega_j) = \sum_{\sigma_1=L,U}\sum_{\mu_1=L,U}\sum_{\sigma_2=L,U}\sum_{\mu_2=L,U}$$
$$\{\varepsilon(\mu_1)\ \varepsilon(\mu_2)\ \alpha_{\sigma_1\sigma_2}\ (m,n;\ m',n')$$
$$+ \varepsilon(\mu_1)\ \varepsilon(\sigma_2)\ \beta_{\sigma_1\mu_2}\ (m,n;\ m',n')$$
$$+ \varepsilon(\sigma_1)\ \varepsilon(\mu_2)\ \gamma_{\mu_1\sigma_2}\ (m,n;\ m',n')$$
$$+ \varepsilon(\sigma_1)\ \varepsilon(\sigma_2)\ \delta_{\mu_1\mu_2}\ (m,n;\ m',n')\}V_{k_i}(\sigma_1,\mu_1)\ V_{k_j}(\sigma_2,\mu_2) \quad , \tag{74}$$

where

$$\alpha_{\sigma_1\sigma_2}(m,n;m',n') = \Delta\phi^i \, \Delta\phi^j \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{r_1 r_2}{\rho_{12}}$$

$$\times \, e_{\phi^i} \cdot e_{\phi^j} \, P(\sigma_1) \, P(\sigma_2) \, dX_1 \, dY_1 \, dX_2 \, dY_2 \, , \qquad (75\text{-a})$$

$$\beta_{\sigma_1\mu_2}(m,n;m',n') = \Delta\phi^i \, \Delta\ell^j \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{r_1}{\rho_{12}}$$

$$\times \, e_{\phi^i} \cdot e_{\ell^j} \, P(\sigma_1) \, Q(\mu_2) \, dX_1 \, dY_1 \, dX_2 \, dY_2 \, , \qquad (75\text{-b})$$

$$\gamma_{\mu_1\sigma_2}(m,n;m',n') = \Delta\ell^i \, \Delta\phi^j \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{r_2}{\rho_{12}}$$

$$\times \, e_{\ell^i} \cdot e_{\phi^j} \, Q(\mu_1) \, P(\sigma_2) \, dX_1 \, dY_1 \, dX_2 \, dY_2 \, , \qquad (75\text{-c})$$

$$\delta_{\mu_1\mu_2}(m,n;m',n') = \Delta\ell^i \, \Delta\ell^j \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{1}{\rho_{12}}$$

$$\times \, e_{\ell^i} \cdot e_{\ell^j} \, Q(\mu_1) \, Q(\mu_2) \, dX_1 \, dY_1 \, dX_2 \, dY_2 \, . \qquad (75\text{-d})$$

$\alpha_{\sigma_1\sigma_2}$, $\beta_{\sigma_1\mu_2}$, $\gamma_{\mu_1\sigma_2}$ and $\delta_{\mu_1\mu_2}$ do not depend on the eigen function of eddy current and but only on the system geometry. We call these double area integrals shape function between the finite element $\Omega_i(m,n)$ and $\Omega_j(m',n')$. Although one must evaluate a great number of the mutual inductance corresponding to the eddy current modes, the use of shape functions (75) has an advantage to save the computational cost. That is, once the shape functions between a set of tori are obtained by the double area integral, then all of the mutual inductance can be easily evaluated by the simple summations of the product of shape function and eigen function using Eq. (74).

Since the current function, which is either even or odd with respect to an equatorial plane, is supposed to be symmetric with respect to $\phi = 0$ and periodic N_ϕ times in a toroidal direction, Eq. (69) is rewritten as:

$$M_{k_i k_j} = \frac{\mu_0 N_\phi}{\pi} \sum_{\Omega_i(m,n)}^{\substack{1\le m\le M \\ 1\le n\le N}} \sum_{\Omega_j(m',n')} \Delta M_{k_i k_j}(\Omega_i,\Omega_j)$$

$$= \frac{\mu_0 N_\phi}{\pi} \sum_{\Omega_i(m,n)} \sum_{\Omega_j(m',n')}^{\substack{1\le m'\le M' \\ 1\le n'\le N'}} \Delta M_{k_i k_j}(\Omega_i,\Omega_j) \quad . \tag{76}$$

Eq. (76) holds for both parity parts of the current function with respect to the equatorial plane. Furthermore, let us notice the following relations of current function, i.e.

$$V(m',n') = V(2M'i_\phi + m',n')$$
$$= V(2M'i_\phi - m'+1,n') \ , \qquad (i_\phi = 0,\dots, N_\phi - 1) \tag{77}$$

and

$$V^{even}(m',n') = V^{even}(m',-n'+1) \quad , \tag{78-a}$$

$$V^{odd}(m',n') = -V^{odd}(m',-n'+1) \quad . \tag{78-b}$$

By making use of these relations, we can redefine shape functions $\alpha_{\sigma_1\sigma_2}^{\{even \atop odd\}}$, $\beta_{\sigma_1\mu_2}^{\{even \atop odd\}}$, $\gamma_{\mu_1\sigma_2}^{\{even \atop odd\}}$ and $\delta_{\mu_1\mu_2}^{\{even \atop odd\}}$ corresponding to the respective parity parts of current function. Here, two lines in the brace of superscript denote the considered parity. Those can be shown for both parity parts

$$\alpha_{\sigma_1\sigma_2}^{\{even \atop odd\}}(m,n;m',n')$$

$$= \sum_{i_\phi=0}^{N_\phi-1} \{(\alpha_{\sigma_1\sigma_2}(m,n;2M'i_\phi + m',n') + \alpha_{\sigma_1\sigma_2}^*(m,n;2M'i_\phi - m'+1,n'))$$

$$\mp (\alpha_{\sigma_1\sigma_2}(m,n;2M'i_\phi + m',-n'+1) + \alpha_{\sigma_1\sigma_2}^*(m,n;2M'i_\phi - m'+1,-n'+1))\},$$

(79-a)

$$\beta_{\sigma_1\mu_2}^{\{\substack{\text{even}\\\text{odd}}\}}(m,n;m',n')$$

$$= \sum_{i_\phi=0}^{N_\phi-1} \{(\beta_{\sigma_1\mu_2}(m,n;2M'i_\phi+m',n')+\beta_{\sigma_1\mu_2}(m,n;2M'i_\phi-m'+1,n'))$$

$$\pm(\beta_{\sigma_1\mu_2^*}(m,n;2M'i_\phi+m',-n'+1)-\beta_{\sigma_1\mu_2^*}(m,n;2M'i_\phi-m'+1,-n'+1))\},$$

(79-b)

$$\gamma_{\mu_1\sigma_2}^{\{\substack{\text{even}\\\text{odd}}\}}(m,n;m',n')$$

$$= \sum_{i_\phi=0}^{N_\phi-1} \{(\gamma_{\mu_1\sigma_2}(m,n;2M'i_\phi+m',n')+\gamma_{\mu_1\sigma_2^*}(m,n;2M'i_\phi-m'+1,n'))$$

$$\mp(\gamma_{\mu_1\sigma_2}(m,n;2M'i_\phi+m',-n'+1)+\gamma_{\mu_1\sigma_2^*}(m,n;2M'i_\phi-m'+1,-n'+1))\},$$

(79-c)

$$\delta_{\mu_1\mu_2}^{\{\substack{\text{even}\\\text{odd}}\}}(m,n;m',n')$$

$$= \sum_{i_\phi=0}^{N_\phi-1} \{(\delta_{\mu_1\mu_2}(m,n;2M'i_\phi+m',n')-\delta_{\mu_1\mu_2}(m,n;2M'i_\phi-m'+1,n'))$$

$$\pm(\delta_{\mu_1\mu_2^*}(m,n;2M'i_\phi+m',-n'+1)-\delta_{\mu_1\mu_2^*}(m,n;2M'i_\phi-m'+1,-n'+1))\},$$

(79-d)

Where, σ_2^*, μ_2^* = L when σ_2, μ_2 = U or σ_2^*, μ_2^* = U when σ_2, μ_2 = L. The upper sign in brace of a right-hand side is for even parity, and the lower sign is for odd parity. Consequently, Eq. (76) is rewritten in the following final form

$$M_{k_i k_j}^{\{\substack{\text{even}\\\text{odd}}\}} = \frac{\mu_0 N_\phi}{\pi} \sum_{\substack{1\leq m\leq N\\1\leq n\leq N\\ \Omega_i(m,n)}} \sum_{\substack{1\leq m'\leq M'\\1\leq n'\leq N'\\ \Omega_j(m',n')}} \Delta M_{k_i k_j}^{\{\substack{\text{even}\\\text{odd}}\}}(\Omega_i,\ \Omega_j) \quad . \tag{80}$$

Here, $\Delta M_{k_i k_j}^{\{even \atop odd\}}(\Omega_i,\Omega_j)$ can be immediately obtained by substituting Eqs. (79) into Eq. (74).

