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A RIGOROUS ALGORITHM FOR SOLVING THE INVERSE KINEMATICS OF A MANIPULATOR ARM

July 1986

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- A Rigorous Algorithm for Solving the Inverse Kinematics of a Manipulator Arm -

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In this report, the kinematics of an articulated manipulator are studied, and an algorithm, based on a polynomial expression, is proposed which allows exact solutions of the inverse problem for a type of manipulator structure.

Compared to the traditional methods, the present approach permits to find all feasible solutions of the inverse problem of the manipulator for a specified location and orientation.

The algorithm for the solution was implemented in the computer code ARM2 for a six degree-of-freedom manipulator.
The results of computer simulation of the direct and inverse kinematics showed that numerical solutions were sufficiently reliable.
This approach will be applicable to other types of manipulator configurations with rotary and revolutionary joints.

Keywords: Articulated Manipulator, Kinematics, Computer Code ARM2, Arm Configulations

マニプレータ・アームの逆運動学解法の厳密なアルゴリズム

日本原子力研究所東海研究所原子炉工学部
佐々木　忍

（1986年7月7日受理）

本報告では，多関節型マニプレータの運動学を取り扱っている。特にマニプレータ関節構造の一タイプに対して，その逆問題を多項式表示に基づき厳密に解くアルゴリズムが提案されている。通常の手法に比べ，ここに紹介した方法は，指定した位置・方向に対するマニプレータ逆運動学の可能解を全て与えることができる。6－自由度マニプレータに対する解法のアルゴリズムは，計算コードARM 2の中に組み込んだ。運動学及び逆運動学の計算機シミュレーションの結果により，数値解が十分に信頼できることが立証された。本手法は，回転・旋回ジョイントをもつ他のマニプレータにも適用可能であろう。

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1. Introduction

The kinematics of articulated robot manipulators is concerned with the geometrical relationships between joint co-ordinates and the position/orientation in the work space. In planning a movement of a manipulator, we are primarily interested in the cartesian position of the end-effector in the work space, which is called the cartesian space of the manipulator. In order to place the end of the manipulator at a given position in the cartesian space, we must solve for the corresponding joint angles.

The relationships between manipulator joint angles, called joint space, and the cartesian space are usually complicated due to a transcendental functions involved.

Mathematical transformation from the joint co-ordinates into the cartesian space co-ordinates, called " the direct(or forward) problem", is easily obtained by successive multiplication of homogeneous transformation matrices. On the other hand, mathematical description of the joint co-ordinates in terms of the cartesian space co-ordinates called " the inverse problem ", is generally difficult and can be derived only for the special manipulator configurations. In reality, there exist many manipulators for which explicit analytical solutions of the inverse kinematics cannot be derived.
Some cases arise in which a somewhat intuitive idea of geometry leads to a promising clue for the inverse problem analysis.[(2)~(4)]

So far, a linearization and iteration technique for solving the inverse problem has been used from the practical standpoint.

In this iterative procedure, however, substantial drawbacks inherent in the linearization approximation are pointed out such as a high dependency of the solutions on the initial guess values and singularity of the Jacobian matrix.(5)

In order to obtain exact solutions, therefore, we have previously introduced a new approach to the inverse problem for a certain type of manipulator arm as a special case.(6),(7)
Here, we also derive in a similar manner a rigorous solution procedure for a different configuration of manipulator with six DOF(Degree-of-Freedom). The foremost idea is to derive a polynomial with a single variable from the kinematic relationships describing a linkage mechanism and to solve it. Thus, the proposed polynomial model plays an important role to determine solutions and, in that respect, requires precise calculation of that equation. Once the desired roots are found, individual articulated variables are straightforwardly determined from the relations between joint angles.

Admitting that this method of solutions does not seem easy to generalize for a variety of manipulator configurations, the analytical model proposed is indicative of a constructive step towards generalization for a certain class of six-link manipulators.
Furthermore, a computer code (ARM2) to obtain a complete transformation between the cartesian and joint co-ordinates for the current manipulator studied is presented. For later use, an example of test runs is demonstrated in Appendix.

In the next section, we present a mathematical background to derive the present algorithm.

2. Model Description

2.1 Kinematic Equations

In this section, a new approach for solving the inverse kinematic equations is presented in detail. First, we explain the co-ordinate transformations necessary for the kinematics analysis of a manipulator.

For an arm having n degrees of freedom, the cartesian co-ordinates of position and orientation at the final point of the open link (i.e., manipulator end-point) are represented in terms of the non-commutative product of n homogeneous transformation matrices A.

This method devised by Denavit & Hartenberg is widely used as a convenient approach to derive the kinematic equations.[1],[2]
In the case of a six-link manipulator, it can be expressed as follows.

$$T_6 = \prod_{i=1}^{6} A_i$$

In order to determine this matrix T_6 from the given arm configuration, six geometrical co-ordinate transformations are required. Figure 1 shows the schematic view of the new manipulator to be studied here. Before specifying A_i matrices and T_6 matrix, the following assumptions and notations are made.

(1) The connection between links (joints) has only one degree of freedom, either rotational or revolutionary.

(2) The motion of revolute or rotational joints about z-axis follows the " right screw rule ".

(3) A right handed system is used to represent the co-ordinate systems of each link as shown in Fig.1.

(4) The notations used throughout the derivation are :

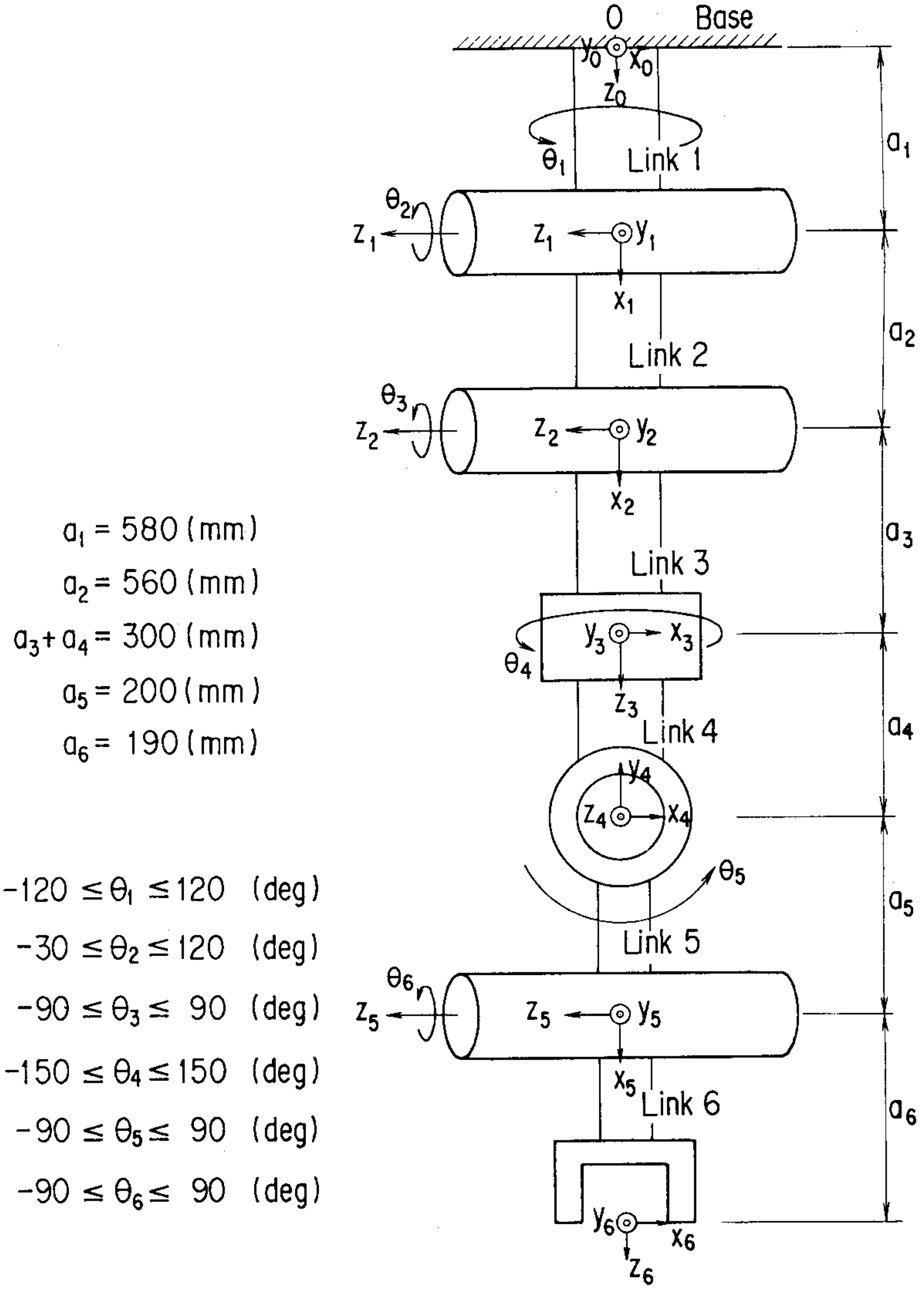

Fig.1 Link Co-ordinate Systems for a Six-link Robot Manipulator

(i) $c_i = \cos\theta_i$; $s_i = \sin\theta_i$; $c_{ij} = \cos(\theta_i + \theta_j)$;
$s_{ij} = \sin(\theta_i + \theta_j)$

(ii) $$\mathrm{Rot}(x,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}\ (p,q,r) = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we make the link description pertaining to the relation between two adjacent axes.

(1) A_1-matrix

Consider a right-handed orthonormal link co-ordinate system (x_0, y_0, z_0) defined at the supporting base, as shown in Fig. 2.

The first link is assumed connected to this base by the first joint.

We first rotate by θ_1 about the Z_0-axis, then make a translation by a_1 along the Z_0-direction up to the next joint of the manipulator. Further, producing a rotation of -90° about the y_0-axis, we establish a new co-ordinate system (x_1, y_1, z_1).

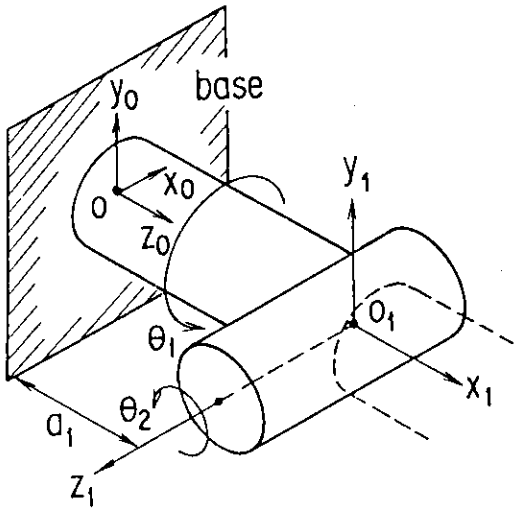

Fig.2 Schematics of Base to Link 1

A_1-matrix representing this operation (rotational and translational transformation) is described as follows:

$$A_1 = Rot(z_0,\theta_1)Trans(0,0,a_1)Rot(y_0,-\frac{\pi}{2})$$

$$= \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -s_1 & -c_1 & 0 \\ 0 & c_1 & -s_1 & 0 \\ 1 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In other words, this matrix exhibits the position and orientation of the coordinate frame (x_1, y_1, z_1) with respect to the base (or reference) co-ordinate system.

(2) A_2-matrix

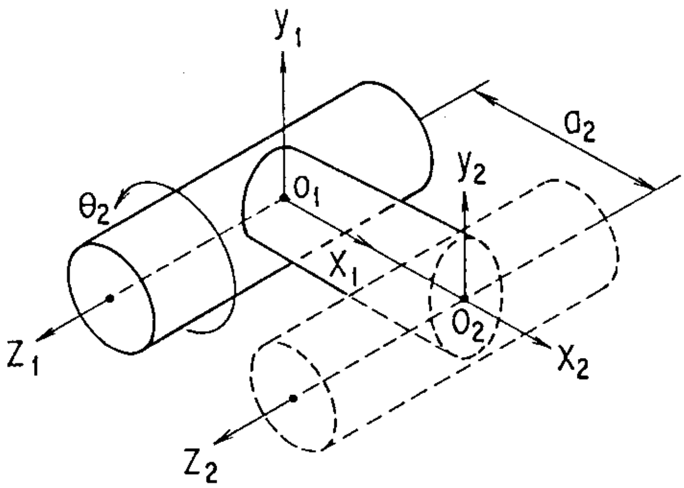

Fig.3 Schematics of Link 1 to Link 2

Referring to Fig. 3, we determine the second transformation matrix from the link 1 to the link 2 of the manipulator. It contains a rotation about the z_1-axis by θ_2 and a translation by a_2 along the x_1-axis. After this transformation, the old co-ordinate system is replaced by (x_2, y_2, z_2).

$$A_2 = \mathrm{Rot}(z_1,\theta_2)\mathrm{Trans}(a_2,0,0)$$

$$= \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) A_3-matrix

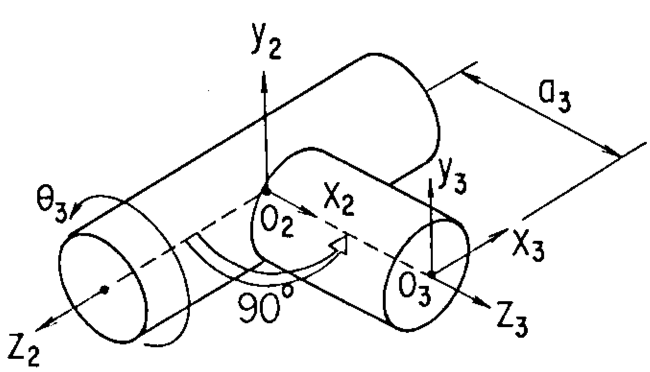

Fig.4 Schematics of Link 2 to Link 3

Now, we rotate by θ_3 about the z_2-axis, move by a_3 along the x_2-axis, then rotate by 90° in a clockwise direction about the y_2-axis so that the transformation is made from the link 2 to the link 3 of the manipulator.