Now, letting M_{sys} be the inductance matrix of the multi-torus system, which has a structure shown in Fig. 8, consider a following eigenvalue problem

$$M_{sys}\, X = \mathrm{diag}(\lambda) X \quad , \tag{81}$$

here, λ means the eigenvalue of eddy current in the multi-torus system. The solution $\Psi \varepsilon R^{\Sigma K_i^* \times \Sigma K_i^*}$ of modal matrix of Eq. (81) in the multi-torus system provides the eigen function $W_i \varepsilon R^{N_i' \times \Sigma K_i^*}$ of eddy current mode on i-th torus together with the reduced modal matrix $\Phi_i^* \varepsilon R^{N_i' \times K_i^*}$ of i-th torus alone. Here, N_i' denotes the number of independent nodal points and K_i^* denotes the mode number of selected eddy current after mode reduction on i-th torus. That is,

$$W_i = \Phi_i^* \Psi_i \qquad (i = 1, \ldots, N_{cond}) \quad , \tag{82}$$

here, $\Psi_i \varepsilon R^{K_i^* \times \Sigma K_i^*}$ is a following submatrix meaning it is a modal matrix only of i-th torus taking Φ_i^* as a representative basis

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \cdot \\ \vdots \\ \cdot \\ \Psi'_{N_{cond}} \end{pmatrix} \quad . \tag{83}$$

Now, we can simulate the transient eddy current in a general multi-torus system by the linear combination of solutions of the first order ordinary differential equations, which is in the same form with Eq. (6). When the electromotive force ε_k on k-th mode is applied by a transient current source J(t), ε_k is represented

$$\varepsilon_k = S_k \dot{J}(t) \quad , \tag{84}$$

where $\dot{(\ )}$ means time differentiation of ( ). The time independent constant S_k denotes the mutual inductance between k-th eddy current mode in the multi-torus system and a current-controlled coil system. Here, the coil system is supposed to be axisymmetric and to be symmetrically arranged with respect to an equatorial plane of the multi-torus system. Therefore, the vector potential due to unit current of the axisymmetric coil system has only a toroidal component $A_\phi^i(\ell)$ on i-th torus

$$A_\phi^i(\ell) \quad = \quad \sum_{\tau=L,U} Q(\tau)\ A_\phi^i(\tau) \qquad\qquad \text{in } \Omega_i(m,n) \quad . \tag{85}$$

Where $A_\phi^i(\tau)$ is the corresponding nodal value of vector potential. On the other, suppose that the current function $W_i^k(\phi,\ell)$ in the finite $\Omega_i(m,n)$ is described in terms of four vertices $W_i^k(\sigma,\mu)$ $(\sigma,\mu{=}L,U)$ in the similer form to Eq. (70). Putting Eq. (85) and the current function $W_i^k(\phi,\ell)$ into Eq. (23), we can obtain S_k in the following concrete form as:

$$S_k \quad = \quad 4N_\phi \sum_{i=1}^{N_{cond}} \sum_{\substack{\Omega_i(m,n) \\ 1\le m\le M \\ 1\le n\le N}} \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U}$$

$$\varepsilon(\mu)\ \chi_{\sigma\tau}\ A_\phi^i(\tau)\ W_i^k(\sigma,\mu) \qquad , \tag{86}$$

here

$$\chi_{\sigma\tau}(m,n) \quad = \quad \Delta\phi^i \int_0^1\!\!\int_0^1 r_i\ P(\sigma)\ Q(\tau)\ dX\ dY \quad . \tag{87}$$

$\chi_{\sigma\tau}(m,n)$ does not depend on the eigen mode of eddy current but only on the geometry of considered system. Although one must evaluate a great number of the mutual inductance S_k, the use of shape function (87) has an advantage to save computational cost. Once the shape function are obtained by the area integral (87), then one can easily evaluate all of the mutual inductance S_k by the simple summation of Eq. (86). Eq. (86) holds for both parity parts of current function with respect to the equatorial plane. In the case that an usual vector potential can be decomposed into its even parity part and odd parity part, the even parity part of current function couples only with the odd parity part of externally applied vector potential, on the contrary, the odd parity part of current function couples only with the even parity part of the vector potential.

Although a suitable numerical method such as Gauss-Legendre quadrature form is most commonly used to carry out the integrals (75) and (87), the following conventional method is available if the finite element $\Omega_i(m,n)$ is sufficiently small comparing with the size and radius of curvature of the torus and the finite elements concerned are kept away from the other. In that case, the variables of integrands except for the interpolation function (71) can be approximated to be constant as $\bar{r}$, $\bar{\rho}$, $\bar{e}_\phi$ and $\bar{e}_\ell$ in the respective rectanguler finite element, so that Eqs. (75) and Eq. (87) are respectively rewritten in the following simple forms

$$\alpha_{\sigma_1\sigma_2}(m,n;m',n') = \Delta\phi^i \, \Delta\phi^j \, \frac{\bar{r}_1 \, \bar{r}_2}{4\,\bar{\rho}_{12}} \, \bar{e}_{\phi^i} \cdot \bar{e}_{\phi^j} \quad , \tag{75-a'}$$

$$\beta_{\sigma_1\mu_2}(m,n;m',n') = \Delta\phi^i \, \Delta\ell^j \, \frac{\bar{r}_1}{4\rho_{12}} \, \bar{e}_{\phi^i} \cdot \bar{e}_{\ell^j} \quad , \tag{75-b'}$$

$$\gamma_{\mu_1\sigma_2}(m,n;m',n') = \Delta\ell^i \, \Delta\phi^j \, \frac{\bar{r}_2}{4\bar{\rho}_{12}} \, e_{\ell^i} \cdot e_{\phi^j} \quad , \tag{75-c'}$$

$$\delta_{\mu_1\mu_2}(m,n;m',n') = \Delta\ell^i \, \Delta\ell^j \, \frac{1}{4\bar{\rho}_{12}} \, \bar{e}_{\ell^i} \cdot \bar{e}_{\ell^j} \quad , \tag{75-d'}$$

and

$$\chi_{\sigma\tau}(m,n) = \Delta\phi^i \, \frac{\bar{r}_i}{4} \quad . \tag{87'}$$

Throughout the paper, the multi-torus system is assumed to be arranged symmetrically with respect to an equatorial plane. Therefore, formulations to solve a general eddy current problem have been presented separetely for the individual parity part of current function, since both parity parts decouple with each other. To the contrary, if the considered system is arranged asymmetry then both parity parts must be treated simultaneously because they couple with each other.

5.3 Introduction of an external coil system

If there exists N_{coil} voltage-controlled coil systems in the considered multi-torus system, the circuit equation is governed together with the K^* eigen modes of eddy current obtained in the previous subsection as follows:

$$M \dot{X} + R X = E \quad . \tag{88}$$

Where $X \in R^{K^*+N_{coil}}$ is the current vector including N_{coil} voltage-controlled coil systems, which is represented by

$$X = (x^T \; y^T)^T \quad , \tag{89}$$

here, $x \in R^{K^*}$ is a current subvector of the eddy current modes in the multi-torus system and $y \in R^{N_{coil}}$ is the current subvector of voltage-controlled coil systems. K^* denotes the total number of eddy current modes as $\sum_{i=1}^{N_{cond}} K_i^*$. $M, R \in R^{(K^*+N_{coil}) \times (K^*+N_{coil})}$ are the inductance and resistance matrices, respectively, which are given as follows:

$$M = \begin{pmatrix} \lambda_1 & & 0 & m_{11} & \cdots & m_{1N_{coil}} \\ & \ddots & & \vdots & & \vdots \\ 0 & & \lambda_{K^*} & m_{K^*1} & \cdots & m_{K^*N_{coil}} \\ & & & L_1 & \cdots & M_{1N_{coil}} \\ & & & & \ddots & \vdots \\ & \text{Sym.} & & \text{Sym.} & & L_{N_{coil}} \end{pmatrix} , \tag{90-a}$$

$$R = \begin{pmatrix} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ & & & R_1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & R_{N_{coil}} \end{pmatrix} . \tag{90-b}$$

In which, m_{kn} means the mutual inductance between k-th eddy current mode and n-th voltage-controlled coil system $M_{nn'}$ denotes the mutual inductance between n-th and n'-th voltage-controlled coil systems. $E \in R^{K^*+N_{coil}}$ is the electromotive force vector. Let $\varepsilon_c \in R^{N_{coil}}$ and $e_c \in R^{N_{coil}}$ be the electromotive force of voltage-controlled coil systems due to the externally applied vector potential and the voltage source of voltage-controlled coil systems, respectively. Then E is represented by

$$E = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{K^*} \\ \varepsilon_c + e_c \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} K^* \\ \} N_{coil} \end{matrix} \quad . \tag{91}$$

Now, the resistance matrix (90-b) is already diagonalized, so that one can easily obtain the following eigenvalue problem

$$\bar{M}\, W = \mathrm{diag}(\tau)\, W \quad , \tag{92}$$

here $\bar{M}$ is real symmetric and positive-definite

$$\bar{M} = R^{-\frac{1}{2}}\, M\, R^{-\frac{1}{2}} \quad . \tag{93}$$

Let $\bar{\Phi}$ be the modal matrix of the eigenvalue problem (92). Then, k-th eigenfunction W^k is described by

$$W^k = (R^{-\frac{1}{2}}\, \bar{\Phi})_k \quad . \tag{94}$$

Here, the subscript $(\ )_k$ denotes k-th column vector of the matrix ( ). The electromotive force vector E' for the newly obtained eigen modes is given

$$E' = \bar{\Phi}^T\, R^{-\frac{1}{2}}\, E \quad . \tag{95}$$

So far, the circuit constants are supposed to be time-invariant, therefore, the finite element circuit method is not applicable to the time-variable problem. However, we frequently encounter a time-variable eddy current problem in tokamak, where the plasma resistance changes every moment during a plasma discharge. In spite of the restriction, if a time varying electrical resistivity is given, one can simulate the transient eddy current including the time-variable coil system by use of the approximation descretizing a time interval into a set of finite time domains. The detailed discussion on the approximate method is described in Appendix C.