Thereby, we establish the fourth co-ordinate frame (x_3, y_3, z_3) as indicated in Fig. 4.

$$A_3 = Rot(Z_2,\theta_3)Trans(a_3,0,0)Rot(y_2,\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -s_3 & c_3 & a_3c_3 \\ 0 & c_3 & s_3 & a_3s_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(4) A_4-matrix

For the motion from the link 3 to the link 4, we make a rotation of angle θ_4 about the z_3-axis, followed by a translation of a_4 along the z_3-direction, then the second rotation about the x_3-axis by -90°. With reference to Fig. 5, the transformation matrix A_4 up to the co-ordinate frame (x_4, y_4, z_4) assigned is:

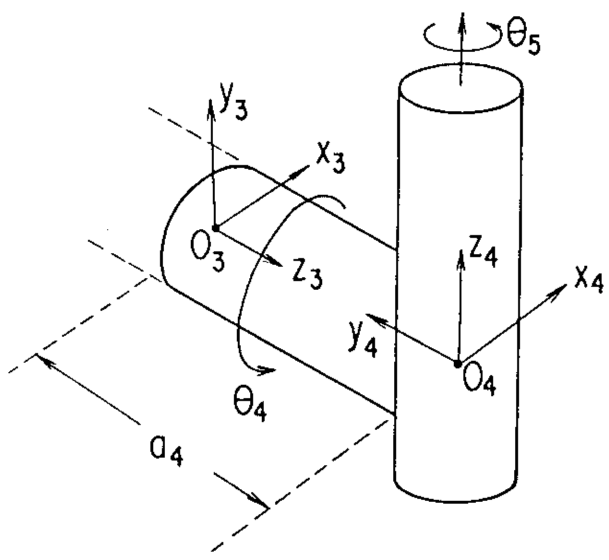

Fig.5 Schematics of Link 3 to Link 4

$$A_4 = Rot(z_3,\ \theta_4)Trans(o,o,a_4)Rot(x_3,-\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) A_5-matrix

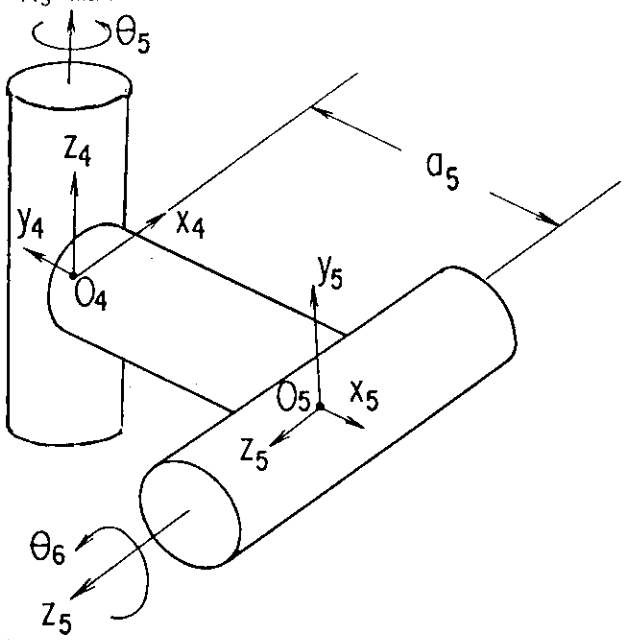

Fig.6 Schematics of Link 4 to Link 5

From the illustration of Fig. 6, the forward and backward motion of the wrist parts, are realized by the combination of: a rotation about the z_4-axis by the angle θ_5 and furthermore by -90°, a translation by a_5 along the x_4-direction, and a rotation about the x_4-axis by 90°. Hence, we obtain a new co-ordinate system $(x_5,\ y_5,\ z_5)$ and A_5-matrix.

$$A_5 = Rot(z_4,\theta_5)Rot(z_4,-\tfrac{\pi}{2})Trans(a_5,0,0)Rot(x_4,\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_5 & c_5 & 0 & 0 \\ -c_5 & s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_5 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_5 & 0 & -c_5 & a_5 s_5 \\ -c_5 & 0 & -s_5 & -a_5 c_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(6) A_6-matrix

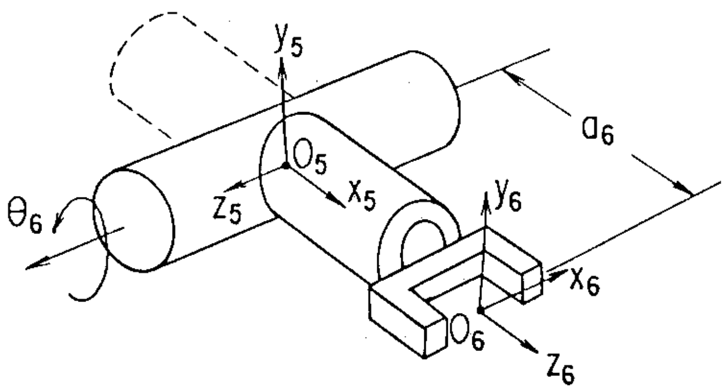

Fig.7 Schematics of Link 5 to Link 6

As shown in Fig. 7, the last link is free and includes the hand. A rotation about the z_5-axis by the angle θ_6, and a translation a_6 along the x_5-axis are made. In addition, for the convenience of later analysis, a new hand co-ordinate system is defined by the second rotation about the y_5-axis by 90° so that its orientation agrees with that of the reference co-ordinate system.

$$A_6 = Rot(z_5,\theta_6)Trans(a_6,0,0)Rot(y_5,\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_6 & -s_6 & 0 & a_6 c_6 \\ s_6 & c_6 & 0 & a_6 s_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -s_6 & c_6 & a_6 c_6 \\ 0 & c_6 & s_6 & a_6 s_6 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now that we specified the individual A-matrices for a serial link manipulator, six chain products of these homogeneous transformations are postmultiplied successively. In other words,

$$A_1A_2 = \begin{pmatrix} 0 & -s_1 & -c_1 & 0 \\ 0 & c_1 & -s_1 & 0 \\ 1 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -s_1s_2 & -s_1c_2 & -c_1 & -a_2s_1s_2 \\ s_2c_1 & c_1c_2 & -s_1 & a_2c_1s_2 \\ c_2 & -s_2 & 0 & a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1)$$

$$A_1A_2A_3 = \begin{pmatrix} -s_1s_2 & -s_1c_2 & -c_1 & -a_2s_1s_2 \\ s_2c_1 & c_1c_2 & -s_1 & a_2c_1s_2 \\ 0 & -s_2 & 0 & a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -s_3 & c_3 & a_3c_3 \\ 0 & c_3 & s_3 & a_3s_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & s_1s_2s_3-s_1c_2c_3 & -s_1s_2c_3-s_1c_2s_3 & -a_3s_1s_2c_3-a_3s_1c_2s_3-a_2s_1s_2 \\ s_1 & -s_2s_3c_1+c_1c_2c_3 & s_2c_1c_3+c_1c_2s_3 & a_3s_2c_1c_3+a_3c_1c_2s_3+a_2c_1s_2 \\ 0 & -c_2s_3-s_2c_3 & c_2c_3-s_2s_3 & a_3c_2c_3-a_3s_2s_3+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & -s_1c_{23} & -s_1s_{23} & -a_3s_1s_{23}-a_2s_1s_2 \\ s_1 & c_1c_{23} & c_1s_{23} & a_3c_1s_{23}+a_2c_1s_2 \\ 0 & -s_{23} & c_{23} & a_3c_{23}+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

$$A_1A_2A_3A_4 = \begin{pmatrix} c_1 & -s_1c_{23} & -s_1s_{23} & -a_3s_1s_{23}-a_2s_1s_2 \\ s_1 & c_1c_{23} & c_1s_{23} & a_3c_1s_{23}+a_2c_1s_2 \\ 0 & -s_{23} & c_{23} & a_3c_{23}+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1c_4-s_1s_4c_{23} & s_1s_{23} & -(c_1s_4+s_1c_4c_{23}) & -a_{34}s_1s_{23}-a_2s_1s_2 \\ s_1c_4+c_1s_4c_{23} & -c_1s_{23} & -s_1s_4+c_1c_4c_{23} & a_{34}c_1s_{23}+a_2c_1s_2 \\ -s_4s_{23} & -c_{23} & -c_4s_{23} & a_{34}c_{23}+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (3)$$

$A_1A_2A_3A_4A_5 =$

$$= \begin{pmatrix} C_1C_4-S_1S_4C_{23} & S_1S_{23} & -(C_1S_4+S_1C_4C_{23}) & -a_{34}S_1S_{23}-a_2S_1S_2 \\ S_1C_4+C_1S_4C_{23} & -C_1S_{23} & -S_1S_4+C_1C_4C_{23} & a_{34}C_1S_{23}+a_2C_1S_2 \\ -S_4S_{23} & -C_{23} & -C_4S_{23} & a_{34}C_{23}+a_2C_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_5 & 0 & -C_5 & a_5S_5 \\ -C_5 & 0 & -S_5 & -a_5C_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (C_1C_4-S_1S_4C_{23})S_5-S_1S_{23}C_5 & -(C_1S_4+S_1C_4C_{23}) & -C_5(C_1C_4-S_1S_4C_{23})-S_1S_5S_{23} & a_5S_5(C_1C_4-S_1S_4C_{23})-a_5C_5S_1S_{23}-a_{34}S_1S_{23}-a_2S_1S_2 \\ (S_1C_4+C_1S_4C_{23})S_5+C_1C_5S_{23} & -S_1S_4+C_1C_4C_{23} & -C_5(S_1C_4+C_1S_4C_{23})+S_5(C_1S_{23}) & a_5S_5(S_1C_4+C_1S_4C_{23})+a_5C_5C_1S_{23}+a_{34}C_1S_{23}+a_2C_1S_2 \\ -S_4S_5S_{23}+C_5C_{23} & -C_4S_{23} & S_4S_{23}C_5+C_{23}S_5 & -a_5S_5S_4S_{23}+a_5C_5C_{23}+a_{34}C_{23}+a_2C_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

$$T_6 = A_1A_2A_3A_4A_5A_6$$

$$= \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.a)$$

where

$$\sigma_{11} = C_5(C_1C_4-S_1S_4C_{23})+S_1S_5S_{23}$$

$$\sigma_{21} = C_5(S_1C_4+C_1S_4C_{23})-C_1S_5S_{23}$$

$$\sigma_{31} = -(S_4S_{23}C_5+C_{23}S_5)$$

$$\sigma_{12} = -\{(C_1C_4-S_1S_4C_{23})S_5-S_1S_{23}C_5\}S_6-(C_1S_4+S_1C_4C_{23})C_6$$

$$\sigma_{22} = -\{(S_1C_4+C_1S_4C_{23})S_5+C_1C_5S_{23}\}S_6+(-S_1S_4+C_1C_4C_{23})C_6$$

$$\sigma_{32} = S_6(S_4S_5S_{23}-C_5C_{23})-C_4S_{23}C_6$$

$$\sigma_{13} = \{(C_1C_4-S_1S_4C_{23})S_5-S_1S_{23}C_5\}C_6-(C_1S_4+S_1C_4C_{23})S_6$$

$$\sigma_{23} = \{(S_1C_4+C_1S_4C_{23})S_5+C_1C_5S_{23}\}C_6+(-S_1S_4+C_1C_4C_{23})S_6$$

$$\sigma_{33} = (-S_4S_5S_{23}+C_5C_{23})C_6-(C_4S_{23})S_6$$

$$\sigma_{14} = a_6C_6\{(C_1C_4-S_1S_4C_{23})S_5-S_1S_{23}C_5\}-a_6S_6(C_1S_4+S_1C_4C_{23})$$
$$+a_5S_5(C_1C_4-S_1S_4C_{23})-a_5C_5S_1S_{23}-a_{34}S_1S_{23}-a_2S_1S_2$$
$$\sigma_{24} = a_6C_6\{(S_1C_4+C_1S_4C_{23})S_5+C_1C_5S_{23}\}+a_6S_6\{-S_1S_4+C_1C_4C_{23}\}$$
$$+a_5S_5(S_1C_4+C_1S_4C_{23})+a_5C_5C_1S_{23}+a_{34}C_1S_{23}+a_2C_1S_2$$
$$\sigma_{34} = a_6C_6[-S_4S_5S_{23}+C_5C_{23}]-a_6S_6C_4S_{23}-a_5S_5S_4S_{23}+a_5C_5C_{23}$$
$$+a_{34}C_{23}+a_2C_2+a_1$$

From this representation, we can find the position and orientation of the end-point of a manipulator with reference to the base co-ordinate system.

2.2 Computational Algorithm

Following the fundamental computation of matrices in the preceding section, we proceed to the main subject of this report. The first part is devoted to derive a non-linear equation from kinematic representations. The second part is concerned with a determination of joint angle variables.

2.2.1 Derivation of an Algebraic Equation

Consider the matrix T_6 to be of the form:

$$T_6 = \begin{pmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5.b)$$

In Eq.(5.b), the left upper 3 × 3 matrix is a rotation matrix made with orthogonal unit vectors —— n, o, and a. n is the normal vector of the hand. Assuming a parallel-jaw hand, it is perpendicular to the fingers of the manipulator. o is orientation vector of the hand, which indicates the direction of the finger motion as the finger opens and closes. a is the approach vector of the hand, which indicates the direction normal to the palm of the hand. p is the position vector describing the location of the hand and is also referred to as a translation vector.

Equating the corresponding elements of the T_6 matrix in Eq.(5.a) and Eq.(5.b), we obtain kinematic expressions of the multi-joint arm shown in Fig. 1, where each component of T_6 in Eq.(5.b) is postulated to be known.

$$o_x = -\{C_1C_4S_5-S_1S_4S_5C_{23}-S_1S_{23}C_5\}S_6-(C_1S_4+S_1C_4C_{23})C_6 \quad (6)$$

$$o_y = -(S_1C_4S_5+C_1S_4S_5C_{23}+C_1C_5S_{23})S_6+(-S_1S_4+C_1C_4C_{23})C_6 \quad (7)$$

$$o_z = (S_4S_5S_{23}-C_5C_{23})S_6-C_4S_{23}C_6 \quad (8)$$

$$a_x = (C_1C_4S_5-S_1S_4S_5C_{23}-S_1S_{23}C_5)C_6-(C_1S_4+S_1C_4C_{23})S_6 \quad (9)$$

$$a_y = (S_1C_4S_5+C_1S_4S_5C_{23}+C_1C_5S_{23})C_6+(-S_1S_4+C_1C_4C_{23})S_6 \quad (10)$$

$$a_z = -(S_4S_5S_{23}-C_5C_{23})C_6-C_4S_{23}S_6 \quad (11)$$

$$p_x = a_6C_6(C_1C_4S_5-S_1S_4S_5C_{23}-S_1S_{23}C_5)-a_6S_6(C_1S_4+S_1C_4C_{23})$$
$$+a_5\{C_1C_4S_5-S_1S_4S_5C_{23}-S_1C_5S_{23}\}-(a_3+a_4)S_1S_{23}-a_2S_1S_2 \quad (12)$$

$$p_y = a_6C_6\{S_1C_4S_5+C_1S_4S_5C_{23}+C_1C_5S_{23})+a_6S_6(-S_1S_4+C_1C_4C_{23})$$
$$+a_5(S_1S_5C_4+C_1S_4S_5C_{23}+C_1C_5S_{23})+(a_3+a_4)C_1S_{23}+a_2C_1S_2 \quad (13)$$

$$p_z = a_6C_6(-S_4S_5S_{23}+C_5C_{23})-a_6S_6C_4S_{23}+a_5(C_5C_{23}-S_5S_4S_{23})$$
$$+(a_3+a_4)C_{23}+a_2C_2+a_1 \quad (14)$$

Here, each element of the first column of the T_6 matrix in Eqs.(5a) and (5b) is not included in a group of Eqs.(6) to (11), because it is uniquely determined from the orthonormal co-ordinate relation, that is, $n = o \times a$. Now, we introduce the following parameters in view of the apparent features of the equations (6) through (11).

$$A = C_1C_4S_5-S_1S_4S_5C_{23}-S_1S_{23}C_5 \quad (15)$$

$$B = C_1S_4+S_1C_4C_{23} \quad (16)$$

$$C = S_1C_4S_5+C_1S_4S_5C_{23}+C_1C_5S_{23} \quad (17)$$

$$D = -S_1S_4+C_1C_4C_{23} \quad (18)$$

$$E = S_4S_5S_{23}-C_5C_{23} \quad (19)$$

$$F = -C_4S_{23} \quad (20)$$

$$a_{34} = a_3+a_4$$

Consequently, the original equations (6) to (11) are simplified as follows.

$$-A\ s_6 - B\ c_6 = o_x \tag{21}$$

$$-C\ s_6 + D\ c_6 = o_y \tag{22}$$

$$E\ s_6 + F\ c_6 = o_z \tag{23}$$

$$A\ c_6 - B\ s_6 = a_x \tag{24}$$

$$C\ c_6 + D\ s_6 = a_y \tag{25}$$

$$-E\ c_6 + F\ s_6 = a_z \tag{26}$$

Similarly, the position vector in Eqs.(12), (13) and (14) is represented by:

$$p_x = a_6 c_6 A - a_6 s_6 B + a_5 A - a_{34} s_1 s_{23} - a_2 s_1 s_2 \tag{27}$$

$$p_y = a_6 c_6 C + a_6 s_6 D + a_5 C + a_{34} c_1 s_{23} + a_2 c_1 s_2 \tag{28}$$

$$p_z = -a_6 c_6 E + a_6 s_6 F - a_5 E + a_{34} c_{23} + a_2 c_2 + a_1 \tag{29}$$

As can be noticed from Eqs.(21) through (26), each parameter defined in Eqs.(15) to (20) can be represented by a trigonometric function of only c_6 and s_6. Namely,

$$A = a_x c_6 - o_x s_6 \tag{30}$$

$$B = -(a_x s_6 + o_x c_6) \tag{31}$$

$$C = a_y c_6 - o_y s_6 \tag{32}$$

$$D = o_y c_6 + a_y s_6 \tag{33}$$

$$E = o_z s_6 - a_z c_6 \tag{34}$$

$$F = a_z s_6 + o_z c_6 \tag{35}$$

From the Eqs.(27) and (28), we obtain

$$\begin{aligned} p_x c_1 + p_y s_1 &= a_6 c_6 (A\ c_1 + C\ s_1) + a_6 s_6 (-B\ c_1 + D\ s_1) + a_5 (A\ c_1 + C\ s_1) \\ &= a_6 \{c_1 (A\ c_6 - B\ s_6) + s_1 (C\ c_6 + D\ s_6)\} + a_5 (A\ c_1 + C\ s_1) \\ &= a_6 \{a_x c_1 + a_y s_1\} + a_5 (A\ c_1 + C\ s_1) \end{aligned} \tag{36}$$

On simplification, we have

$$(p_x-a_6a_x-a_5A)c_1 = (a_6a_y+a_5C-p_y)s_1 \tag{37}$$

that is, $$\tan\theta_1 = \frac{s_1}{c_1} = \frac{p_x-a_6a_x-a_5A}{a_6a_y+a_5C-p_y} = \frac{XX-a_5A}{a_5C-YY}$$

$$= \frac{XX-a_5(a_xc_6-o_xs_6)}{a_5(a_yc_6-o_ys_6)-YY} \tag{38}$$

where $$\begin{cases} XX = p_x-a_6a_x , \\ YY = p_y-a_6a_y , \end{cases} \tag{39}$$

Let $\tan\frac{\theta_6}{2} = t$. Then $c_6 = \frac{1-t^2}{1+t^2}$, $s_6 = \frac{2t}{1+t^2}$ and $\tan\theta_6 = \frac{2t}{1-t^2}$.

Therefore, Eq.(38) can be rewritten as follows.

$$\tan\theta_1 = \frac{a_5o_x\left(\frac{2t}{1+t^2}\right) - a_5a_x\left(\frac{1-t^2}{1+t^2}\right) + XX}{a_5a_y\left(\frac{1-t^2}{1+t^2}\right) - a_5o_y\left(\frac{2t}{1+t^2}\right) - YY}$$

$$= \frac{x_n + 2a_5o_xt + x_pt^2}{y_n - 2a_5o_yt + y_pt^2} \tag{40}$$

where $$\left.\begin{aligned} x_p &= a_5a_x + XX \\ x_n &= -a_5a_x + XX \\ y_p &= -(a_5a_y + YY) \\ y_n &= a_5a_y - YY \end{aligned}\right\} \tag{41}$$

From Eq.(29) we have

$$p_z - a_6a_z - a_1 = a_5E + a_{34}c_{23} + a_2c_2$$

$$= a_5(a_zc_6 - o_zs_6) + a_{34}c_{23} + a_2c_2$$

$$a_5c_6(a_z - o_z\tan\theta_6) - zz + a_{34}c_{23} = -a_2c_2 \tag{42}$$

where $zz = p_z - a_6 a_z - a_1.$

Furthermore, Eq.(27) × $(-s_1)$ + Eq.(28) × c_1 reduces to

$$a_5 c_6 \{a_y c_1 - a_x s_1 + (s_1 o_x - o_y c_1)\tan\theta_6\} - \{-p_x s_1 + p_y c_1 + a_6(a_x s_1 - a_y c_1)\} + a_{34} s_{23} = -a_2 s_2 \qquad (43)$$

As described above, we obtained two principal equations (42) and (43) from kinematic equations.

From here on, we will make further simplification so that these equations may be unified in one single mathematical relationship.

By putting $a_5 c_6(a_z - o_z \tan\theta_6) - zz = \psi$, (44)

and
$$a_5 c_6 \{a_y c_1 - a_x s_1 + (s_1 o_x - o_y c_1)\tan\theta_6\} - \{-p_x s_1 + p_y c_1 + a_6(a_x s_1 - a_y c_1)\} = \eta \qquad (45)$$

Eqs.(42) and (43) are rewritten as:

$$\psi + a_{34} c_{23} = -a_2 c_2 \qquad (46)$$

$$\eta + a_{34} s_{23} = -a_2 s_2 \qquad (47)$$

Adding the square of Eq.(46) and Eq.(47), we get the following simple form.

$$\psi^2 + \eta^2 - a = -2a_{34}(\psi c_{23} + \eta s_{23}) \qquad (48)$$

where $a = a_2{}^2 - a_{34}{}^2$

Calculation of η

Now, we derive an expression of η using the Eqs.(45) and (40).

$$\eta = a_5 c_6 \{a_y c_1 - a_x s_1 + (s_1 o_x - o_y c_1)\tan\theta_6\}$$

$$- \{-p_x s_1 + p_y c_1 + a_6(a_x s_1 - a_y c_1)\}$$

$$= a_5 c_6 c_1 \{a_y - a_x \tan\theta_1 + (o_x \tan\theta_1 - o_y)\tan\theta_6\}$$

$$+ c_1\{p_x \tan\theta_1 - p_y - a_6(a_x \tan\theta_1 - a_y)\}$$

$$= a_5 c_6 c_1 \{a_y - o_y \tan\theta_6 + (o_x \tan\theta_6 - a_x)\tan\theta_1\}$$

$$+ c_1\{(p_x - a_6 a_x)\tan\theta_1 - (p_y - a_6 a_y)\}$$

$$= a_5 c_6 c_1 \{(a_y - o_y \tan\theta_6) + (o_x \tan\theta_6 - a_x)\tan\theta_1\}$$

$$+ c_1(XX\tan\theta_1 - YY)$$

$$= a_5 c_6 c_1 \{a_y - o_y \frac{2t}{1-t^2} + (o_x \frac{2t}{1-t^2} - a_x)\tan\theta_1\}$$

$$+ c_1(XX\tan\theta_1 - YY)$$

$$= a_5 c_1 \left(\frac{1-t^2}{1+t^2}\right)\left[\frac{a_y(1-t^2) - 2o_y t + \{2o_x t - a_x(1-t^2)\}\tan\theta_1}{1-t^2}\right]$$

$$+ c_1(XX\tan\theta_1 - YY)$$

$$= a_5 c_1 \frac{1}{1+t^2}[a_y(1-t^2) - 2o_y t + (a_x t^2 + 2o_x t - a_x)\tan\theta_1]$$

$$+ c_1(XX\tan\theta_1 - YY)$$

$$\eta = a_5 c_1 \left(\frac{1}{1+t^2}\right)[(a_y - 2o_y t - a_y t^2)(y_n - 2a_5 o_y t + y_p t^2)$$

$$+ (a_x t^2 + 2o_x t - a_x)(x_n + 2a_5 o_x t + x_p t^2)]\frac{1}{y_n - 2a_5 o_y t + y_p t^2}$$

$$+ c_1\left[\frac{XX(x_n + 2a_5 o_x t + x_p t^2) - YY(y_n - 2a_5 o_y t + y_p t^2)}{y_n - 2a_5 o_y t + y_p t^2}\right]$$

$$= \frac{a_5 c_1}{1+t^2} \cdot \frac{1}{y_n - 2a_5 o_y t + y_p t^2} [a_y y_n - 2a_5 a_y o_y t + a_y y_p t^2$$

$$- 2o_y y_n t + 4a_5 o_y{}^2 t^2 - 2o_y y_p t^3 - a_y y_n t^2 + 2a_5 a_y o_y t^3 - a_y y_p t^4$$

$$+ a_x x_n t^2 + 2a_5 a_x o_x t^3 + a_x x_p t^4 + 2x_n o_x t + 4a_5 o_x{}^2 t^2 + 2x_p o_x t^3$$

$$- a_x x_n - 2a_5 a_x o_x t - a_x x_p t^2] + \left(\frac{c_1}{y_n - 2a_5 o_y t + y_p t^2}\right) [XX \cdot x_n$$

$$+ 2a_5 o_x \cdot XXt + XX \cdot x_p t^2 - YYy_n + 2a_5 o_y YYt - YYy_p t^2]$$

$$= \frac{c_1}{(1+t^2)(y_n - 2a_5 o_y t + y_p t^2)} \left[\sum_{i=0}^{4} b_i t^i + \sum_{i=5}^{9} b_i t^{i-5} \right]$$

$$= \frac{c_1}{(1+t^2)(y_n - 2a_5 o_y t + y_p t^2)} \sum_{i=0}^{4} \overline{cc}_i \ t^i \qquad (49)$$

(See the Appendix 4 as to the values of respective coefficients in a power series.)

Calculation of ψ

Based on Eq.(44), we will try to express ψ as a function of variable t.

$$\psi = a_5 c_6 (a_z - o_z \tan\theta_6) - zz$$

$$= a_5 \left(\frac{1-t^2}{1+t^2}\right) \left(a_z - o_z \cdot \frac{2t}{1-t^2}\right) - zz$$

$$= \frac{1}{1+t^2} [a_5(a_z - 2o_z t - a_z t^2) - zz(1+t^2)]$$

$$= \frac{1}{1+t^2} [(a_5 a_z - zz) - 2a_5 o_z t - (a_5 a_z + zz)t^2]$$

$$= \frac{1}{1+t} \sum_{i=1}^{3} do_i \ t^{i-1} \qquad (50)$$

Calculation of $c_1{}^2$

Using the expression of $\tan\theta_1$ given by Eq.(40), it follows that

$$\frac{1}{c_1^2} = 1 + \tan^2\theta_1 = 1 + \left(\frac{x_n + 2a_5o_x t + x_p t^2}{y_n - 2a_5o_y t + y_p t^2}\right)^2$$

thus,

$$c_1{}^2 = \frac{(y_n - 2a_5o_y t + y_p t^2)^2}{(x_n + 2a_5o_x t + x_p t^2)^2 + (y_n - 2a_5o_y t + y_p t^2)^2}$$

$$= \frac{(y_n - 2a_5o_y t + y_p t^2)^2}{\sum_{i=0}^{4} f_i t^i} \quad . \tag{51}$$

Calculation of ψ^2

Taking the square of Eq.(50) results in

$$\psi^2 = \frac{1}{(1+t^2)^2}\left(\sum_{i=1}^{3} do_i \, t^{i-1}\right)^2 = \frac{1}{(1+t^2)^2}\sum_{i=0}^{4} d_i \, t^i \quad . \tag{52}$$

Calculation of $(\eta^2 + \psi^2)$

Referring to Eqs.(50) and (52), the square sum of η and ψ is:

$$\eta^2 + \psi^2 = \frac{c_1^2}{(1+t^2)^2} \cdot \frac{\left(\sum_{i=0}^{4} \overline{cc}_i \, t^i\right)^2}{(y_n - 2a_5o_y t + y_p t^2)^2} + \frac{1}{(1+t^2)^2}\sum_{i=0}^{4} d_i \, t^i$$

$$= \frac{1}{(1+t^2)^2} \cdot \frac{\cancel{(y_n - 2a_5o_y t + y_p t^2)^2}}{\sum_{i=0}^{4} f_i \, t^i} \cdot \frac{\left(\sum_{i=0}^{4} \overline{cc}_i \, t^i\right)^2}{\cancel{(y_n - 2a_5o_y t + y_p t^2)^2}}$$

$$+ \frac{1}{(1+t^2)^2}\sum_{i=0}^{4} d_i \, t^i$$

$$= \frac{1}{(1+t^2)^2} \left\{ \frac{\left(\sum_{i=0}^{4} \overline{cc}_i \; t^i \right)^2}{\sum_{i=0}^{4} f_i \; t^i} + \sum_{i=0}^{4} d_i \; t^i \right\}$$

$$= \frac{1}{(1+t^2)^2} \cdot \frac{1}{\sum_{i=0}^{4} f_i \; t^i} \left\{ \sum_{i=0}^{8} e_i \; t^i + \left(\sum_{i=0}^{4} d_i \; t^i \right) \left(\sum_{i=0}^{4} f_i \; t^i \right) \right\}$$

$$= \frac{1}{(1+t^2)^2} \cdot \frac{1}{\sum_{i=0}^{4} f_i \; t^i} \left\{ \sum_{i=0}^{8} e_i \; t^i + \sum_{i=0}^{8} g_i \; t^i \right\} . \tag{53}$$

Thus,

$$\eta^2 + \psi^2 - a = \frac{1}{(1+t^2)^2 \sum_{i=0}^{4} f_i \; t^i} \left\{ \sum_{i=0}^{8} e_i \; t_i + \sum_{i=0}^{8} g_i \; t^i - a \sum_{i=0}^{8} h_i \; t^i \right\}$$

$$= \frac{1}{(1+t^2)^2 \sum_{i=0}^{4} f_i \; t^i} \sum_{i=0}^{8} J_i \; t^i \tag{54}$$

where $\quad (1 + t^2)^2 \sum_{i=0}^{8} f_i \; t^i = \sum_{i=0}^{8} h_i \; t^i$

In this way, we can express $(\eta^2 + \psi^2 - a)$ as a rational function of t.
Let us return to the original equation (48)

$$\eta^2 + \psi^2 - a = -2a_{34}(\psi c_{23} + \eta s_{23}).$$

After taking the square of both sides, and putting $\tan\theta_{23} = k$ in this equation, we obtain

$$(\psi^2 + \eta^2 - a)^2 = \frac{4a_{34}{}^2}{1+k^2} (\psi + k\eta)^2 \; . \tag{55}$$

As the next step, we consider the expression of $\tan\theta_{23}$ in terms of θ_6. Using Eqs.(16), (18) and (20) for s_4c_{23} and s_4s_{23}, we obtain

$$\left.\begin{aligned} s_4c_{23} &= (-a_x s_1 + a_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6 \\ s_4s_{23} &= (-a_z s_6 + o_z c_6) \end{aligned}\right\} . \tag{56}$$

Accordingly,

$$k = \tan\theta_{23} = \frac{-(a_z s_6 + o_z c_6)}{(-a_x s_1 + a_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6} \tag{57}$$

Using the expression of $\tan\theta_6$, Eq.(40) and Eq.(57), k_n can be represented as follows:

Calculation of k_n

$$k_n = \frac{c_1 \sum_{i=0}^{4} \overline{cc}_i\, t^i}{(1+t^2)(y_n - 2a_5o_y t + y_p t^2)} \cdot \frac{-(a_z\tan\theta_6 + o_z)}{(-a_x s_1 + a_y c_1)\tan\theta_6 + (o_y c_1 - o_x s_1)}$$

$$= \frac{c_1\left(\sum_{i=0}^{4} \overline{cc}_i\, t^i\right)}{(1+t^2)(y_n - 2a_5o_y t + y_p t^2)} \cdot \frac{-(o_z + 2a_z t - o_z t^2)(y_n - 2a_5o_y t + y_p t^2)}{c_1\left(\sum_{i=0}^{4} I_i\, t^i\right)}$$

$$= \frac{-\left(\sum_{i=0}^{4} \overline{cc}_i\, t^i\right)(o_z + 2a_z t - o_z t^2)}{(1+t^2)\left(\sum_{i=0}^{4} I_i\, t^i\right)} \tag{58}$$

Calculation of $c_{23}{}^2$

With the aid of Eq.(57), it is possible to represent $c_{23}{}^2$ as a function of t.