5.4 Resultant magnetic field

In this subsection, a magnetic field due to eddy current is described. Fig. 10 shows the concept of magnetic field structures relevant to parity of the eddy current. Since the assumption of even parity of current function in the toroidal direction, the magnetic field structures shown only by the subfigures (a) and (b) are discussed in the report. The odd parity part of current function with respect to the equatorial plane produces the magnetic field perpendicular to the equatorial plane. On the contrary, the even parity part produces the magnetic field tangent to the equatorial plane.

Consider a cylindrical coordinate system (R,θ,Z). Letting $A_k^i = (A_{kR}^i, A_{k\theta}^i, A_{kZ}^i)$ $(i = 1,\ldots, N_{cond})$ and $A_k{}^j = (0, A_{k\theta}^j, 0)$ $(j = 1,\ldots, N_{coil})$ be the vector potential due to the k-th eigen mode on the i-th torus and on the j-th voltage-controlled coil system, the vector potential A_k of the k-th eddy current mode in the multi-torus system is given by

$$A_k = \sum_{i=1}^{N_{cond}} A_k{}^i + \sum_{j=1}^{N_{coil}} A_k{}^j \quad . \tag{96}$$

Then, the magnetic field $B_k = (B_{kR}, B_{k\theta}, B_{kZ})$ is

$$B_k = \nabla \times A_k \quad . \tag{97}$$

A_k^j is easily represented by making use of the vector component W_j^k of the k-th eigen function as follows:

$$A_k^j\ (R,Z) = \frac{\mu_0 W_j{}^k}{2\pi} \sum_{h_j=1}^{H_j} \sqrt{\frac{R_c}{R}}\ \{(\frac{2}{k} - k)K(k) - \frac{2}{k}E(k)\} \quad , \tag{98}$$

where h_j denotes the axisymmetric loop of the j-th voltage-controlled coil system. (R_c,Z_c) is a coordinate of the h_j-th axisymmetric loop. K and E are the respective complete elliptic integrals of modulus k. k is defined by

$$k = \sqrt{\frac{4\,R\,R_c}{(R + R_c)^2 + (Z - Z_c)^2}} \quad . \tag{99}$$

A_k^i is now given by

$$A_k^i = \frac{\mu_0}{4\pi} \int_{S_i} \frac{J_i^{\,k}}{\rho} \, d\,S_i \quad . \tag{100}$$

Here, the integral is all over the torus S_i. ρ is the distance between an integral point on S_i and the space point (R,θ,Z). $J_i^{\,k}$ is expressed by Eq. (15) in terms of the k-th eigen function of eddy current on the i-th torus $W_i^{\,k}$. In the cylindrical coordinate system, each component of $A_k^{\,i}$ are represented

$$A_{kR}^i(R,\theta,Z) = \frac{\mu_0}{4\pi} \int_{\phi i} \int_{\ell i} \frac{1}{\rho} \left\{ \frac{\partial W_i^{\,k}}{\partial \ell} \sin(\theta - \phi) - \frac{\partial W_i^{\,k}}{r\partial\phi} \frac{dr}{d\ell} \cos(\theta - \phi) \right\} d\phi^i \, d\ell^i \quad , \tag{101-a}$$

$$A_{k\theta}^i(R,\theta,Z) = \frac{\mu_0}{4\pi} \int_{\phi i} \int_{\ell i} \frac{1}{\rho} \left\{ \frac{\partial W_i^{\,k}}{\partial \ell} \cos(\theta - \phi) + \frac{\partial W_i^{\,k}}{r\partial\phi} \frac{dr}{d\ell} \sin(\theta - \phi) \right\} d\phi^i \, d\ell^i \quad , \tag{101-b}$$

$$A_{kZ}^i(R,\theta,Z) = \frac{\mu_0}{4\pi} \int_{\phi i} \int_{\ell i} \frac{1}{\rho} \frac{\partial W_i^{\,k}}{r\partial\phi} \frac{dZ}{d\ell} \, d\phi^i \, d\ell^i \quad . \tag{101-c}$$

Now, we here discuss a symmetry and periodicity of vector potential. It is evident that the vector potential Eq. (96) is periodic in θ direction with an interval $[\frac{2}{N_\phi} i_\phi, \frac{2}{N_\phi}(i_\phi+1)]$ $(i_\phi = 0,\ldots, N_\phi - 1)$, which is same periodicity of the multi-torus system. Furthermore, the radial component A_{kR} and axial component A_{kZ} of vector potential are both odd with respect to $\theta = \frac{(2i_\phi+1)}{N_\phi}\pi$ because of the summetry of current function. But the component along θ-direction is even with respect to both of $\theta = \frac{2\pi}{N_\phi} i_\phi$ and $\frac{(2i_\phi+1)}{N_\phi}\pi$. When the current function in the multi-torus system is decomposed into odd and even parity parts with respect to the equatorial plane, the parity of vector potential is classified as Table 1. From the above-mentioned, it is sufficient to evaluate the vector potential in a range only of $\theta = [0, \frac{\pi}{N_\phi}]$ and $Z \geqq 0$.

Since the individual torus is divided into finite elements, the area integral given by Eq. (101) can be rewritten in the discrete form. Now, introduce the shape functions $\xi_{\sigma\mu}(m,n)$, $\omega_{\sigma\mu}(m,n)$ and $\psi_{\sigma\mu}(m,n)$, which are denoted by

$$\xi_{\sigma\mu}(m,n) = \int_0^1\int_0^1 \frac{1}{\rho} \{r_i \, \Delta\phi^i \, \varepsilon(\mu) \, P(\sigma) \, \sin(\theta - \phi) - \Delta\ell^i \, \varepsilon(\sigma) \, Q(\mu) \frac{dr_i}{d\ell^i} \cos(\theta - \phi)\} \, dX \, dY \quad , \tag{102-a}$$

$$\omega_{\sigma\mu}(m,n) = \int_0^1\int_0^1 \frac{1}{\rho} \{r_i \, \Delta\phi^i \, \varepsilon(\mu) \, P(\sigma) \, \cos(\theta - \phi) + \Delta\ell^i \, \varepsilon(\sigma) \, Q(\mu) \frac{dr_i}{d\ell^i} \sin(\theta - \phi)\} \, dX \, dY \quad , \tag{102-b}$$

$$\psi_{\sigma\mu}(m,n) = \int_0^1\int_0^1 \frac{\Delta\ell^i}{\rho} \, \varepsilon(\sigma) \, Q(\mu) \frac{dr_i}{d\ell^i} \, dX \, dY \quad . \tag{102-c}$$

Where, X and Y are the respective local coordinates in the finite element $\Omega(m,n)$, which is already given by Eq. (72). Then Eqs. (101) become to be the following summation over the finite elements on the i-th torus

$$A^i_{kR}(R,\theta,Z) = \frac{\mu_0}{4\pi} \sum_{\Omega_i(m,n)} \sum_{\sigma=L,U} \sum_{\mu=L,U} \xi_{\sigma\mu}(m,n) \, W_i{}^k(\sigma,\mu) \quad , \tag{103-a}$$

$$A^{i}_{k\theta}(R,\theta,Z) = \frac{\mu_0}{4\pi} \sum_{\Omega_i(m,n)} \sum_{\sigma=L,U} \sum_{\mu=L,U} \omega_{\sigma\mu}(m,n)\ W_i^{\ k}(\sigma,\mu), \tag{103-b}$$

$$A^{i}_{kZ}(R,\theta,Z) = \frac{\mu_0}{4\pi} \sum_{\Omega_i(m,n)} \sum_{\sigma=L,U} \sum_{\mu=L,U} \psi_{\sigma\mu}(m,n)\ W_i^{\ k}(\sigma,\mu). \tag{103-c}$$

where $W_i^k(\sigma,\mu)$ denotes the nodal value of the k-th eigen function on the vertex (σ,μ) of the finite element $\Omega_i(m,n)$. The summation for the finite element is all over the i-th torus.

Furthermore, we can redefine the shape function by making use of the symmetry and periodicity of current function in a toroidal direction and the parity with respect to the equatorial plane, that is

$$\xi_{\sigma\mu}^{\{\substack{\text{even}\\ \text{odd}}\}}(m,n) = \sum_{i_\phi=0}^{N_\phi-1} \{\xi_{\sigma\mu}(\frac{2\pi i_\phi}{N_\phi}+m,n) + \xi_{\sigma*\mu}(\frac{2\pi i_\phi}{N_\phi}-m+1,\ n)$$
$$\pm (\xi_{\sigma\mu*}(\frac{2\pi i_\phi}{N_\phi}+m,\ -n+1) + \xi_{\sigma*\mu*}(\frac{2\pi i_\phi}{N_\phi}-m+1,\ -n+1))\} \quad , \tag{104-a}$$

$$\omega_{\sigma\mu}^{\{\substack{\text{even}\\ \text{odd}}\}}(m,n) = \sum_{i_\phi=0}^{N_\phi-1} \{\omega_{\sigma\mu}(\frac{2\pi i_\phi}{N_\phi}+m,n) + \omega_{\sigma*\mu}(\frac{2\pi i_\phi}{N_\phi}-m+1,\ n)$$
$$\pm (\omega_{\sigma\mu*}(\frac{2\pi i_\phi}{N_\phi}+m,\ -n+1) - \omega_{\sigma*\mu*}(\frac{2\pi i_\phi}{N_\phi}-m+1,\ -n+1))\} \ , \tag{104-b}$$

$$\psi_{\sigma\mu}^{\{\substack{\text{even}\\ \text{odd}}\}}(m,n) = \sum_{i_\phi=0}^{N_\phi-1} \{\psi_{\sigma\mu}(\frac{2\pi i_\phi}{N_\phi}+m,n) + \psi_{\sigma*\mu}(\frac{2\pi i_\phi}{N_\phi}-m+1,\ n)$$
$$\pm (\psi_{\sigma\mu*}(\frac{2\pi i_\phi}{N_\phi}+m,\ -n+1) + \psi_{\sigma*\mu*}(\frac{2\pi i_\phi}{N_\phi}-m+1,\ -n+1))\} \ , \tag{104-c}$$

where $\sigma^*, \mu^* = L$ when $\sigma,\mu = U$ and $\sigma^*, \mu^* = U$ when $\sigma,\mu = L$. The sign in brace of a right-hand side must be taken "+" and "-" for the shape functions of "even parity" and "odd parity", respectively. Introducing Eq. (104) into Eq. (103) leads to the following simple forms

$$A_{kR}^{i\left\{\substack{even\\odd}\right\}}(R,\theta,Z) = \frac{\mu_0}{4\pi} \sum_{\substack{1\leq m\leq M\\1\leq n\leq N\\ \Omega_i(m,n)}} \sum_{\sigma=L,U} \sum_{\mu=L,U} \xi_{\sigma\mu}^{\left\{\substack{even\\odd}\right\}}(m,n)\, W_i^{k\left\{\substack{even\\odd}\right\}}(\sigma,\mu) \quad , \tag{105-a}$$

$$A_{k\theta}^{i\left\{\substack{even\\odd}\right\}}(R,\theta,Z) = \frac{\mu_0}{4\pi} \sum_{\substack{1\leq m\leq M\\1\leq n\leq N\\ \Omega_i(m,n)}} \sum_{\sigma=L,U} \sum_{\mu=L,U} \omega_{\sigma\mu}^{\left\{\substack{even\\odd}\right\}}(m,n)\, W_i^{k\left\{\substack{even\\odd}\right\}}(\sigma,\mu) \quad , \tag{105-b}$$

$$A_{kZ}^{i\left\{\substack{even\\odd}\right\}}(R,\theta,Z) = \frac{\mu_0}{4\pi} \sum_{\substack{1\leq m\leq M\\1\leq n\leq N\\ \Omega_i(m,n)}} \sum_{\sigma=L,U} \sum_{\mu=L,U} \psi_{\sigma\mu}^{\left\{\substack{even\\odd}\right\}}(m,n)\, W_i^{k\left\{\substack{even\\odd}\right\}}(\sigma,\mu) \quad . \tag{105-c}$$

Lastly, magnetic field due to eddy current is described by putting Eq. (105) into Eq. (97). Now, we make use a box-shaped finite element Ω obtained by descretizing the three-dimensional space (R,θ,Z) into finite elements as shown in Fig. 11. Supposed that $A_k(R,\theta,Z)$ in the finite element Ω can be expressed in terms of the interpolation function $P(\sigma)$, $Q(\mu)$ and $R(\tau)$

$$A_k(R,\theta,Z) = \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U} P(\sigma)\, Q(\mu)\, R(\tau)\, A_k(\sigma,\mu,\tau) \quad , \tag{106}$$

where, $A_k(\sigma,\mu,\tau)$ is a nodal value of the vector potential on the vertex (σ,μ,τ) of the finite element Ω. The interpolation functions are respectively approximated by

$$
\begin{aligned}
P(L) &= 1 - X, & P(U) &= X \;, \\
Q(L) &= 1 - Y, & Q(U) &= Y \;, \\
R(L) &= 1 - V, & R(U) &= V \;.
\end{aligned}
\tag{107}
$$

in which, X, Y and V are the following local coordinates

$$
\begin{aligned}
X &= \frac{1}{\Delta R}(R - R_L) \;, \\
Y &= \frac{1}{\Delta\theta}(\theta - \theta_L) \;, \\
V &= \frac{1}{\Delta Z}(Z - Z_L) \;.
\end{aligned}
\tag{108}
$$

Introducing Eq. (106) into Eq. (97) leads to the following equation as:

$$
B_{kR}(R,\theta,Z) = \sum_{\sigma=L,U}\sum_{\mu=L,U}\sum_{\tau=L,U}\left(\frac{1}{R\Delta\theta}\varepsilon(\mu)\,P(\sigma)\,R(\tau)\,A_{kZ}(\sigma,\mu,\tau) - \frac{1}{\Delta Z}\varepsilon(\tau)\,P(\sigma)\,Q(\mu)\,A_{k\theta}(\sigma,\mu,\tau)\right) \;, \tag{109-a}
$$

$$
B_{k\theta}(R,\theta,Z) = \sum_{\sigma=L,U}\sum_{\mu=L,U}\sum_{\tau=L,U}\left(\frac{1}{\Delta Z}\varepsilon(\tau)\,P(\sigma)\,Q(\mu)\,A_{kR}(\sigma,\mu,\tau) - \frac{1}{\Delta R}\varepsilon(\sigma)\,Q(\mu)\,R(\tau)\,A_{kZ}(\sigma,\mu,\tau)\right) \;, \tag{109-b}
$$

$$
B_{kZ}(R,\theta,Z) = \sum_{\sigma=L,U}\sum_{\mu=L,U}\sum_{\tau=L,U}\left\{\left(\frac{1}{R}P(\sigma)+\varepsilon(\sigma)\right)Q(\mu)\,R(\tau)\,A_{k\theta}(\sigma,\mu,\tau) - \frac{1}{R\Delta\theta}\varepsilon(\mu)\,P(\sigma)\,R(\tau)\,A_{kR}(\sigma,\mu,\tau)\right\} \;. \tag{109-c}
$$

At a center of gravity (R_c,θ_c,Z_c) of the box-shaped finite element Ω, Eq. (109) becomes

$$B_{kR} = \frac{1}{4} \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U} \{ \frac{1}{R_c \Delta\theta} \varepsilon(\mu) A_{kZ}(\sigma,\mu,\tau) - \frac{1}{\Delta Z} \varepsilon(\tau) A_{k\theta}(\sigma,\mu,\tau) \} \quad , \tag{110-a}$$

$$B_{k\theta} = \frac{1}{4} \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U} \{ \frac{1}{\Delta Z} \varepsilon(\tau) A_{kR}(\sigma,\mu,\tau) - \frac{1}{\Delta R} \varepsilon(\sigma) A_{kZ}(\sigma,\mu,\tau) \} \quad , \tag{110-b}$$

$$B_{kZ} = \frac{1}{4} \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U} \{ (\frac{1}{2R_c} + \varepsilon(\sigma)) A_{k\theta}(\sigma,\mu,\tau) - \frac{1}{R_c \Delta\theta} \varepsilon(\mu) A_{kR}(\sigma,\mu,\tau) \} \quad . \tag{110-c}$$

6. Concluding Remarks

In order to develop a numerical code for solving the general eddy current problem in a multi-torus system, the formulations based on the finite element circuit method, which has been applied to a relatively simple system so far, are given and described in detail. It has been anticipated that the practice of eddy current analysis in the multi-torus system considerably enlarges its computational scale comparing with the actual capacity of computer. Therefore, the investigation has been performed taking sufficient care of the problems in what follows.

(a) Mode reduction of eddy current without destroying accuracy of computation.

(b) Good performance of multiple integrals. Otherwise, a great deal of computational cost should be indispensable.

For the purpose (a), the eddy current in a multi-torus system is firstly expressed in terms of the basis of eigen function, which is localized only on a individual torus. Next, a set of the dominant

$$B_{kR} = \frac{1}{4} \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U} \{ \frac{1}{R_c \Delta\theta} \varepsilon(\mu) A_{kZ}(\sigma,\mu,\tau) - \frac{1}{\Delta Z} \varepsilon(\tau) A_{k\theta}(\sigma,\mu,\tau) \} \quad , \qquad (110\text{-a})$$

$$B_{k\theta} = \frac{1}{4} \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U} \{ \frac{1}{\Delta Z} \varepsilon(\tau) A_{kR}(\sigma,\mu,\tau) - \frac{1}{\Delta R} \varepsilon(\sigma) A_{kZ}(\sigma,\mu,\tau) \} \quad , \qquad (110\text{-b})$$

$$B_{kZ} = \frac{1}{4} \sum_{\sigma=L,U} \sum_{\mu=L,U} \sum_{\tau=L,U} \{ (\frac{1}{2R_c} + \varepsilon(\sigma)) A_{k\theta}(\sigma,\mu,\tau) - \frac{1}{R_c \Delta\theta} \varepsilon(\mu) A_{kR}(\sigma,\mu,\tau) \} \quad . \qquad (110\text{-c})$$

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In order to develop a numerical code for solving the general eddy current problem in a multi-torus system, the formulations based on the finite element circuit method, which has been applied to a relatively simple system so far, are given and described in detail. It has been anticipated that the practice of eddy current analysis in the multi-torus system considerably enlarges its computational scale comparing with the actual capacity of computer. Therefore, the investigation has been performed taking sufficient care of the problems in what follows.

(a) Mode reduction of eddy current without destroying accuracy of computation.

(b) Good performance of multiple integrals. Otherwise, a great deal of computational cost should be indispensable.