$$c_{23}^{2} = \frac{1}{1+\tan^{2}\theta_{23}} = \frac{1}{1+k^{2}} = \frac{1}{1+\left(\dfrac{a_z s_6 + o_z c_6}{(-a_x s_1 + a_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6}\right)^{2}}$$

$$= \frac{[(-a_x s_1 + a_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6]^2}{[(-a_x s_1 + a_y c_1)s_6 + (o_y c_1 - o_x s_1)c_6]^2 + (a_z s_6 + o_z c_6)^2}$$

$$= \frac{[(-a_x s_1 + a_y c_1)\tan\theta_6 + (o_y c_1 - o_x s_1)]^2}{[(-a_x s_1 + a_y c_1)\tan\theta_6 + (o_y c_1 - o_x s_1)]^2 + (o_z + a_z \tan\theta_6)^2}$$

$$= \frac{c_1^{2}[(-a_x \tan\theta_1 + a_y)\tan\theta_6 + (o_y - o_x \tan\theta_1)]^2}{c_1^{2}[(-a_x \tan\theta_1 + a_y)\tan\theta_6 + (o_y - o_x \tan\theta_1)]^2 + (o_z + a_z \tan\theta_6)^2}$$

$$= \frac{c_1^2\ [(-a_x \tan\theta_1 + a_y)\frac{2t}{1-t^2} + (o_y - o_x \tan\theta_1)]^2}{c_1^{2}[(-a_x \tan\theta_1 + a_y)\frac{2t}{1-t^2} + (o_y - o_x \tan\theta_1)]^2 + \left(o_z + a_z \frac{2t}{1-t^2}\right)^2}$$

$$= \frac{c_1^{2}[2t(-a_x \tan\theta_1 + a_y) + (o_y - o_x \tan\theta_1)(1-t^2)]^2}{c_1^{2}[2t(-a_x \tan\theta_1 + a_y) + (o_y - o_x \tan\theta_1)(1-t^2)]^2 + (o_z(1-t^2) + 2a_z t)^2} \tag{59}$$

$$c_{23}^{2} = \frac{c_1^{2}(\sum_{i=0}^{4} I_i\ t^i)^2}{c_1^{2}\ (\sum_{i=0}^{4} I_i\ t^i)^2 + (2a_z t + o_z - o_z t^2)^2 (y_n - 2a_5 o_y t + y_p t^2)^2}$$

$$= \frac{\left[(y_n - 2a_5 o_y t - y_p t^2)^2 \Big/ \sum_{i=0}^{4} f_i\ t^i\right]\left(\sum_{i=0}^{4} I_i\ t^i\right)^2}{\left[(y_n - 2a_5 o_y t + y_p t^2)^2 \Big/ \sum_{i=0}^{4} f_i\ t^i\right]\left(\sum_{i=0}^{4} I_i\ t^i\right)^2 + \Delta}$$

where, $\Delta = (2a_z t + o_z - o_z t^2)^2 (y_n - 2a_5 o_y t + y_p t^2)^2$

Thus,

$$c_{23}{}^{2} = \frac{\left(\sum_{i=0}^{4} I_i\ t^i\right)^2}{\left(\sum_{i=0}^{4} I_i\ t^i\right)^2 + (2a_z + o_z - o_z t^2)^2 \left(\sum_{i=0}^{4} f_i\ t^i\right)}$$

$$= \frac{\sum_{i=0}^{8} I_{2i}\ t^i}{\sum_{i=0}^{8} I_{2i}\ t^i + (2a_z t + o_z - o_z t^2)^2 \left(\sum_{i=0}^{4} f_i\ t^i\right)} \tag{60}$$

$$= \frac{\left(\sum_{i=0}^{8} I_{2i}\ t^i\right)}{\sum_{i=0}^{8} I_{2i}\ t^i + \left(\sum_{i=0}^{4} LL_i\ t^i\right)\left(\sum_{i=0}^{4} f_i\ t^i\right)}$$

$$= \frac{\left(\sum_{i=0}^{8} I_{2i}\ t^i\right)}{\sum_{i=0}^{8} I_{2i}\ t^i + \sum_{i=0}^{8} PL_i\ t^i}$$

$$= \frac{\sum_{i=0}^{8} I_{2i}\ t^i}{\sum_{i=0}^{8} L_i\ t^i} \tag{60'}$$

Calculation of $(k\eta + \psi)$

With reference to Eqs.(50) and (58), we have

$$k\eta + \psi = \frac{-(o_z + 2a_z t - o_z t^2)\left(\sum_{i=0}^{4} \overline{cc}_i\ t^i\right)}{(1 + t^2)\left(\sum_{i=0}^{4} I_i\ t^i\right)} + \frac{\sum_{i=0}^{3} do_i\ t^{i-1}}{1 + t^2}$$

$$= \frac{1}{(1+t^2)\left(\sum_{i=0}^{4} I_i\ t^i\right)} \left\{ -(o_z + 2a_z t - o_z t^2)\left(\sum_{i=0}^{4} \overline{cc}_i\ t^i\right) + \left(\sum_{i=1}^{3} do_i\ t^{i-1}\right)\left(\sum_{i=0}^{4} I_i\ t^i\right)\right\}$$

$$= \frac{1}{(1+t^2)\left(\sum_{i=0}^{4} I_i\ t^i\right)} \left\{ \sum_{i=0}^{6} K_i\ t^i + \sum_{i=0}^{6} II_i\ t^i \right\}$$

$$= \frac{1}{(1+t^2)\left(\sum_{i=0}^{4} I_i\ t_i\right)} \left\{ \sum_{i=0}^{6} PI_i\ t^i \right\} \tag{61}$$

Taking the square of both sides in Eq.(61) and substituting it into the right hand side of Eq.(55), then we have

$$4a_{34}{}^2 c_{23}{}^2 (kn + \psi)^2 = 4a_{34}{}^2 \frac{\cancel{\left(\sum_{i=0}^{8} I_{2i}\ t^i\right)}}{\sum_{i=0}^{8} L_i\ t^i} \cdot \frac{\left(\sum_{i=0}^{6} PI_i\ t^i\right)^2}{(1+t^2)^2 \cancel{\left(\sum_{i=0}^{8} I_{2i}\ t^i\right)}}$$

$$= 4a_{34}{}^2 \frac{\left(\sum_{i=0}^{6} PI_i\ t^i\right)^2}{\left(\sum_{i=0}^{8} L_i\ t^i\right) \cdot (1+t^2)^2}$$

$$= \frac{\left(\sum_{i=0}^{12} m_i\ t^i\right)}{\left(\sum_{i=0}^{8} L_i\ t^i\right)(1+t^2)^2} \tag{62}$$

On the other hand, using the result of Eq.(54), the left hand side of Eq.(55) gives:

$$(\eta^2 + \psi^2 - a)^2 = \frac{1}{(1+t^2)^4\left(\sum_{i=0}^{4} f_i\, t^i\right)^2}\left(\sum_{i=0}^{8} J_i\, t^i\right)^2$$

$$= \frac{1}{(1+t^2)^4\left(\sum_{i=0}^{4} f_i\, t^i\right)^2}\left(\sum_{i=0}^{16} N_i\, t^i\right) \tag{63}$$

When equating Eqs.(62) and (63), we obtain

$$\frac{\left(\sum_{i=0}^{12} m_i\, t^i\right)}{\left(\sum_{i=0}^{8} L_i\, t^i\right)(1+t^2)^2} = \frac{\left(\sum_{i=0}^{16} N_i\, t^i\right)}{(1+t^2)^4\left(\sum_{i=0}^{4} f_i\, t^i\right)^2} = \frac{\sum_{i=0}^{16} N_i\, t^i}{\left(\sum_{i=0}^{12} f_i\, t^i\right)(1+t^2)^2}$$

or

$$\frac{\left(\sum_{i=0}^{12} m_i\, t^i\right)}{\left(\sum_{i=0}^{8} L_i\, t^i\right)} = \frac{\left(\sum_{i=0}^{16} N_i\, t^i\right)}{(1+t^2)^2\left(\sum_{i=0}^{4} f_i\, t^i\right)^2} = \frac{\left(\sum_{i=0}^{16} N_i\, t^i\right)}{\left(\sum_{i=0}^{12} f_i\, t^i\right)} \tag{64}$$

Following the cancellation of the denominator in Eq.(64), we define:

$$\left(\sum_{i=0}^{12} m_i\, t^i\right)\left(\sum_{i=0}^{12} f_i\, t^i\right) = \sum_{i=0}^{24} p_i\, t^i \tag{65}$$

and

$$\left(\sum_{i=0}^{8} L_i\, t^i\right)\left(\sum_{i=0}^{16} N_i\, t^i\right) = \sum_{i=0}^{24} q_i\, t^i\,. \tag{66}$$

As a final description, we get

$$\sum_{i=0}^{24} (p_i - q_i)t^i = \sum_{i=0}^{24} r_i\, t^i = 0\,. \tag{67}$$

As derived above, the problem of finding the articular angles of a manipulator was reduced to a non-linear high-order algebraic equation $f(t) = \sum_{i=0}^{24} r_i\, t^i = 0$ with respect to $\tan(\theta_6/2)$.

In order to compute roots of this polynomial as exactly as possible, we use the Bairstow's method, which is outlined in Appendix. Since $|\theta_6| \leq 90°$ is specified as the operation domain of the finger tip, it is required for us to find the feasible solutions of f(t) on the interval $|t| \leq 1$.

2.2.2 Determination of Individual Joint Angles

(1) Calculation of θ_6

Once the desired solutions t are found from the algebraic equation (67), a joint angle θ_6 can easily be calculated.

That is, $\tan \frac{\theta_6}{2} = t$

thus, we have

$$\theta_6 = 2 \tan^{-1} t \,. \tag{68}$$

Corresponding trigonometric function is

$$s_6 = \sin\theta_6$$

$$c_6 = \cos\theta_6 \,.$$

Using these values, we can determine the parameters given by Eqs.(30) through (35).

(2) Calculation of θ_1

Let $X_1 = XX - a_5A$

and $Y_1 = a_5C - YY$,

then, Eq.(38) reduces to

$$X_1 \cdot c_1 = Y_1 \cdot s_1 \quad \text{(See Eq.(39) for XX and YY)}$$

Thus,

$$\theta_1 = \tan^{-1}\left(\frac{X_1}{Y_1}\right) \tag{69}$$

$$s_1 = \sin\theta_1$$

$$c_1 = \cos\theta_1$$

(3) Calculation of θ_{23}

From Eq.(48), we have

$$\psi^2 + \eta^2 - a = -2a_{34}(\psi c_{23} + \eta s_{23})$$
$$= -2a_{34}\sqrt{\psi^2 + \eta^2}\ \sin(\theta_{23} + \varepsilon) \ .$$

where $\varepsilon = \tan^{-1}(\psi/\eta), \quad a = a_2{}^2 - a_{34}{}^2$

Thus, we have

$$\sin(\theta_{23} + \varepsilon) = (\psi^2 + \eta^2 - a)/-2a_{34}\sqrt{\psi^2 + \eta^2}$$
$$\cos(\theta_{23} + \varepsilon) = \pm\sqrt{1 - \sin^2(\theta_{23} + \varepsilon)} \ .$$

Hence, $\tan(\theta_{23} + \varepsilon) = \dfrac{\sin(\theta_{23} + \varepsilon)}{\cos(\theta_{23} + \varepsilon)}$

or $$\theta_{23} = \tan^{-1}\left(\frac{\pm(\psi^2 + \eta^2 - a)/(-2a_{34}\sqrt{\psi^2 + \eta^2})}{\sqrt{1 - \{(\psi^2 + \eta^2 - a)/(-2a_{34}\sqrt{\psi^2 + \eta^2})\}^2}}\right) - \tan^{-1}\left(\frac{\psi}{\eta}\right) \tag{70}$$

$$s_{23} = \sin\theta_{23}$$
$$c_{23} = \cos\theta_{23}$$

(4) Calculation of θ_4

Making use of Eqs.(16) and (18), c_4 and s_4 are described as follows.

$$s_4 = Bc_1 - Ds_1$$
$$c_4 = (Bs_1 + Dc_1)/c_{23} \qquad (c_{23} \neq 0)$$

if $c_{23} = 0$, we use Eq.(20) to determine s_4.

or $c_4 = -F/s_{23}$

Thus, we have

$$\theta_4 = \tan^{-1}\left(\frac{s_4}{c_4}\right) \ . \tag{71}$$

(5) Calculation of θ_5

By Eq.(15) $\times$ c_1 - Eq.(17) $\times$ s_1, we obtain

$$s_5 = (Ac_1 + Cs_1)/c_4$$
$$= \{c_1(a_x c_6 - o_x s_6) + s_1(a_y c_6 - o_y s_6)\}/c_4 \quad (72)$$
$$(c_4 \neq 0)$$

On the other hand, Eq.(15) $\times$ s_1 + Eq.(17) $\times$ c_1 leads to

$$c_5 s_{23} + s_5 s_4 c_{23} = -As_1 + Cc_1 \; . \quad (73)$$

In the case of $c_4 = 0$, Eq.(73) $\times$ c_{23} - Eq.(19) $\times$ s_{23} results in

$$s_5 = \{c_{23}(Cc_1 - As_1) + E\, s_{23}\}/s_4 \; . \quad (74)$$

By making Eq.(73) $\times$ s_{23} + Eq.(17) $\times$ c_{23}, it holds that

$$c_5 = (Cc_1 - As_1)s_{23} - E\, c_{23} \; . \quad (75)$$

Hence,

$$\theta_5 = \tan^{-1}\left(\frac{s_5}{c_5}\right) \quad (76)$$

(6) Calculation of θ_2

Eqs.(42) and (43) give:

$$c_2 = \{p_z - a_6 a_z - a_1 - a_5(a_z c_6 - o_z s_6) - a_{34} c_{23}\}/a_2$$
$$s_2 = \{-p_x s_1 + p_y c_1 + a_6(a_x s_1 - a_y c_1) - a_5(a_y c_1 - a_x s_1)c_6$$
$$- a_5(s_1 o_x - o_y c_1)s_6 - a_{34} s_{23}\}/a_2$$

Therefore, we have

$$\theta_2 = \tan^{-1}\left(\frac{s_2}{c_2}\right) \; . \quad (77)$$

(7) Calculation of θ_3

From the calculation of θ_{23} and θ_2, we can obtain

$$\theta_3 = \theta_{23} - \theta_2 \quad (78)$$

3. Outline of Computer Program ARM2

ARM2 is a computer program which was developed to solve the inverse kinematics of a six-link robot manipulator for any spatial motion in the work space.

In this section, only overall framework of the code is briefed. The program is written in FORTRAN 77 using double precision version and needs approximately 140 KB core storage, two intermediate areas of disk storage, and printer output. This program is operable on the FACOM-380 computer (with EBCDIC code).

Two auxiliary computer programs used in conjunction with the ARM2 are the GRH51 program which produces plots of polynomial behaviors to examine the solutions of an algebraic equation, and the GRH52 which produces individual profiles of joint angles calculated in ARM2.

The running time of the ARM2 for kinematic analysis can vary depending primarily on the degree of detail of the problem to be solved. However, it was very short in our test runs.
(The case shown in Appendix is about 18 msec.)
By means of the implementation of the variable dimension method, the expansion of the program size is allowed.

As explained in the previous section, the proposed method represents a rigorous approach to solutions. Accordingly, much attention was directed to the computational techniques necessary to solve the algebraic equation as exactly as possible. As a numerical procedure of this kind, the Bairstow's (or Hitchcock's) method was used.

For the equation derived from the kinematic relationships, however, individual real coefficients were, at large, of an extremely complicated form. On close examination of these values, they are too large, in some cases, and too small, in other cases, depending on the given position and orientation of the manipulator.

In order to cope with such situations, we first take the absolute values of respective coefficients of polynomial and assume that C be the logarithm of their geometric mean. That is,

$$C = \frac{1}{n} \log_{10} |a_0||a_1| \cdots |a_n|$$

where n is a degree of polynomial f(x).

If $|a_i| = 0$, we set C = C + 1.0.

(i.e., we use 1.0 for $\log|a_i|$ in the formulation.)

Using the values of C obtained , each co-efficient is defined as :

$$a_i = \frac{a_i}{10^C} \qquad (i = 0, \cdots, n)$$

If the absolute value of new coefficients made through this procedure is less than 10^{-30}, that term is regarded as to be zero, and omitted from the present equation system.

After establishment of the algebraic equation, the next thing we must do is to find a quadratic factor ($x^2 + px + q$) according to the Bairstow's method. The initial values in the process to determine both p and q are specified as follows.

(1) Initialization of p and q

(a) First case (IR = 0)

. If $a_n = 0$, then we assume p = 1 and q = 0.

. If $a_n \neq 0$,

$$\text{then}\begin{cases} \text{(i)}\ \left|\dfrac{a_{n-2}}{a_n}\right| \geq 0.2, & \begin{cases} p = 0.01 \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| < 10^{-5}\right) \\ q = \dfrac{a_n}{a_{n-2}} \end{cases} \\ \quad\text{or} & \begin{cases} p = \dfrac{a_{n-1}}{a_{n-2}} \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| \geq 10^{-5}\right) \\ q = \dfrac{a_n}{a_{n-2}} \end{cases} \\ \text{(ii)}\ \left|\dfrac{a_{n-2}}{a_n}\right| < 0.2, & p = 0.5 \text{ and } q = -1.0 \end{cases}$$

(b) Second case (IR = 1)

When the starting values of case (a) are bad and the resulting convergence of solution is not accomplished within iteration numbers required, the second option (b) is used for the initial values p and q.

. If $a_n = 0$, then we assume p = -1.0 and q = 0.

. If $a_n \neq 0$,

$$\text{then}\begin{cases} \text{(i)}\ \left|\dfrac{a_{n-2}}{a_n}\right| \geq 0.2, & \begin{cases} p = -0.01 \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| < 10^{-5}\right) \\ q = \dfrac{a_n}{a_{n-2}} \end{cases} \\ & \begin{cases} p = -\dfrac{a_{n-1}}{a_{n-2}} \left(\text{for } \left|\dfrac{a_{n-1}}{a_{n-2}}\right| \geq 10^{-5}\right) \\ q = \dfrac{a_n}{a_{n-2}} \end{cases} \\ \text{(ii)}\ \left|\dfrac{a_{n-2}}{a_n}\right| < 0.2, & p = -0.5 \text{ and } q = -1.0 \end{cases}$$

Using the above initial conditions, increments Δp and Δq in the Bairstow's routine are computed by Gaussian elimination method.(8),(9) Final determination of them is made under the specified convergence condition. (EPS $< \varepsilon$, ε ; input data)
Following the determination of a quadratic factor, the same procedures are repeated for the polynomial with degrees (n-2, n-4, ..., 4, 2).
When the absolute value of imaginary part in complex root is less than 10^{-3}, that root is regarded as real.
Once real roots of the equation are determined in this manner, respective joint solutions can be obtained easily in accordance with explicit formulation described in the preceding section.
The remaining problem is that accuracy to numerical results.