For the purpose (a), the eddy current in a multi-torus system is firstly expressed in terms of the basis of eigen function, which is localized only on a individual torus. Next, a set of the dominant modes is individually selected from the entire modes of eddy current by eliminating the uncontrollable and higher modes. Validity of the idea depends on the fact that the uncontrollable-higher mode of eddy current usually decouples with the controllable mode, that is, any external electromotive force can not affect the uncontrollable-higher mode. For the purpose (b), the shape function is introduced to avoid a reckless practice given by Eq. (74) paying for a great deal of computational cost.

In the paper, each torus conductor is supposed to be infinitely thin although the thickness of torus components in an actual tokamak device is finite. Our hypothesis requires a moderate variation of the externally-applied field comparing with the skin time τ_{skin} of the conductor. To avoid this restriction, it becomes to be necessary to take three-dimensional eddy current problem for the finitely-thick conductor into account. Although the procedure stated in the paper is applicable to three dimensional problem with a finite thickness of conductor, it seems to be difficult to get the results with good accuracy because of the restriction of computer storage and computing time. Even though a torus conductor is assumed to be infinitely thin, our procedure to evaluate the eddy current in a multi-torus system is regarded as one of the realistic and effective approach in the general eddy current problem in tokamaks. The method can provides the three-dimensional solution in a multi-torus system neglecting a detail of the penetration of induced eddy current into conductor. From a veiwpoint of control analysis of tokamak equilibrium influenced by the eddy current field, the method described in the paper is considered to have the advantage that the eigen mode of eddy current provides a linear control model of the lumped parameter system after combining with the equation of plasma motion. On the other hand, the finite element method[5,6] or the finite difference method[8,9] are not directly applicable to the plasma control problem with a help of the linear control theory in the lumped parameter system, since these method are basically for the problem governed by a partial differential equation. In the design studies of JT-60 tokamak, the use of numerical code EDDYMULT has demonstrated that our procedure is effective in the general eddy current problem in the multi-torus system including most of the main structural components such as a vacuum vessel, toroidal magnets, support structures and poloidal field coils[15,16].

Acknowledgment

The authors wish to express many thanks to Mr. A. Kameari of Mitsubishi Atomic Power Industries, Inc. for his useful advice and fruitful discussion. They would like to express their sincere gratitude to Drs. T. Iijima and M. Ohta for the continuing encouragement during the course of the work. The authers appreciation also goes to Drs. Y. Suzuki, H. Maeda and M. Shiho for their kind encouragement and support in preparation of this manuscript.

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Acknowledgment

The authors wish to express many thanks to Mr. A. Kameari of Mitsubishi Atomic Power Industries, Inc. for his useful advice and fruitful discussion. They would like to express their sincere gratitude to Drs. T. Iijima and M. Ohta for the continuing encouragement during the course of the work. The authers appreciation also goes to Drs. Y. Suzuki, H. Maeda and M. Shiho for their kind encouragement and support in preparation of this manuscript.

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	W^{even}	W^{odd}
A_R	odd	even
R_0	odd	even
A_Z	even	odd

Table 1 Parity of vector potential with respect to the equatorial plane of the multi-torus system. W denotes a current function.

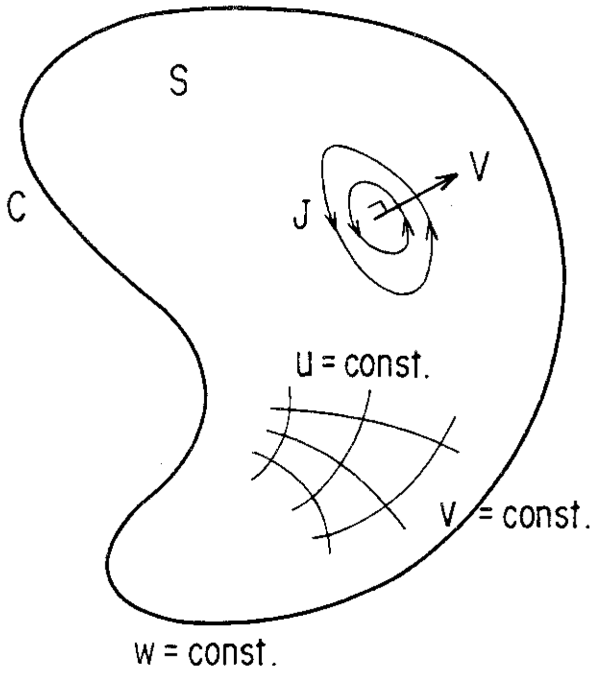

Fig. 1 Current function V on a conductor S and coordinate system (u,v,w). The conductor S lies on the coordinate surface w. C is a simple closed boundary.

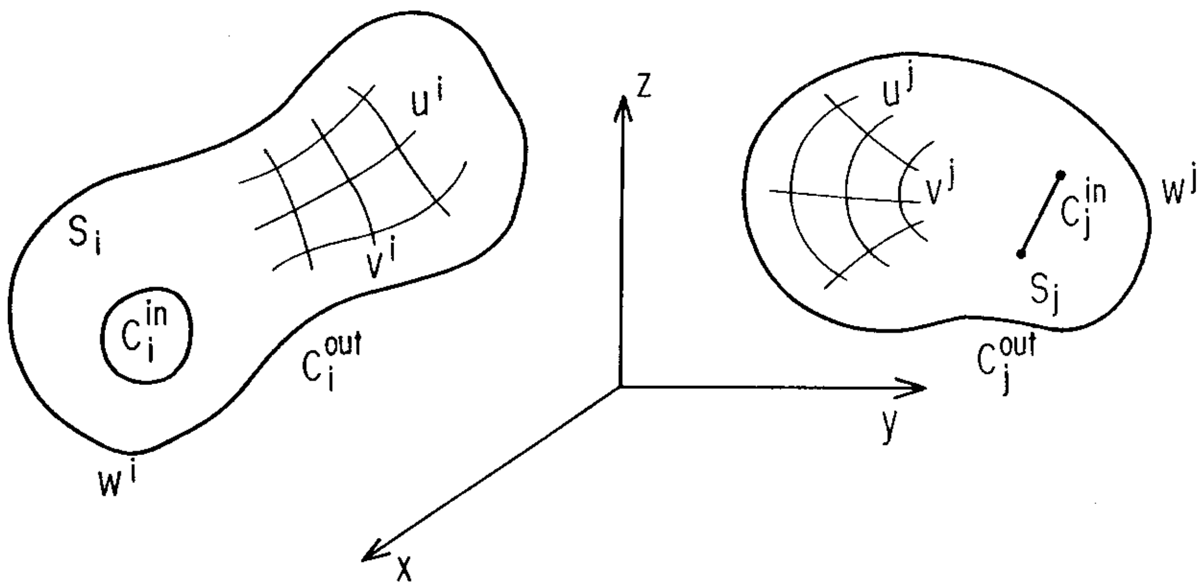

Fig. 2 Multi-conductor system with hole and electrical cut C^{in}. Conductor S_i lies on the coordinate surface w^i. The curvilinear coordinate system (u^i,v^i,w^i) is defined separetely corresponding to the individual conductor. (u^i,v^i) denotes the two dimensional orthogonal coordinate system on S_i.

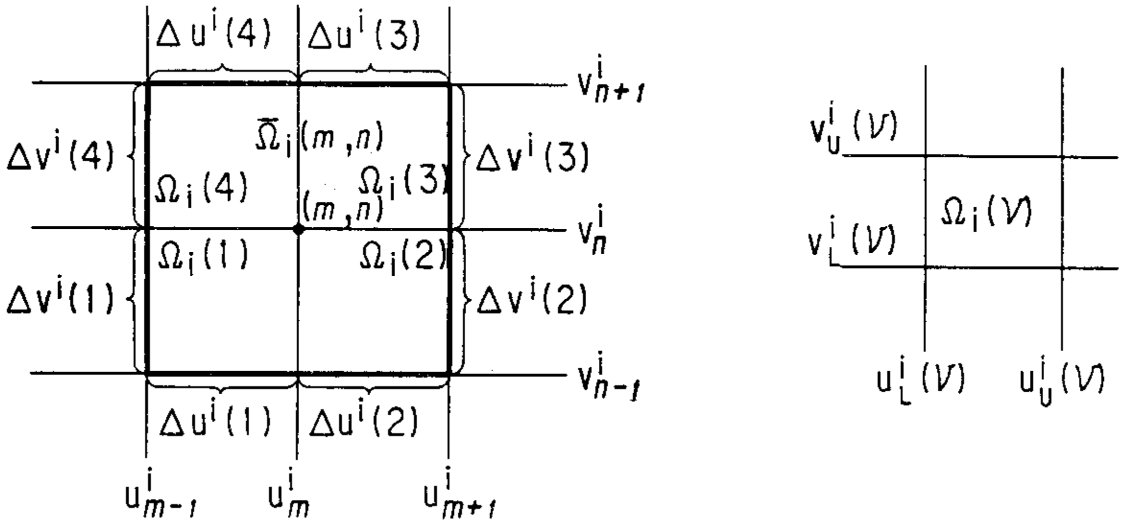

Fig. 3 Partition of the conductor S_i into finite elements $\Omega_i(\nu)$ ($\nu = 1,2,3,4$). There are four rectanguler finite elements in the finite element circuit $\bar{\Omega}_i(m,n)$ corresponding to the node (m,n), where these are numbered in a counterclockwise sense.

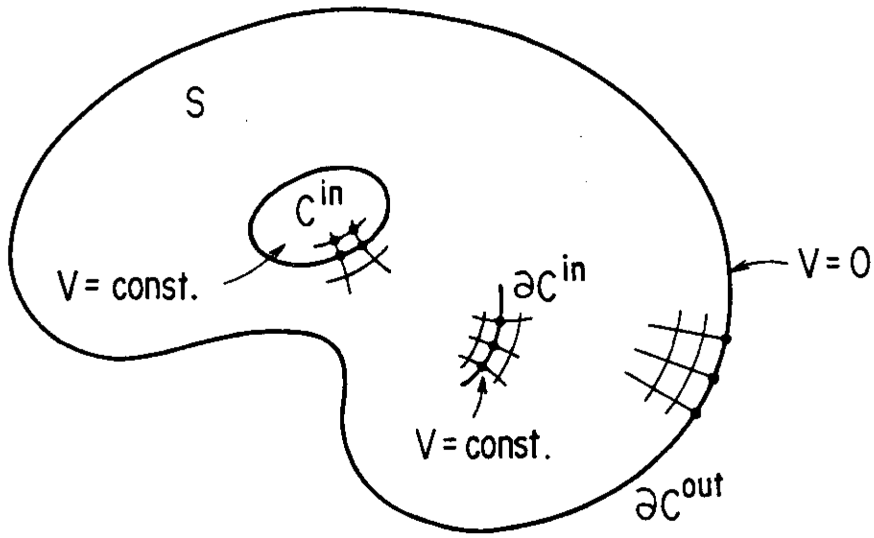

Fig. 4 Boundary conditions in conductor surface S bounded by ∂C^{out}. C^{in} and ∂C^{in} mean the hole and the electrical cut, respectively. On the outer boundary ∂C^{out}, V can be zero because an arbitrary constant can be added to V.

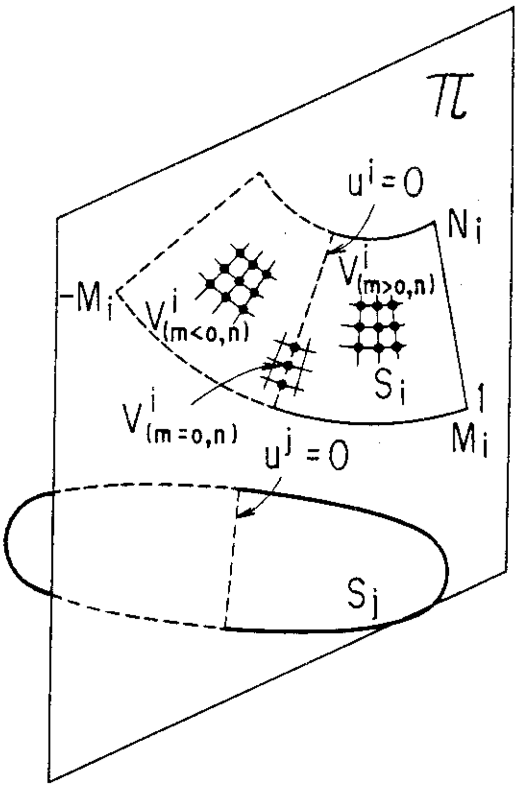

Fig. 5 Multi-conductor system symmetric with respect to a common symmetric plane Π, which is given by the coordinates $u^i = 0$ ($i = 1, 2, \ldots, N_{cond}$).

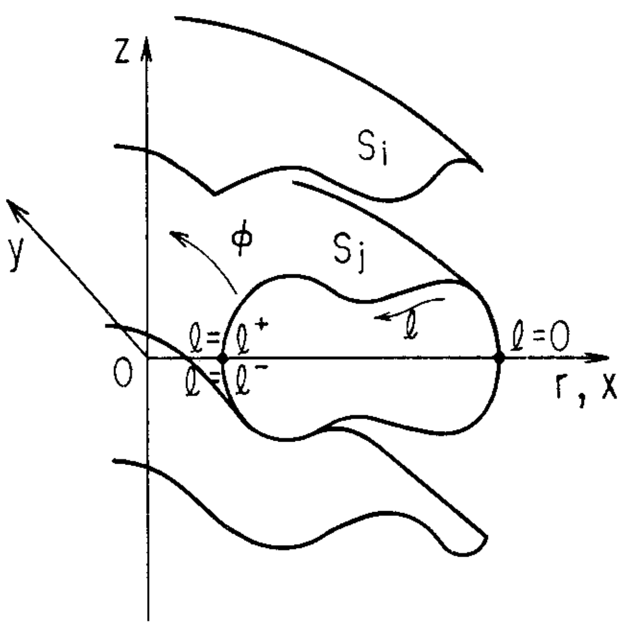

Fig. 6 Multi-torus system. (ϕ, ℓ) is a two-dimensional orthogonal coordinates on the torus S. (r, ϕ, z) and (x, y, z) are the usual cylindrical coordinates and a Cartesian coordinate system, respectively.

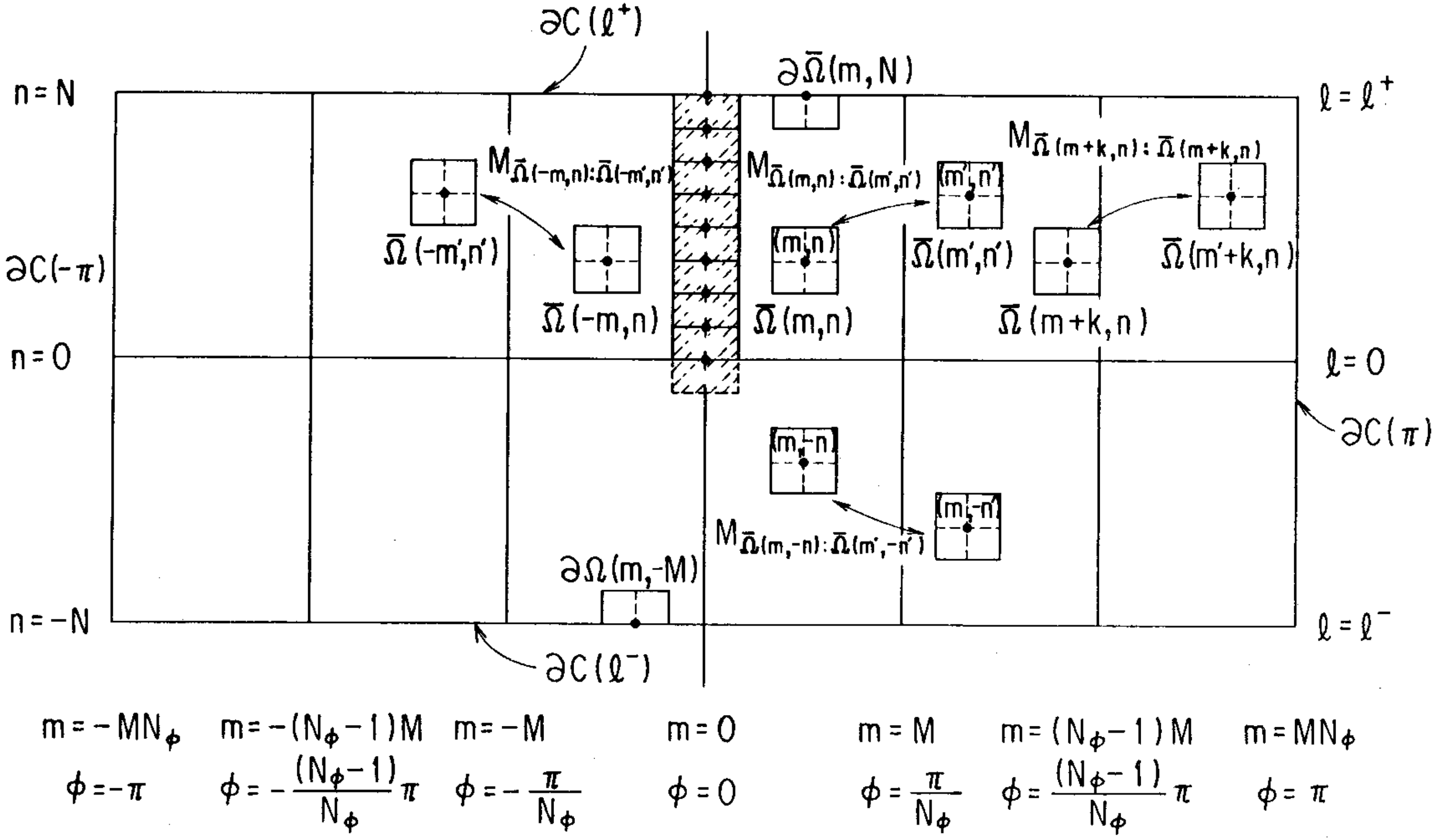

Fig. 7 Finite element mesh of the extended torus. In this figure, the equivalence relation of the mutual inductances is illustrated schematically.

diag (λ_1)	$(m_{1,2})$	--------	$(m_{1,N_{cond}})$
$(m_{1,2})^T$	diag (λ_2)	--------	$(m_{2,N_{cond}})$
⋮	⋮	⋱	⋮
$(m_{1,N_{cond}})^T$	$(m_{2,N_{cond}})^T$	--------	diag$(\lambda_{N_{cond}})$

Fig. 8 Structure of the inductance matrix in the multi-torus system. λ_i denotes the eigenvalve of eigen mode on the i-th torus conductor. $(m_{i,j})$ means a submatrix of mutual inductance between the eigen modes on the i-th and j-th torus.