Thus, we have incorporated the direct kinematic routine in the code in order to verify the validity or reliability of joint angle solutions. Namely, when the maximum absolute error, between reproduced values and original ones used to determine the joint angles, is less than some tolerance (EPS1 specified by input data), the joint solutions ($\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, and θ_6) are regarded as feasible.

In order to make easier for everyone the practical use of this code, the program is described in the simplest form without particular restrictions. All the data to be used are set in the main memory.
The user must specify the control parameter related to the size of capacity, which is prepared in the main program according as the size of problem to be solved.

Pertaining to the input or output information, the program description is as follows.

(1) Input requirements included

title of problem, position and orientation vector at the end-point of manipulator, position numbers, convergence factor and so on.

(2) Output information included

(a) a complete copy of input data

(b) print out of the calculation

(for each position advancement)

The output of the calculated results consists of a complete edit and a short edit.

The short edit is limited only to the joint solutions. In this edit, the following notations may appear when the calculation is abnormal.

Namely,

-999.D0 : denotes no real roots for the polynomial in question. (All the roots are complex.)

-888.D0 : denotes no desired joint solutions within the mechanical constraints.

999.D0 : denotes no convergence for Δp and Δq in determining the coefficients of the first quadratic factor. It is indicative of no joint angle solutions.

**** contained in the complete edit is : ****

(i) real roots of polynomial

(ii) corresponding joint angle solutions

(iii) component values of T_6 matrix

(iv) absolute errors

(c) graphics

(at the end of calculation, the followings are plotted.)

(i) plots of polynomial

(ii) plots of individual joint angle variables

4. Concluding remarks

The joint calculational method and its computer program have been designed, developed, and tested on a type of manipulator configuration. Some checkout runs were successfully completed, and thereby, the ARM2 model worked as intended. Since the code is free of major programming errors, it is available for the manipulator arm studies.
With respect to the accuracy of solutions and running time, the capability of the code is adequate from the viewpoint of off-line calculation. Together with technical experience accumulated in the ARM1 or ARM2, it is anticipated that a generic and elegant method for any articulated configuration will be progressively developed with the help of the present idea and further extension.

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Appendix 1. Test run example

In this section, one of the test runs is described.
The purpose is to show that the present code :

(1) works as intended,

(2) runs within an acceptable time for a typical application.

For the verification calculation, a trajectory was given such that the hand of the manipulator moves along a straight line with the initial point A and the terminal point B. As an approximation of this path segment $\overline{AB}$, n equidistant points, named " position numbers " are used in the calculation. At the present analysis, 201 spatial points were selected and calculated by means of a linear interpolation method. The orientation of the hand was kept constant during a movement.

A set of data necessary for the calculation is specified as follows.

(a) position co-ordinates of the initial point A

(-0.50, 0.69, 1.14) (m in unit)

(b) position co-ordinates of the terminal point B

(0.50, 0.69, 1.14) (m in unit)

(c) number of points (position numbers) = 201

(d) direction cosines

OX = 0.0, OY = 1.0, OZ = 0.0

AX = 0.0, AY = 0.0, AZ = 1.0

(The same data is used at the terminal point because a constant direction is intended.)

(e) convergence condition

EPS = 10^{-4}, EPS1 = 10^{-3}

Under the above motion scheme, the calculated results of individual joint angles are plotted in Fig.a.1 through Fig.a.6 .
Concerning the sensitivity check of EPS, calculated results were almost similar for $10^{-5} < \text{EPS} \leq 10^{-4}$. Beyond that range, some of solutions were unrealistic, however.
As illustrated in the figures, in the present test case, each joint angle was uniquely determined. The accuracies of solutions were of the order of less than 10^{-6} on the average.
And the running time was about 18 msec on the FACOM-380 computer.