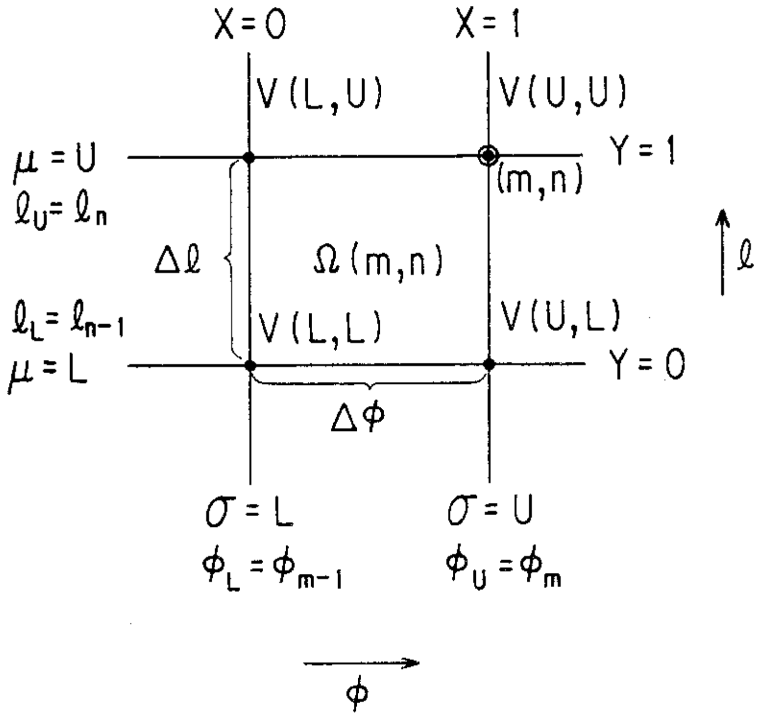

Fig. 9 Finite element $\Omega(m,n)$ corresponding to the node (m,n), the nodal value of current function $V(\sigma,\mu)$ on the vertex (σ,μ) and the local coordinate (X,Y).

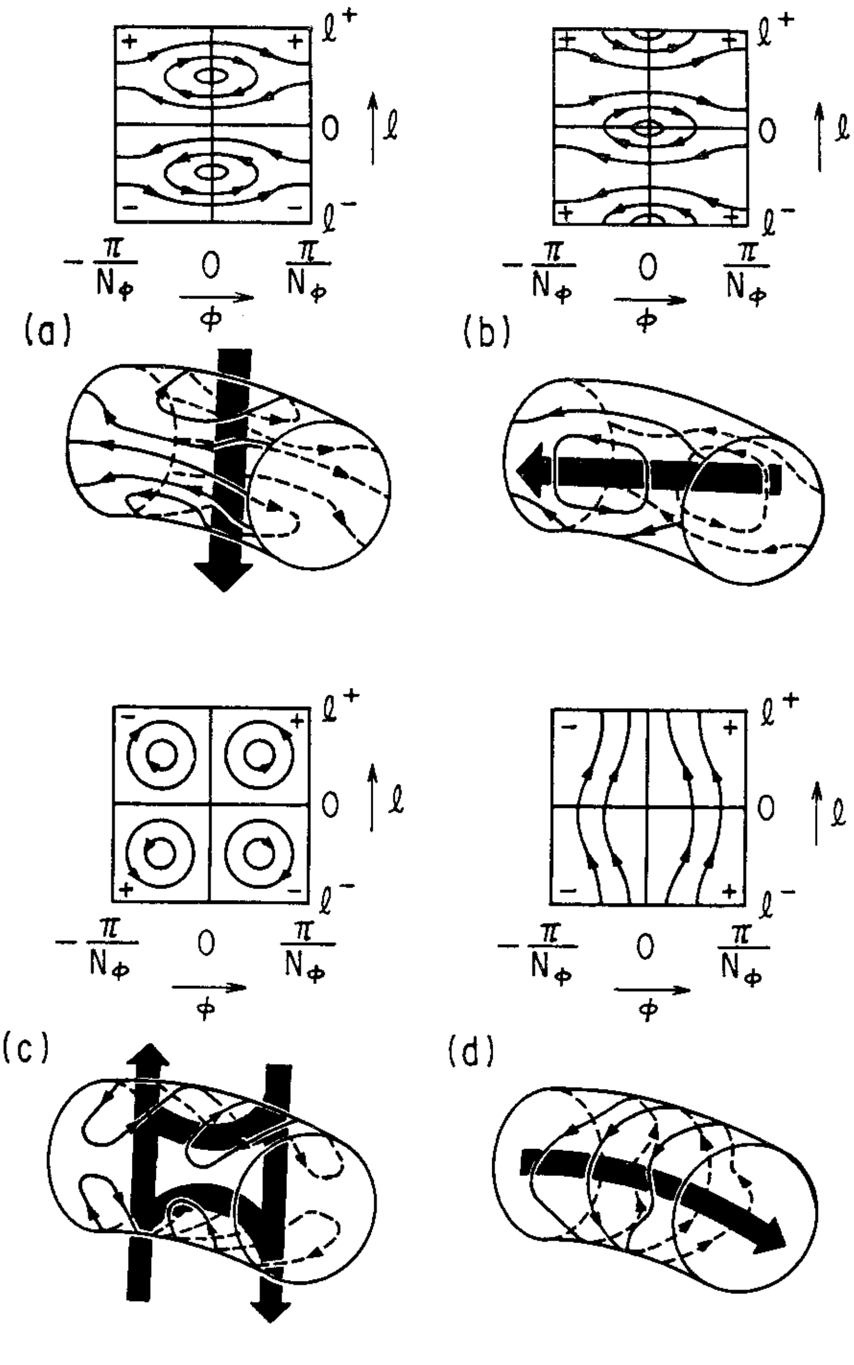

Fig. 10 Parity of eddy current (illustrated by a thin arrow) and structure of its corresponding magnetic field (a thick arrow). The magnetic fields of subfigures (a) and (b), only of which are discussed in the paper, couple with the axisymmetric poloidal field coils. The eddy current with a parity of subfigure (d) can interact with toroidal field coils.

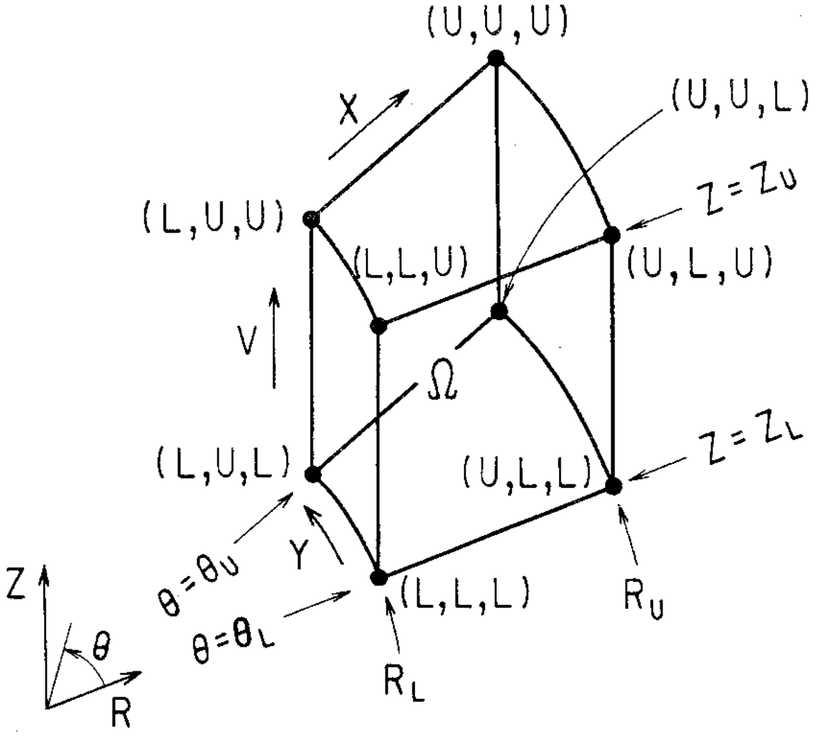

Fig. 11 Box-shaped finite element Ω in the cylindrical coordinate (R,θ,Z). X, Y and V are the respective local coordinate in the finite element. Eight node indices are given by (σ,μ,τ) (σ,μ,τ = L, U).

Appendix A. Quadruple integral including a singular point

Here, we will carry out the following integrals including singular points

$$S_1 = \int_0^a\int_0^b\int_0^a\int_0^b \frac{x_1 x_2}{r}\, dx_1 dy_1 dx_2 dy_2 \ , \tag{A.1}$$

$$S_2 = \int_0^a\int_0^b\int_0^a\int_0^b \frac{x_1}{r}\, dx_1 dy_1 dx_2 dy_2 \ . \tag{A.2}$$

It is well known that if one can find an analytic function $g(x_1, x_2, y_1, y_2)$, which satisfies the equation

$$f(x_1, x_2, y_1, y_2) = \frac{\partial^2 g}{\partial x_1 \partial x_2} + \frac{\partial^2 g}{\partial y_1 \partial y_2} \ , \tag{A.3}$$

then, the following relation holds

$$\iint_{\Omega_1}\iint_{\Omega_2} f dx_1 dy_1 dx_2 dy_2 = \oint_{\Omega_1}\oint_{\Omega_2} g\, d\ell_1 \cdot d\ell_2 \ . \tag{A.4}$$

Where, the right-hand side of Eq. (A.4) is a double line integral, and $d\ell_1 \cdot d\ell_2 = dx_1 dx_2 + dy_1 dy_2$. The analytic functions are obviously

$$g_1 = -\frac{1}{15}\{4(x_1^2 + x_2^2) + (y_2 - y_1)^2 + 7x_1 x_2\} \times \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{for Eq. (A.1)} \ , \tag{A.5}$$

and

$$g_2 = -\frac{1}{3}(2x_1 + x_2)\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{for Eq. (A.2)} \ . \tag{A.6}$$

Using these functions g_1 and g_2, the double line integral given by the right-hand side of Eq. (A.4) can be straightforwordly performed and the respective results are expressed as:

$$S_1 = \frac{2}{15} a^5 + \frac{1}{3} a^2 b^3 + \frac{1}{45} b^5 - \left(\frac{2}{15} a^4 + \frac{29}{90} a^2 b^2 + \frac{1}{45} b^4\right) \sqrt{a^2 + b^2} + \frac{1}{2} a^4 b \sinh^{-1} \frac{b}{a} + \frac{2}{3} a^3 b^2 \sinh^{-1} \frac{a}{b} \quad , \tag{A.7}$$

$$S_2 = -a^4 - ab^3 + (a^3 + ab^2) \sqrt{a^2 + b^2} - 3a^3 b \sinh^{-1} \frac{b}{a} - 3a^2 b^2 \sinh^{-1} \frac{a}{b} \quad . \tag{A.8}$$

While, if the finite element Ω_1 overlaps Ω_2, the double area integral including the singular point $\rho_{12} = 0$ is expressed in terms of S_1 and S_2, which are given by Eqs. (A.7) and (A.8). In this case, Eq. (56) can be approximated as follows:

$$A^1_{\nu\mu} = \frac{\bar{r}_1 \bar{r}_2}{a^2 b^2} \int_0^a \int_0^b \int_0^a \int_0^b \frac{P(\nu)P(\mu)}{\rho_{12}} dX_\nu dY_\nu dX_\mu dY_\mu \quad ,$$

$$A^2_{\nu\mu} = A^3_{\nu\mu} = 0 \quad ,$$

$$A^4_{\nu\mu} = \left(\frac{dr_1}{d\ell_1}\frac{dr_2}{d\ell_2} + \frac{dZ_1}{d\ell_1}\frac{dZ_2}{d\ell_2}\right) \times \frac{1}{a^2 b^2} \int_0^a \int_0^b \int_0^a \int_0^b \frac{Q(\nu)Q(\mu)}{\rho_{12}} dX_\nu dY_\nu dX_\mu dY_\mu \quad . \tag{A.9}$$

By putting Eqs. (A.7) and (A.8) for the double area integrals in Eq. (A.9), $A^j_{\nu\mu}$ ($j = 1, 2, 3, 4$) are evaluated immediately.

Appendix B. Mode reduction and "structural controllability"

Here is shown the investigation of the mode elimination technique with the help of "structural controllability" of a time-invariant system[21], which is familiar in the field of linear control theory. For the purpose, we use the following representation called an intermediate standard form, descriptor form, or semistate equation as:

$$K \dot{x} + A x = B u \quad , \tag{B.1}$$

Here (K,A,B) denotes a "structurized system". Now, let classify x as x_1, x_2 and x_3. Where, x_1 indicates an equivalent class of controllable state variable coupled with x_2 but x_3 and x_2 is an equivalent class of state variable coupled with x_1 and decoupled with the input. Lastly, x_3 indicates an equivalent class of uncontrollable state variable decoupled with x_1 and x_2. Then, a directed graph G(K,A,B) can be drawn in Fig. B.1. One of the necessary condition of (K,A,B) structural controllability is

(a) (K,A,B) is irreducible.

or

(b) Node x_i is accessible in the directed graph G(K,A,B).

In our case, the equivalent class x_3 is structurally uncontrollable. Moreover, since all of the members of x_3 are asymptotically stable, therefore, x_3 may be eliminated from the entire system if the initial states of x_3 are zero.

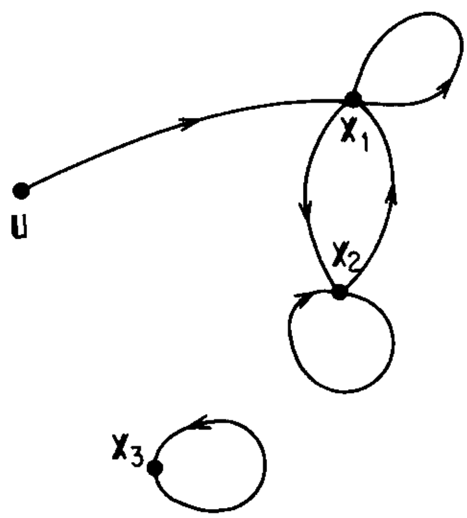

Fig. B.1 Directed graph G(K,A,B)

Appendix C. Approximate method for the eddy current problem with time-variable parameters

The paper has described the eddy current problem with time-invariant parameters. In the appendix, the procedure based on the finite element circuit method is extended to the eddy current problem, where the electrical resistance of voltage-controlled coil system is time-variable and the variation is also known beforehand.

Let divide a time into a set of finite time intervale $[\tau_k, \tau_{k+1}]$ $(k = 0, 1, \dots)$ as shown in Fig. C.1. Here, the change of coil resistance R(k) in a time interval $[\tau_k, \tau_{k+1}]$ can be well approximated to be constant. Consequently, the procedure mentioned in the paper for the time-invariant parameter problem is available to the individual time interval, where the solution of eddy current is given by Eq. (8). It is obvious that the initial value $\xi_\ell{}^k(t_i)$ in the time interval $[\tau_k, \tau_{k+1}]$ is the terminal value $\xi_\ell{}^{k-1}(t_f)$ in the previous time interval $[\tau_{k-1}, \tau_k]$. However, one must notice that the basis vectors of eigen modes of eddy current in respective time intervals are different each other because the coil resistance changes as time interval. Now, we represent a linear transformation $Q_k \in R^{(k^*+N_{coil}) \times (K^*+N_{coil})}$ of basis vectors from the time intervals $[\tau_{k-1}, \tau_k]$ to $[\tau_k, \tau_{k+1}]$. Let $\boldsymbol{\xi}^{k-1}(t_f)$ and $\boldsymbol{\xi}^k(t_i) \in R^{K^*+N_{coil}}$ be the terminal vector of eddy current in the time interval $[\tau_{k-1}, \tau_k]$ and the initial vector in the time interval $[\tau_k, \tau_{k+1}]$ each of which is respectively defined in terms of the basis of each time interval. Then, $\boldsymbol{\xi}^k(t_i)$ is represented

$$\boldsymbol{\xi}^k(t_i) = Q_k\ \boldsymbol{\xi}^{k-1}(t_f) \quad . \tag{C.1}$$

For more explanation, consider the circuit equation (88). By solving those respective eigenvalue problems for the time intervals $[\tau_{k-1}, \tau_k]$ and $[\tau_k, \tau_{k+1}]$, the linear transformation Q_k can be given as:

$$Q_k = \bar{\Phi}_k{}^T\ (R(k)/R(k-1))^{\frac{1}{2}}\ \bar{\Phi}_{k-1} \ . \tag{C.2}$$

Where, $\bar{\Phi}_k$ denotes the modal matrix of the eigenvalue problem (92) corresponding to the time interval $[\tau_k, \tau_{k+1}]$, and R(k) is the constant resistance matrix (90-b) of the time interval $[\tau_k, \tau_{k+1}]$. By successively using Eq. (C.2) for time intervals, we can simulate the eddy current in the multi-torus system including the voltage-controlled coil system with time-variable resistance.

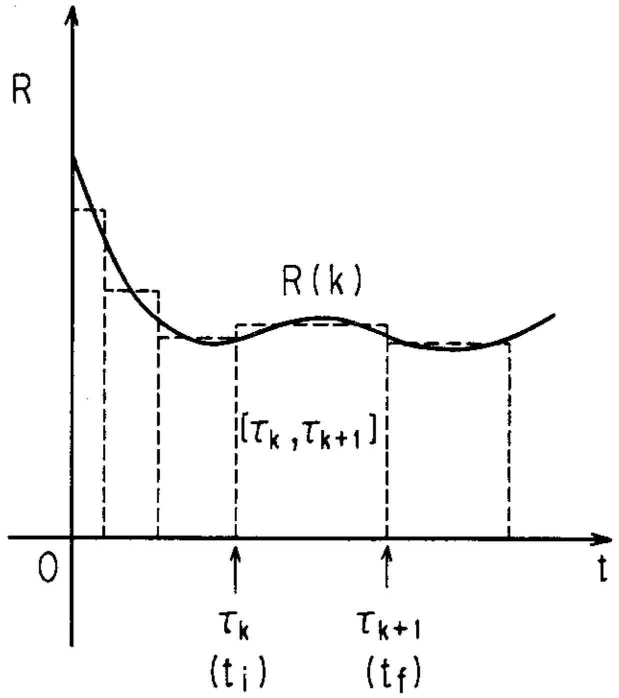

Fig. C.1 Division of a time into a set of finite time intervals. The continuous change of the electric resistivity is approximated by the descretized ones.