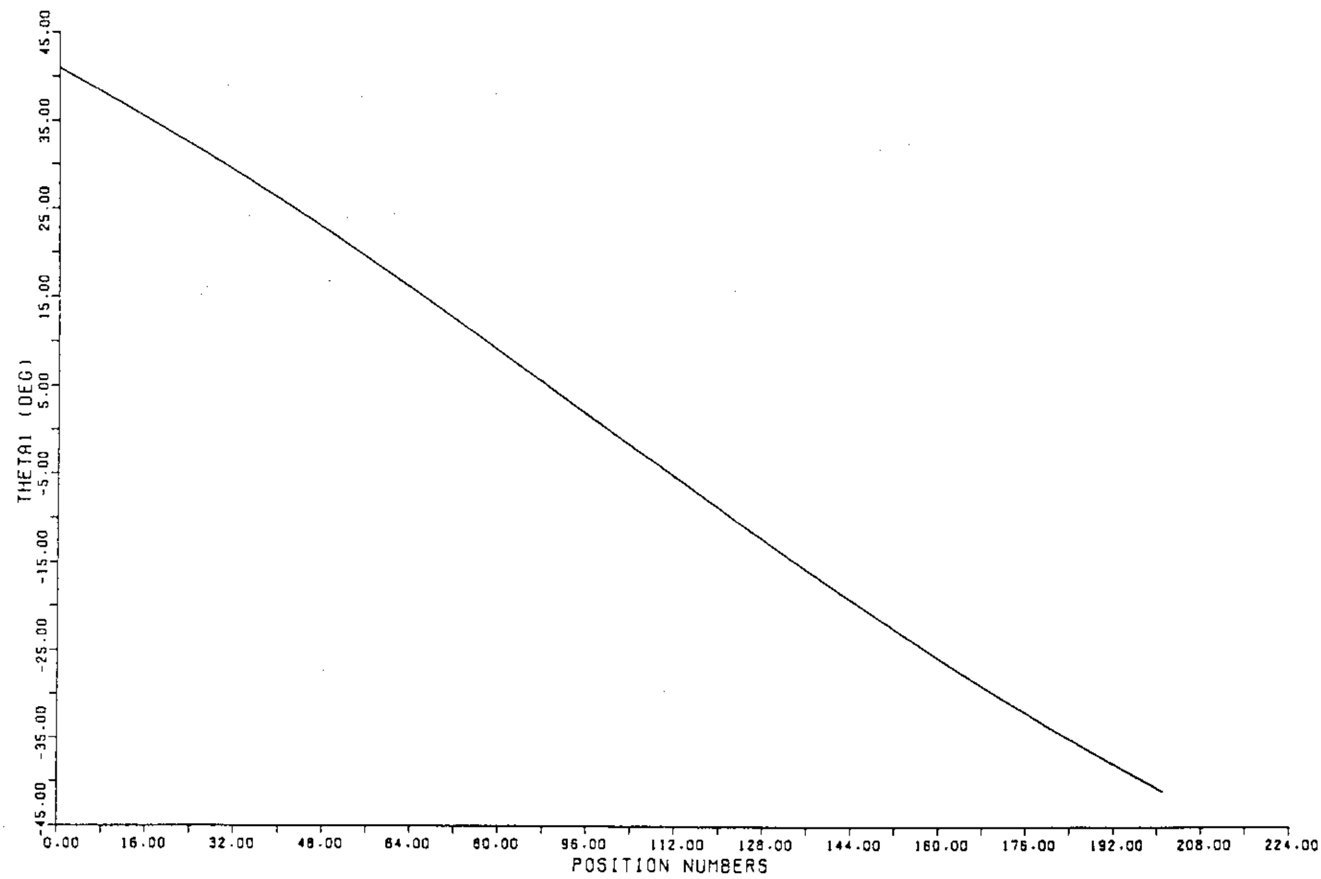

Fig.a.1 Calculated Results of Joint Angle THETA 1

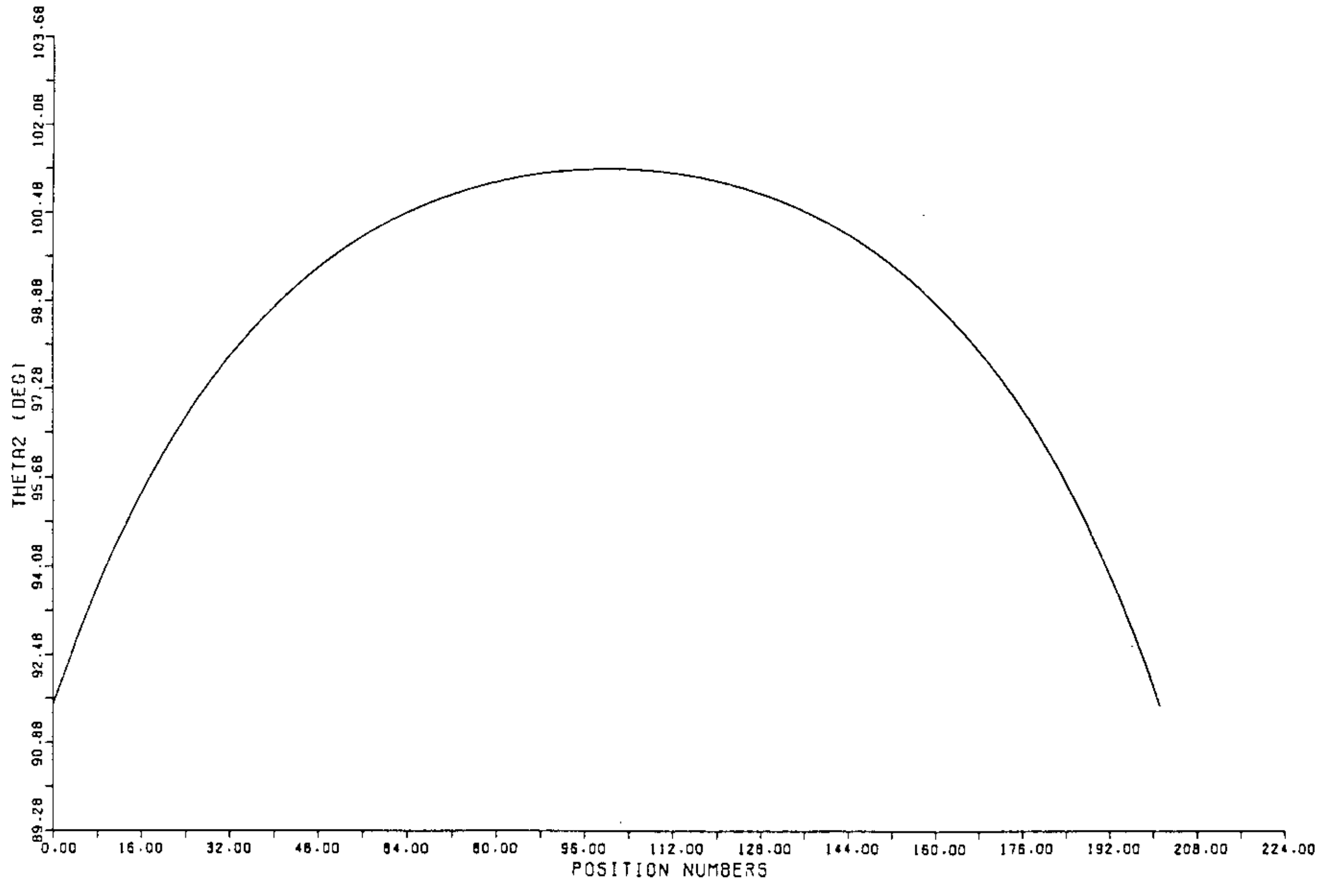

Fig.a.2 Calculated Results of Joint Angle THETA 2

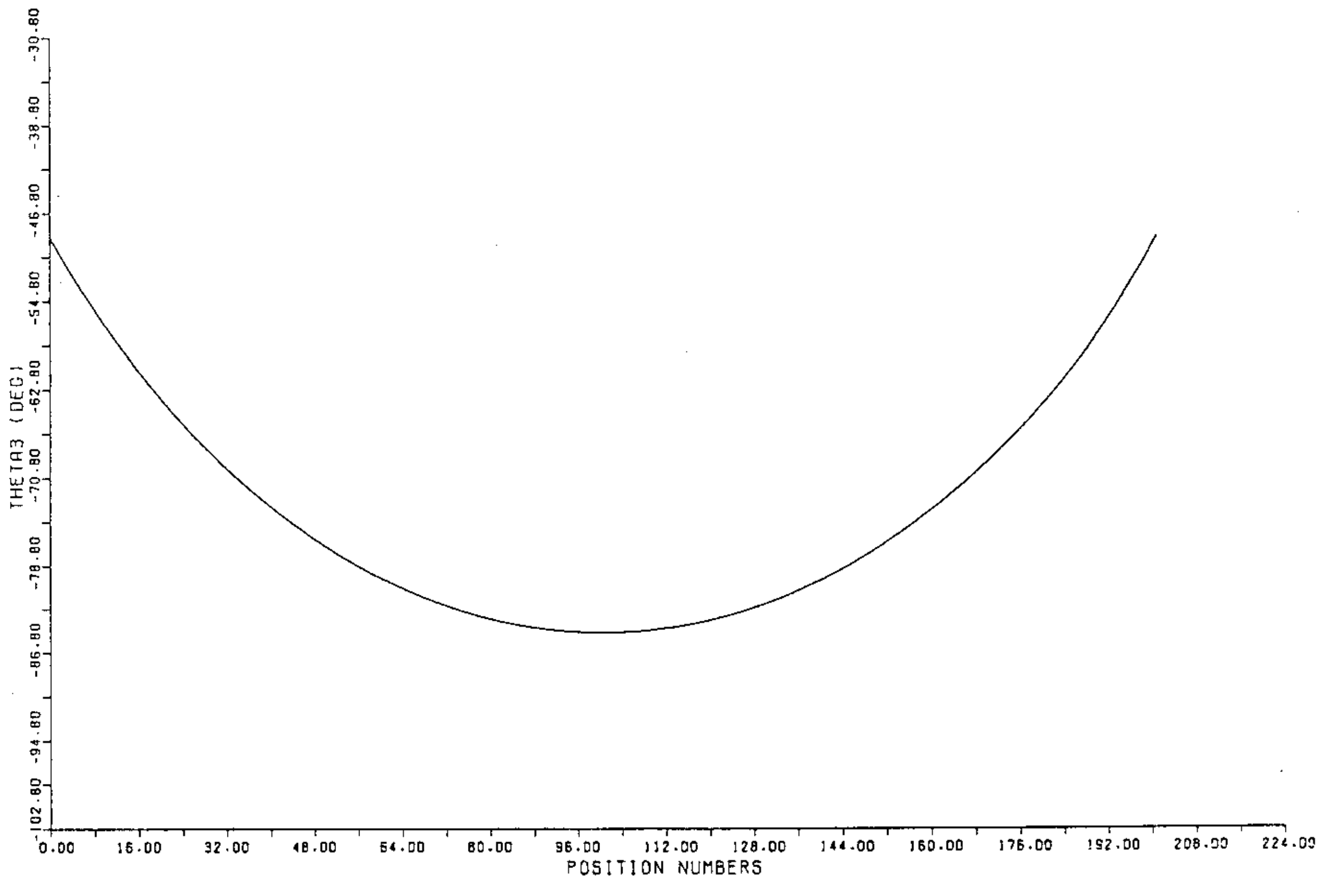

Fig.a.3 Calculated Results of Joint Angle THETA 3

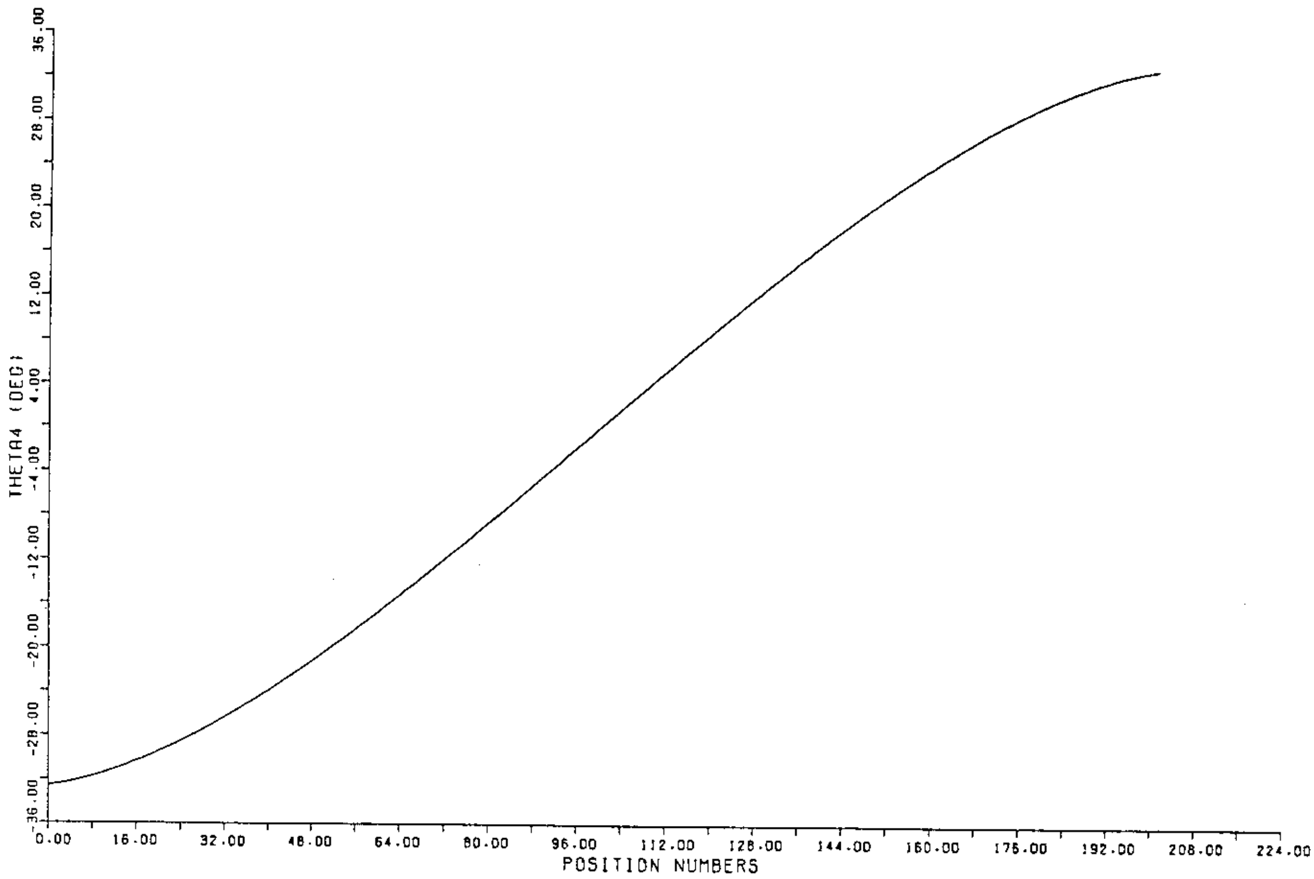

Fig.a.4 Calculated Results of Joint Angle THETA 4

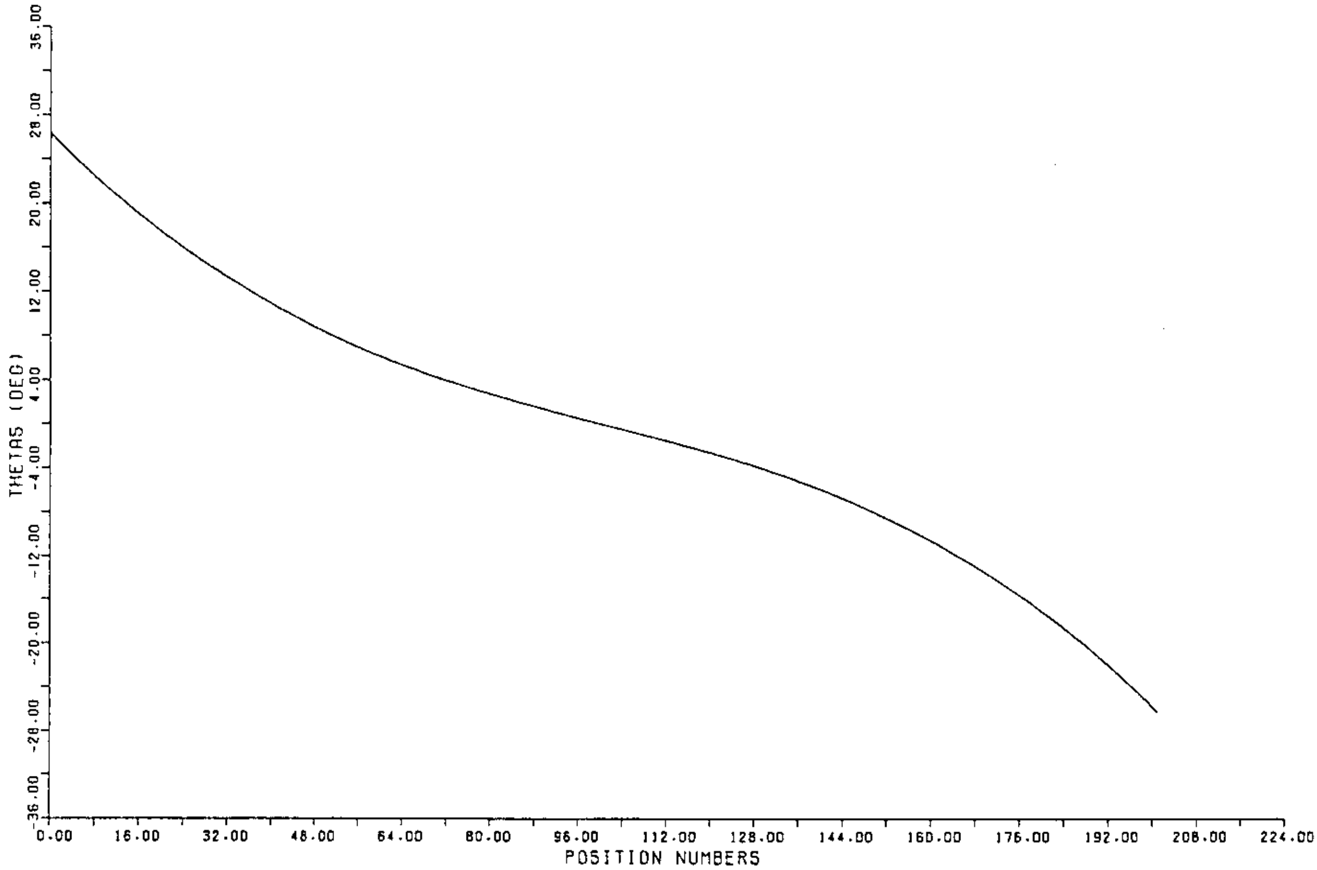

Fig.a.5 Calculated Results of Joint Angle THETA 5

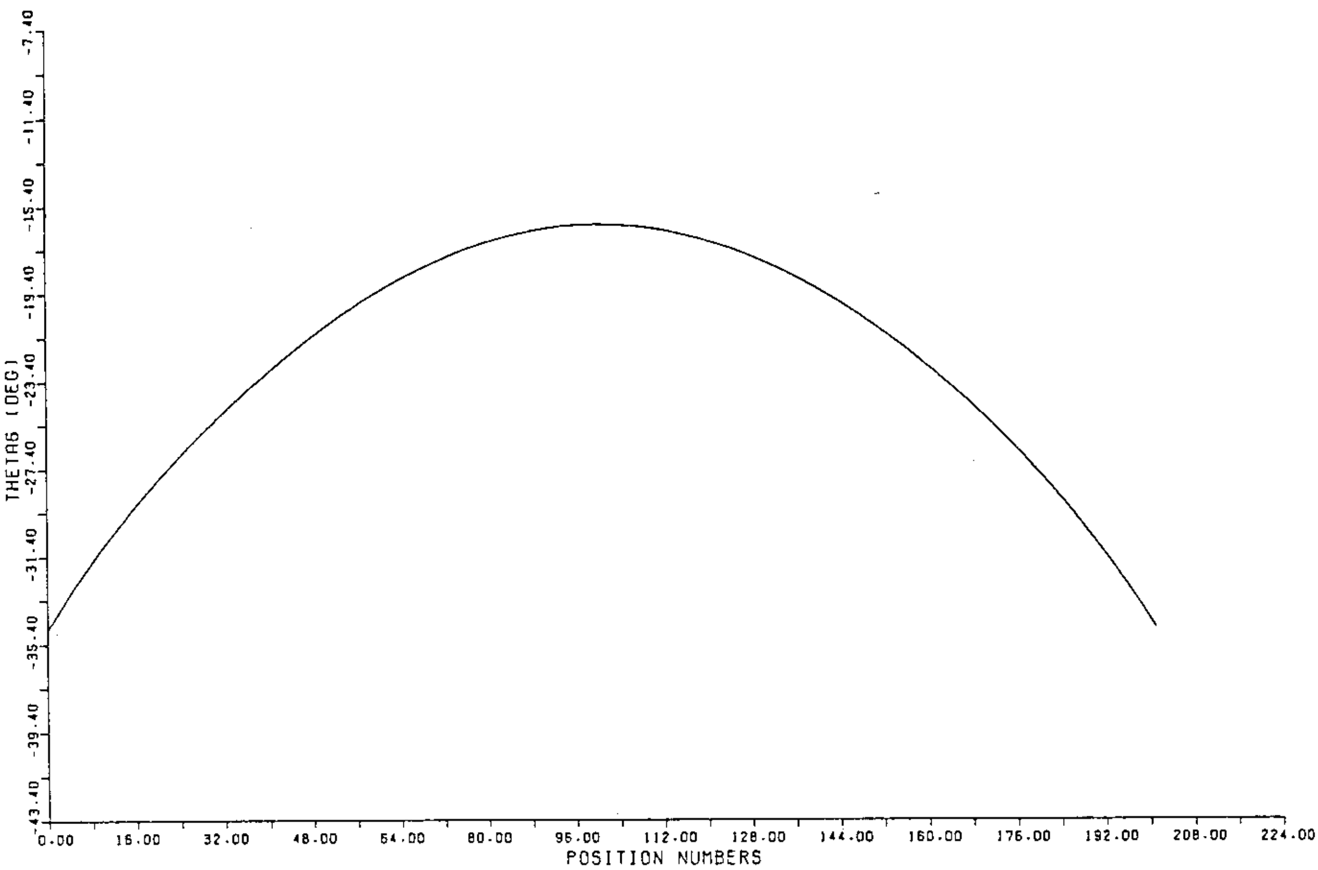

Fig.a.6 Calculated Results of Joint Angle THETA 6

***** List of Calculated Joint Angles *****

+++ TEST CALC. 4 +++
THETA (DEG)

NO.	1	2	3	4	5	6
NO. -- 0 --- 18	40.9595	91.5616	-48.7896	-32.5888	26.3243	-34.7480
NO. -- 1 --- 18	40.6183	91.8561	-49.8709	-32.5179	25.8173	-34.3387
NO. -- 2 --- 18	40.2757	92.1416	-50.7310	-32.4355	25.3164	-33.9368
NO. -- 3 --- 18	39.9324	92.4186	-51.5739	-32.3439	24.8251	-33.5471
NO. -- 4 --- 18	39.5872	92.6873	-52.3950	-32.2404	24.3369	-33.1598
NO. -- 5 --- 18	39.2413	92.9483	-53.2009	-32.1288	23.8587	-32.7850
NO. -- 6 --- 18	38.8951	93.2017	-53.9934	-32.0103	23.3918	-32.4241
NO. -- 7 --- 18	38.5456	93.4479	-54.7614	-31.8781	22.9219	-32.0561
NO. -- 8 --- 18	38.1961	93.6873	-55.5193	-31.7407	22.4654	-31.7051
NO. -- 9 --- 18	37.8451	93.9199	-56.2613	-31.5947	22.0148	-31.3596
NO. -- 10 --- 18	37.4929	94.1462	-56.9895	-31.4413	21.5717	-31.0223
NO. -- 11 --- 18	37.1383	94.3663	-57.6994	-31.2780	21.1304	-30.6841
NO. -- 12 --- 18	36.7837	94.5805	-58.4011	-31.1103	20.7020	-30.3619
NO. -- 13 --- 18	36.4268	94.7889	-59.0858	-30.9335	20.2759	-30.0395
NO. -- 14 --- 18	36.0690	94.9918	-59.7599	-30.7510	19.8587	-29.7268
NO. -- 15 --- 18	35.7088	95.1893	-60.4179	-30.5598	19.4438	-29.4137
NO. -- 16 --- 18	35.3474	95.3815	-61.0646	-30.3626	19.0361	-29.1078
NO. -- 17 --- 18	34.9848	95.5687	-61.7004	-30.1595	18.6359	-28.8092
NO. -- 18 --- 18	34.6201	95.7510	-62.3222	-29.9490	18.2391	-28.5116

+++ TEST CALC. 4 +++
THETA (DEG)

NO.	1	2	3	4	5	6
NO. -- 19 --- 18	34.2542	95.9286	-62.9343	-29.7332	17.8502	-28.2222
NO. -- 20 --- 18	33.8870	96.1015	-63.5359	-29.5118	17.4679	-27.9388
NO. -- 21 --- 18	33.5173	96.2699	-64.1222	-29.2825	17.0869	-27.6526
NO. -- 22 --- 18	33.1465	96.4340	-64.6999	-29.0485	16.7139	-27.3750
NO. -- 23 --- 18	32.7740	96.5938	-65.2668	-28.8088	16.3463	-27.1012
NO. -- 24 --- 18	32.4000	96.7495	-65.8236	-28.5638	15.9846	-26.8323
NO. -- 25 --- 18	32.0255	96.9011	-66.3746	-28.3154	15.6330	-26.5753
NO. -- 26 --- 18	31.6467	97.0489	-66.9038	-28.0564	15.2748	-26.3021
NO. -- 27 --- 18	31.2681	97.1928	-67.4314	-27.7961	14.9305	-26.0471
NO. -- 28 --- 18	30.8877	97.3330	-67.9478	-27.5300	14.5900	-25.7936
NO. -- 29 --- 18	30.5056	97.4695	-68.4547	-27.2590	14.2548	-25.5441
NO. -- 30 --- 18	30.1223	97.6025	-68.9536	-26.9838	13.9262	-25.3007
NO. -- 31 --- 18	29.7371	97.7320	-69.4417	-26.7031	13.6013	-25.0585
NO. -- 32 --- 18	29.3497	97.8581	-69.9183	-26.4169	13.2793	-24.8162
NO. -- 33 --- 18	28.9610	97.9810	-70.3868	-26.1265	12.9633	-24.5789
NO. -- 34 --- 18	28.5708	98.1006	-70.8466	-25.8317	12.6525	-24.3458
NO. -- 35 --- 18	28.1789	98.2170	-71.2974	-25.5325	12.3465	-24.1158
NO. -- 36 --- 18	27.7852	98.3304	-71.7383	-25.2285	12.0444	-23.8873
NO. -- 37 --- 18	27.3904	98.4408	-72.1723	-24.9210	11.7486	-23.6650

+++ TEST CALC. 4 +++
THETA (DEG)

1	2	3	4	5	6
NO. -- 38	--- 18				
26.9939	98.5482	-72.5975	-24.6092	11.4573	-23.4456
NO. -- 39	--- 18				
26.5958	98.6527	-73.0138	-24.2932	11.1703	-23.2288
NO. -- 40	--- 18				
26.1953	98.7544	-73.4169	-23.9715	10.8841	-23.0074
NO. -- 41	--- 18				
25.7951	98.8533	-73.8214	-23.6494	10.6102	-22.8049
NO. -- 42	--- 18				
25.3924	98.9496	-74.2129	-23.3217	10.3371	-22.5979
NO. -- 43	--- 18				
24.9883	99.0432	-74.5964	-22.9903	10.0684	-22.3943
NO. -- 44	--- 18				
24.5827	99.1342	-74.9718	-22.6553	9.8042	-22.1937
NO. -- 45	--- 18				
24.1755	99.2227	-75.3380	-22.3162	9.5434	-21.9943
NO. -- 46	--- 18				
23.7673	99.3087	-75.6990	-21.9745	9.2889	-21.8024
NO. -- 47	--- 18				
23.3574	99.3923	-76.0511	-21.6290	9.0378	-21.6119
NO. -- 48	--- 18				
22.9461	99.4735	-76.3952	-21.2801	8.7907	-21.4240
NO. -- 49	--- 18				
22.5334	99.5524	-76.7316	-20.9279	8.5477	-21.2393
NO. -- 50	--- 18				
22.1194	99.6290	-77.0606	-20.5725	8.3089	-21.0579
NO. -- 51	--- 18				
21.7040	99.7034	-77.3820	-20.2140	8.0742	-20.8797
NO. -- 52	--- 18				
21.2876	99.7756	-77.6976	-19.8528	7.8444	-20.7072
NO. -- 53	--- 18				
20.8694	99.8456	-78.0027	-19.4878	7.6165	-20.5330
NO. -- 54	--- 18				
20.4499	99.9135	-78.3010	-19.1201	7.3929	-20.3627
NO. -- 55	--- 18				
20.0296	99.9794	-78.5942	-18.7501	7.1744	-20.1992
NO. -- 56	--- 18				
19.6079	100.0432	-78.8796	-18.3771	6.9592	-20.0379

+++ TEST CALC. 4 +++
THETA (DEG)

1	2	3	4	5	6
NO. -- 57	--- 18				
19.1848	100.1051	-79.1564	-18.0011	6.7469	-19.8775
NO. -- 58	--- 18				
18.7607	100.1650	-79.4274	-17.6228	6.5390	-19.7225
NO. -- 59	--- 18				
18.3354	100.2231	-79.6911	-17.2419	6.3344	-19.5703
NO. -- 60	--- 18				
17.9090	100.2792	-79.9476	-16.8584	6.1332	-19.4210
NO. -- 61	--- 18				
17.4815	100.3335	-80.1976	-16.4726	5.9356	-19.2757
NO. -- 62	--- 18				
17.0528	100.3861	-80.4406	-16.0844	5.7413	-19.1336
NO. -- 63	--- 18				
16.6232	100.4368	-80.6766	-15.6939	5.5501	-18.9945
NO. -- 64	--- 18				
16.1924	100.4859	-80.9060	-15.3013	5.3623	-18.8591
NO. -- 65	--- 18				
15.7607	100.5332	-81.1286	-14.9064	5.1776	-18.7269
NO. -- 66	--- 18				
15.3280	100.5788	-81.3447	-14.5095	4.9960	-18.5984
NO. -- 67	--- 18				
14.8942	100.6229	-81.5525	-14.1103	4.8166	-18.4707
NO. -- 68	--- 18				
14.4597	100.6653	-81.7562	-13.7096	4.6413	-18.3504
NO. -- 69	--- 18				
14.0242	100.7061	-81.9524	-13.3068	4.4684	-18.2321
NO. -- 70	--- 18				
13.5881	100.7454	-82.1450	-12.9028	4.2994	-18.1219
NO. -- 71	--- 18				
13.1506	100.7831	-82.3249	-12.4958	4.1307	-18.0054
NO. -- 72	--- 18				
12.7125	100.8194	-82.5013	-12.0877	3.9657	-17.8972
NO. -- 73	--- 18				
12.2736	100.8542	-82.6709	-11.6779	3.8031	-17.7918
NO. -- 74	--- 18				
11.8340	100.8875	-82.8357	-11.2668	3.6435	-17.6922
NO. -- 75	--- 18				
11.3937	100.9194	-82.9931	-10.8540	3.4860	-17.5949

+++ TEST CALC. 4 +++
THETA (DEG)

NO.	1	2	3	4	5	6
NO. -- 76 --- 18	10.9526	100.9499	-83.1443	-10.4398	3.3307	-17.5012
NO. -- 77 --- 18	10.5108	100.9790	-83.2892	-10.0241	3.1776	-17.4111
NO. -- 78 --- 18	10.0684	101.0067	-83.4279	-9.6072	3.0266	-17.3246
NO. -- 79 --- 18	9.6253	101.0331	-83.5598	-9.1889	2.8776	-17.2409
NO. -- 80 --- 18	9.1817	101.0581	-83.6865	-8.7696	2.7307	-17.1624
NO. -- 81 --- 18	8.7374	101.0819	-83.8062	-8.3489	2.5855	-17.0865
NO. -- 82 --- 18	8.2927	101.1043	-83.9202	-7.9273	2.4422	-17.0146
NO. -- 83 --- 18	7.8474	101.1255	-84.0280	-7.5046	2.3005	-16.9466
NO. -- 84 --- 18	7.4016	101.1454	-84.1294	-7.0809	2.1604	-16.8818
NO. -- 85 --- 18	6.9553	101.1641	-84.2252	-6.6563	2.0219	-16.8214
NO. -- 86 --- 18	6.5086	101.1815	-84.3147	-6.2309	1.8848	-16.7643
NO. -- 87 --- 18	6.0615	101.1978	-84.3982	-5.8046	1.7490	-16.7112
NO. -- 88 --- 18	5.6140	101.2128	-84.4755	-5.3776	1.6144	-16.6615
NO. -- 89 --- 18	5.1662	101.2266	-84.5468	-4.9499	1.4810	-16.6157
NO. -- 90 --- 18	4.7181	101.2392	-84.6122	-4.5216	1.3486	-16.5737
NO. -- 91 --- 18	4.2696	101.2506	-84.6716	-4.0927	1.2172	-16.5355
NO. -- 92 --- 18	3.8209	101.2609	-84.7250	-3.6633	1.0867	-16.5011
NO. -- 93 --- 18	3.3719	101.2700	-84.7725	-3.2334	0.9570	-16.4705
NO. -- 94 --- 18	2.9228	101.2780	-84.8140	-2.8031	0.8280	-16.4438

+++ TEST CALC. 4 +++
THETA (DEG)

NO.	1	2	3	4	5	6
NO. -- 95 --- 18	2.4734	101.2848	-84.8495	-2.3725	0.6996	-16.4208
NO. -- 96 --- 18	2.0239	101.2905	-84.8792	-1.9415	0.5717	-16.4017
NO. -- 97 --- 18	1.5743	101.2950	-84.9028	-1.5103	0.4442	-16.3863
NO. -- 98 --- 18	1.1246	101.2984	-84.9206	-1.0789	0.3171	-16.3748
NO. -- 99 --- 18	0.6748	101.3007	-84.9324	-0.6474	0.1902	-16.3671
NO. -- 100 --- 18	0.2249	101.3018	-84.9384	-0.2158	0.0634	-16.3633
NO. -- 101 --- 18	-0.2249	101.3018	-84.9384	0.2158	-0.0634	-16.3633
NO. -- 102 --- 18	-0.6748	101.3007	-84.9324	0.6474	-0.1902	-16.3671
NO. -- 103 --- 18	-1.1246	101.2984	-84.9206	1.0789	-0.3171	-16.3748
NO. -- 104 --- 18	-1.5743	101.2950	-84.9028	1.5103	-0.4442	-16.3863
NO. -- 105 --- 18	-2.0239	101.2905	-84.8792	1.9415	-0.5717	-16.4017
NO. -- 106 --- 18	-2.4734	101.2848	-84.8495	2.3725	-0.6996	-16.4208
NO. -- 107 --- 18	-2.9228	101.2780	-84.8140	2.8031	-0.8280	-16.4438
NO. -- 108 --- 18	-3.3719	101.2700	-84.7725	3.2334	-0.9570	-16.4705
NO. -- 109 --- 18	-3.8209	101.2609	-84.7250	3.6633	-1.0867	-16.5011
NO. -- 110 --- 18	-4.2696	101.2506	-84.6716	4.0927	-1.2172	-16.5355
NO. -- 111 --- 18	-4.7181	101.2392	-84.6122	4.5216	-1.3486	-16.5737
NO. -- 112 --- 18	-5.1662	101.2266	-84.5468	4.9499	-1.4810	-16.6157
NO. -- 113 --- 18	-5.6140	101.2128	-84.4755	5.3776	-1.6144	-16.6615

+++ TEST CALC. 4 +++
THETA (DEG)

1	2	3	4	5	6
NO. -- 114	--- 18				
-6.0615	101.1978	-84.3982	5.8046	-1.7490	-16.7112
NO. -- 115	--- 18				
-6.5086	101.1815	-84.3147	6.2309	-1.8848	-16.7643
NO. -- 116	--- 18				
-6.9553	101.1641	-84.2252	6.6563	-2.0219	-16.8214
NO. -- 117	--- 18				
-7.4016	101.1454	-84.1294	7.0809	-2.1604	-16.8818
NO. -- 118	--- 18				
-7.8474	101.1255	-84.0280	7.5046	-2.3005	-16.9466
NO. -- 119	--- 18				
-8.2927	101.1043	-83.9202	7.9273	-2.4422	-17.0146
NO. -- 120	--- 18				
-8.7374	101.0819	-83.8062	8.3489	-2.5855	-17.0865
NO. -- 121	--- 18				
-9.1817	101.0581	-83.6865	8.7696	-2.7307	-17.1624
NO. -- 122	--- 18				
-9.6253	101.0331	-83.5598	9.1889	-2.8776	-17.2409
NO. -- 123	--- 18				
-10.0684	101.0067	-83.4279	9.6072	-3.0266	-17.3246
NO. -- 124	--- 18				
-10.5108	100.9790	-83.2892	10.0241	-3.1776	-17.4111
NO. -- 125	--- 18				
-10.9526	100.9499	-83.1443	10.4398	-3.3307	-17.5012
NO. -- 126	--- 18				
-11.3937	100.9194	-82.9931	10.8540	-3.4860	-17.5949
NO. -- 127	--- 18				
-11.8340	100.8875	-82.8357	11.2668	-3.6435	-17.6922
NO. -- 128	--- 18				
-12.2736	100.8542	-82.6709	11.6779	-3.8031	-17.7918
NO. -- 129	--- 18				
-12.7125	100.8194	-82.5013	12.0877	-3.9657	-17.8972
NO. -- 130	--- 18				
-13.1506	100.7831	-82.3249	12.4958	-4.1307	-18.0054
NO. -- 131	--- 18				
-13.5881	100.7454	-82.1450	12.9028	-4.2994	-18.1219
NO. -- 132	--- 18				
-14.0242	100.7061	-81.9524	13.3068	-4.4684	-18.2321

+++ TEST CALC. 4 +++
THETA (DEG)

1	2	3	4	5	6
NO. -- 133	--- 18				
-14.4597	100.6653	-81.7562	13.7096	-4.6413	-18.3504
NO. -- 134	--- 18				
-14.8942	100.6229	-81.5525	14.1103	-4.8166	-18.4707
NO. -- 135	--- 18				
-15.3280	100.5788	-81.3447	14.5095	-4.9960	-18.5984
NO. -- 136	--- 18				
-15.7607	100.5332	-81.1286	14.9064	-5.1776	-18.7269
NO. -- 137	--- 18				
-16.1924	100.4859	-80.9060	15.3013	-5.3623	-18.8591
NO. -- 138	--- 18				
-16.6232	100.4368	-80.6766	15.6939	-5.5501	-18.9945
NO. -- 139	--- 18				
-17.0528	100.3861	-80.4406	16.0844	-5.7413	-19.1336
NO. -- 140	--- 18				
-17.4815	100.3335	-80.1976	16.4726	-5.9356	-19.2757
NO. -- 141	--- 18				
-17.9090	100.2792	-79.9476	16.8584	-6.1332	-19.4210
NO. -- 142	--- 18				
-18.3354	100.2231	-79.6911	17.2419	-6.3344	-19.5703
NO. -- 143	--- 18				
-18.7607	100.1650	-79.4274	17.6228	-6.5390	-19.7225
NO. -- 144	--- 18				
-19.1848	100.1051	-79.1564	18.0011	-6.7469	-19.8775
NO. -- 145	--- 18				
-19.6079	100.0432	-78.8796	18.3771	-6.9592	-20.0379
NO. -- 146	--- 18				
-20.0296	99.9794	-78.5942	18.7501	-7.1744	-20.1992
NO. -- 147	--- 18				
-20.4499	99.9135	-78.3010	19.1201	-7.3929	-20.3627
NO. -- 148	--- 18				
-20.8694	99.8456	-78.0027	19.4878	-7.6165	-20.5330
NO. -- 149	--- 18				
-21.2876	99.7756	-77.6976	19.8528	-7.8444	-20.7072
NO. -- 150	--- 18				
-21.7040	99.7034	-77.3820	20.2140	-8.0742	-20.8797
NO. -- 151	--- 18				
-22.1194	99.6290	-77.0606	20.5725	-8.3089	-21.0579

+++ TEST CALC. 4 +++
THETA (DEG)

1	2	3	4	5	6
NO. -- 152	--- 18				
-22.5334	99.5524	-76.7316	20.9279	-8.5477	-21.2393
NO. -- 153	--- 18				
-22.9461	99.4735	-76.3952	21.2801	-8.7907	-21.4240
NO. -- 154	--- 18				
-23.3574	99.3923	-76.0511	21.6290	-9.0378	-21.6119
NO. -- 155	--- 18				
-23.7673	99.3087	-75.6990	21.9745	-9.2889	-21.8024
NO. -- 156	--- 18				
-24.1755	99.2227	-75.3380	22.3162	-9.5434	-21.9943
NO. -- 157	--- 18				
-24.5827	99.1342	-74.9718	22.6553	-9.8042	-22.1937
NO. -- 158	--- 18				
-24.9883	99.0432	-74.5964	22.9903	-10.0684	-22.3943
NO. -- 159	--- 18				
-25.3924	98.9496	-74.2129	23.3217	-10.3371	-22.5979
NO. -- 160	--- 18				
-25.7951	98.8533	-73.8214	23.6494	-10.6102	-22.8049
NO. -- 161	--- 18				
-26.1953	98.7544	-73.4169	23.9715	-10.8841	-23.0074
NO. -- 162	--- 18				
-26.5958	98.6527	-73.0138	24.2932	-11.1703	-23.2288
NO. -- 163	--- 18				
-26.9939	98.5482	-72.5975	24.6092	-11.4573	-23.4456
NO. -- 164	--- 18				
-27.3904	98.4408	-72.1723	24.9210	-11.7486	-23.6650
NO. -- 165	--- 18				
-27.7852	98.3304	-71.7383	25.2285	-12.0444	-23.8873
NO. -- 166	--- 18				
-28.1789	98.2170	-71.2974	25.5325	-12.3465	-24.1158
NO. -- 167	--- 18				
-28.5708	98.1006	-70.8466	25.8317	-12.6525	-24.3458
NO. -- 168	--- 18				
-28.9610	97.9810	-70.3868	26.1265	-12.9633	-24.5789
NO. -- 169	--- 18				
-29.3497	97.8581	-69.9183	26.4169	-13.2793	-24.8162
NO. -- 170	--- 18				
-29.7371	97.7320	-69.4417	26.7031	-13.6013	-25.0585

+++ TEST CALC. 4 +++
THETA (DEG)

1	2	3	4	5	6
NO. -- 171	--- 18				
-30.1223	97.6025	-68.9536	26.9838	-13.9262	-25.3007
NO. -- 172	--- 18				
-30.5056	97.4695	-68.4547	27.2590	-14.2548	-25.5441
NO. -- 173	--- 18				
-30.8877	97.3330	-67.9478	27.5300	-14.5900	-25.7936
NO. -- 174	--- 18				
-31.2681	97.1928	-67.4314	27.7961	-14.9305	-26.0471
NO. -- 175	--- 18				
-31.6467	97.0489	-66.9038	28.0564	-15.2748	-26.3021
NO. -- 176	--- 18				
-32.0255	96.9011	-66.3746	28.3154	-15.6330	-26.5753
NO. -- 177	--- 18				
-32.4000	96.7495	-65.8236	28.5638	-15.9846	-26.8323
NO. -- 178	--- 18				
-32.7740	96.5938	-65.2668	28.8088	-16.3463	-27.1012
NO. -- 179	--- 18				
-33.1465	96.4340	-64.6999	29.0485	-16.7139	-27.3750
NO. -- 180	--- 18				
-33.5173	96.2699	-64.1222	29.2825	-17.0869	-27.6526
NO. -- 181	--- 18				
-33.8870	96.1015	-63.5359	29.5118	-17.4679	-27.9388
NO. -- 182	--- 18				
-34.2542	95.9286	-62.9343	29.7332	-17.8502	-28.2222
NO. -- 183	--- 18				
-34.6201	95.7510	-62.3222	29.9490	-18.2391	-28.5116
NO. -- 184	--- 18				
-34.9848	95.5687	-61.7004	30.1595	-18.6359	-28.8092
NO. -- 185	--- 18				
-35.3474	95.3815	-61.0646	30.3626	-19.0361	-29.1078
NO. -- 186	--- 18				
-35.7088	95.1893	-60.4179	30.5598	-19.4438	-29.4137
NO. -- 187	--- 18				
-36.0690	94.9918	-59.7599	30.7510	-19.8587	-29.7268
NO. -- 188	--- 18				
-36.4268	94.7889	-59.0858	30.9335	-20.2759	-30.0395
NO. -- 189	--- 18				
-36.7837	94.5805	-58.4011	31.1103	-20.7020	-30.3619

+++ TEST CALC. 4 +++
THETA (DEG)

1	2	3	4	5	6
NO. -- 190	--- 18				
-37.1383	94.3663	-57.6994	31.2780	-21.1304	-30.6841
NO. -- 191	--- 18				
-37.4929	94.1462	-56.9895	31.4413	-21.5717	-31.0223
NO. -- 192	--- 18				
-37.8451	93.9199	-56.2613	31.5947	-22.0148	-31.3596
NO. -- 193	--- 18				
-38.1961	93.6873	-55.5193	31.7407	-22.4654	-31.7051
NO. -- 194	--- 18				
-38.5456	93.4479	-54.7614	31.8781	-22.9219	-32.0561
NO. -- 195	--- 18				
-38.8951	93.2017	-53.9934	32.0103	-23.3918	-32.4241
NO. -- 196	--- 18				
-39.2413	92.9483	-53.2009	32.1288	-23.8587	-32.7850
NO. -- 197	--- 18				
-39.5872	92.6873	-52.3950	32.2404	-24.3369	-33.1598
NO. -- 198	--- 18				
-39.9324	92.4186	-51.5739	32.3439	-24.8251	-33.5471
NO. -- 199	--- 18				
-40.2757	92.1416	-50.7310	32.4355	-25.3164	-33.9368
NO. -- 200	--- 18				
-40.6183	91.8561	-49.8709	32.5179	-25.8173	-34.3387
NO. -- 201	--- 18				
-40.9595	91.5616	-48.9896	32.5888	-26.3243	-34.7480

***** List of Input Data for Test Run *****

```
 INPUT --- TITLE
++++++++++++++++++ T  I  T  L  E ++++++++++++++++++
+                                                 +
+ +++  TEST  CALC.  3  +++                        +
+                                                 +
+++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>  201

 INPUT --- EPS , EPS1

 EPS    ===>  1.00000D-04    EPS1    ===>  1.00000D-03

---------- INITIAL  POINT ----------

 INPUT --- PX : PY :PZ

 PX ==> -5.00000D-01   PY ==>  6.90000D-01   PZ ==>  1.14000D+00 ( M )

 KEYIN       0 : USER SPECIFIED
             1 : RPY
             Z : EARLY STAGES
     DEFAULTS : EULER

 INPUT --- OX , OY , OZ

 OX ==>  0.0            OY ==>  1.00000D+00   OZ ==>  0.0

 INPUT --- AX , AY , AZ

 AX ==>  0.0            AY ==>  0.0            AZ ==>  1.00000D+00

---------- TERMINAL  POINT ----------

 INPUT --- PX : PY :PZ

 PX ==>  5.00000D-01   PY ==>  6.90000D-01   PZ ==>  1.14000D+00 ( M )

 KEYIN       0 : USER SPECIFIED
             1 : RPY
             Z : EARLY STAGES
     DEFAULTS : EULER

 INPUT --- OX , OY , OZ

 OX ==>  0.0            OY ==>  1.00000D+00   OZ ==>  0.0

 INPUT --- AX , AY , AZ

 AX ==>  0.0            AY ==>  0.0            AZ ==>  1.00000D+00
```

Appendix 2. Input Data Formats

The current code written in FORTRAN 77 is operational on the FACOM-380 computer. In this section, the input data requirements are presented in the form necessary for computer execution from TSS terminal.

Record 1	TITLE (A50)*	= Title of problem
Record 2	NDEL(I*4) (NDEL > 0)	= Position numbers between initial and terminal position. If NDEL = 0, Record NO.7 through NO.9 are neglected, because a point interpolation is not used.
Record 3	EPS(R*8)	= Convergence condition related to the determination of a quadratic factor ($x^2 + px + q$)
	EPS1(R*8)	= Check of the validity of the calculated joint angles. Joint angles solutions are substituted into the components of the T_6 matrix and compared with the given data.

Record 4 ~ Record 6 ---- Initial point data ----

Record 4	DATA(1,4,1)	= P_x(R*8) = Initial position of x-direction of the manipulator hand (m)
	DATA(2,4,1)	= P_y(R*8) = Initial position of y-direction of the manipulator hand (m)

*) ( ) denotes the type of variable.

DATA(3,4,1) = P_z(R*8) = Initial position of z-direction of the manipulator hand (m)

Record 5 GET('A1') = Option of orientation calculation
= '0' : user specified
= '1' : Roll-Pitch-Yaw transformation
= 'z' : Return to the initial stage Record 1 (re-trial of input data)
= excepting the above letter (default) : Euler transformation

Record 6

(i) If GET = '0', then

DATA(1,2,1) = OX(R*8) = x-component of orientation vector o
DATA(2,2,1) = OY(R*8) = y-component of orientation vector o
DATA(3,2,1) = OZ(R*8) = z-component of orientation vector o
DATA(1,3,1) = AX(R*8) = x-component of approach vector a
DATA(2,3,1) = AY(R*8) = y-component of approach vector a
DATA(3,3,1) = AZ(R*8) = z-component of approach vector a

(ii) If GET = '1', then

w_1(R*8) = Rotation angle about the z-axis (deg)
w_2(R*8) = Rotation angle about the y-axis (deg)
w_3(R*8) = Rotation angle about the x-axis (deg)

The direction cosines are calculated as follows.

DATA(1,2,1) = $\cos w_1 \sin w_2 \sin w_3 - \sin w_1 \cos w_3$
DATA(2,2,1) = $\sin w_1 \sin w_2 \sin w_3 + \cos w_1 \cos w_3$
DATA(3,2,1) = $\cos w_2 \sin w_3$
DATA(1,3,1) = $\cos w_1 \sin w_2 \cos w_3 + \sin w_1 \sin w_3$
DATA(2,3,1) = $\sin w_1 \sin w_2 \cos w_3 - \cos w_1 \sin w_3$
DATA(3,3,1) = $\cos w_2 \cos w_3$

(iii) If GET = Euler option, then

w_1(R*8) = Rotation angle about the z-axis (deg)

w_2(R*8) = Rotation angle about the y-axis (deg)

w_3(R*8) = Rotation angle about the z-axis (deg)

The direction cosines are calculated as follows.

DATA(1,2,1) = $-\cos w_1 \cos w_2 \sin w_3 - \sin w_1 \cos w_3$

DATA(2,2,1) = $-\sin w_1 \cos w_2 \sin w_3 + \cos w_1 \cos w_3$

DATA(3,2,1) = $\sin w_2 \sin w_3$

DATA(1,3,1) = $\cos w_1 \sin w_2$

DATA(2,3,1) = $\sin w_1 \sin w_2$

DATA(3,3,1) = $\cos w_2$

Record 7～Record 9 ---- Terminal point data ----

Record 7 DATA(1,4,2) = P_x(R*8) = Terminal position of x-direction of the manipulator hand (m)

DATA(2,4,2) = P_y(R*8) = Terminal position of y-direction of the manipulator hand (m)

DATA(3,4,2) = P_z(R*8) = Terminal position of z-direction of the manipulator hand (m)

Record 8 GET('A1') = Option of orientation calculation

= '0' : user specified

= '1' : Roll-Pitch-Yaw transformation

= 'z' : Return to the initial stage Record 7 (re-trial of input data)

= excepting the above letter (default) : Euler transformation

Record 9

(i) If GET = '0', then

DATA(1,2,2) = OX(R*8) = x-component of orientation vector o

DATA(2,2,2) = OY(R*8) = y-component of orientation vector o

DATA(3,2,2) = OZ(R*8) = z-component of orientation vector o

DATA(1,3,2) = AX(R*8) = x-component of approach vector a

DATA(2,3,2) = AY(R*8) = y-component of approach vector a

DATA(3,3,2) = AZ(R*8) = z-component of approach vector a

(ii) If GET = '1', then

w_1(R*8) = Rotation angle about the z-axis (deg)

w_2(R*8) = Rotation angle about the y-axis (deg)

w_3(R*8) = Rotation angle about the x-axis (deg)

The direction cosines are calculated as follows.

DATA(1,2,2) = $\cos w_1 \sin w_2 \sin w_3 - \sin w_2 \cos w_3$

DATA(2,2,2) = $\sin w_1 \sin w_2 \sin w_3 + \cos w_1 \cos w_3$

DATA(3,2,2) = $\cos w_2 \sin w_3$

DATA(1,3,2) = $\cos w_1 \sin w_2 \cos w_3 + \sin w_1 \sin w_3$

DATA(2,3,2) = $\sin w_1 \sin w_2 \cos w_3 - \cos w_1 \sin w_3$

DATA(3,3,2) = $\cos w_2 \cos w_3$

(iii) If GET = Euler option, then

w_1(R*8) = Rotation angle about the z-axis (deg)

w_2(R*8) = Rotation angle about the y-axis (deg)

w_3(R*8) = Rotation angle about the z-axis (deg)

The direction cosines are calculated as follows.

DATA(1,2,2) = $-\cos w_1 \cos w_2 \sin w_3 - \sin w_1 \cos w_3$

DATA(2,2,2) = $-\sin w_1 \cos w_2 \sin w_3 + \cos w_1 \cos w_3$

DATA(3,2,2) = $\sin w_2 \sin w_3$

DATA(1,3,2) = $\cos w_1 \sin w_2$

DATA(2,3,2) = $\sin w_1 \sin w_2$

DATA(3,3,2) = $\cos w_2$

Appendix 3. Outline of Bairstow's method

We use Bairstow's iterative method to find an approximation to a quadratic factor of a given polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0 \tag{1}$$

Let $Q(x) = x^2 + px + q$ be any quadratic polynomial with real coefficients. Then

$$\left.\begin{aligned} &f(x) = (x^2 + px + q)Q_1(x) + Rx + S \\ &\text{where a quotient } Q_1(x) = b_0 x^{n-2} + b_1 x^{n-3} + \dots b_{n-2} . \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} &\text{Additionally, assume } Q\ (x) = (x^2 + px + q)Q_2(x) + \bar{R}x + \bar{S}, \\ &\text{where } Q_2(x) = c_0 x^{n-4} + c_1 x^{n-5} + \dots + c_{n-4} \end{aligned}\right\} \tag{3}$$

Thus, we get:

$$\left.\begin{aligned} &b_0 = a_0,\ b_1 = a_1 - pb_0 \\ &b_k = a_k - pb_{k-1} - qb_{k-2} \quad (k = 2,3,\ \dots,\ n) \\ &c_0 = b_0,\ c_1 = b_1 - pb_0 \\ &c_k = b_k - pc_{k-1} - qc_{k-2} \quad (k = 2,3,\ \dots,\ n-1) \end{aligned}\right\} \tag{4}$$

Using the coefficient relations in Eqs.(1) to (4), we have

$$R = b_{n-1},\quad S = b_n + pb_{n-1} \tag{5}$$

$$\bar{R} = C_{n-3},\quad \bar{S} = C_{n-2} + pC_{n-3} . \tag{6}$$

Suppose that $XQ_1(x)$ divided by $(x^2 + px + q)$ gives $R^*x + S^*$ as the remainder.

or

$$\begin{aligned} xQ_1(x) &= Q_3(x)(x^2 + px + q)\ R^*x + S^* \\ &= (x^2 + px + q)xQ_2(x) + \bar{R}x^2 + \bar{S}x \\ &= (x^2 + px + q)xQ_2(x) + c_{n-3}x^2 + (c_{n-2} + pc_{n-3})x \\ &= (x^2 + px + q)(xQ_2(x) + c_{n-3}) + c_{n-2}x - qc_{n-3} \end{aligned} \tag{7}$$

Hence, we obtain

$$\left.\begin{aligned} R^* &= c_{n-2} \\ S^* &= -c_{n-3}\, q = c_{n-1} - b_{n-1} + p\, c_{n-2} \end{aligned}\right\} \tag{8}$$

On the other hand, if we differentiate Eq.(2) with respect to p and q, we have

$$\left.\begin{aligned} xQ_1(x) &= -(x^2 + px + q)\,\frac{\partial Q_1}{\partial p} - x\frac{\partial R}{\partial p} - \frac{\partial S}{\partial p} \\ Q_1(x) &= -(x^2 + px + q)\,\frac{\partial Q_1}{\partial q} - x\frac{\partial R}{\partial q} - \frac{\partial S}{\partial q} \end{aligned}\right\} \tag{9}$$

From these Eqs.(3), (6), (7) and (8),

$$\left.\begin{aligned} \frac{\partial R}{\partial p} &= -R^* = -c_{n-2} \\ \frac{\partial S}{\partial p} &= -S^* = -(c_{n-1} - b_{n-1} + p\, c_{n-2}) \\ \frac{\partial R}{\partial q} &= -\bar{R} = -c_{n-3} \\ \frac{\partial S}{\partial q} &= -\bar{S} = -(c_{n-2} + p\, c_{n-3}) \end{aligned}\right\} \tag{10}$$

If $Q(x) = Q'(x) = (x^2 + p'x + q')$ is a factor of $f(x)$, R and S in Eq.(2) must be zero.

<u>i.e.</u>,

$$\left.\begin{aligned} R(p,q) &= 0 \\ S(p,q) &= 0 \end{aligned}\right\} \tag{11}$$

Suppose that $Q(x)$ is an approximation to $Q'(x)$; then, p and q are assumed to have approximations to p' and q', respectively.

or $$p' = p + \Delta p$$
$$q' = q + \Delta q$$

Therefore,

$$\left.\begin{aligned} R(p + \Delta p,\ q + \Delta q) &= 0 \\ S(p + \Delta p,\ q + \Delta q) &= 0 \end{aligned}\right\} \quad (12)$$

Expanding in Taylor series and truncating after the first-order terms, we get:

$$\left.\begin{aligned} R(p,q) + \frac{\partial R}{\partial p}\Delta p + \frac{\partial R}{\partial q}\Delta q &= 0 \\ S(p,q) + \frac{\partial S}{\partial R}\Delta p + \frac{\partial S}{\partial q}\Delta q &= 0 \end{aligned}\right\} \quad (13)$$

Solving Eq.(13), we find that:

$$\left.\begin{aligned} \Delta p &= \frac{R\cdot c_{n-2} - S\cdot c_{n-3}}{c_{n-2}^{2} - \bar{c}_{n-1}\cdot c_{n-3}} = \frac{b_{n-1}c_{n-2} - b_n c_{n-3}}{c_{n-2}^{2} - \bar{c}_{n-1}\cdot c_{n-3}} \\ \Delta q &= \frac{S\cdot c_{n-2} - R\cdot c_{n-1}}{c_{n-2}^{2} - c_{n-1}\cdot c_{n-3}} = \frac{b_n c_{n-2} - b_{n-1}\bar{c}_{n-1}}{c_{n-2}^{2} - \bar{c}_{n-1}\cdot c_{n-3}} \end{aligned}\right\} \quad (14)$$

, where $c_{n-2}^{2} - \bar{c}_{n-1}c_{n-3} \neq 0$

$$\bar{c}_{n-1} = c_{n-1} - b_{n-1} \quad (15)$$

Now we can replace p by $(p + \Delta p)$, and q by $(q + \Delta q)$, and repeat the above procedure.

Appendix 4. Calculation of Coefficients of the Polynomial

(1) $b_0 = a_5(a_y\, y_n - a_x\, x_n)$

$b_1 = 2a_5\{(o_x\, x_n - o_y\, y_n) - a_5(a_x\, o_x + a_y\, o_y)\}$

$b_2 = a_5\{(a_x\, x_n - a_y\, y_n) + (a_y\, y_p - a_x\, x_p) + 4a_5(o_x^2 + o_y^2)\}$

$b_3 = 2a_5\{(o_x\, x_p - o_y\, y_p) + a_5(a_x\, o_x + a_y\, o_y)\}$

$b_4 = a_5(a_x\, x_p - a_y\, y_p)$

$b_5 = XX \cdot x_n - YY \cdot y_n$

$b_6 = 2a_5(XX \cdot o_x + YY \cdot o_y)$

$b_7 = XX \cdot x_n - YY \cdot y_n + XX \cdot x_p - YY \cdot y_p$

$b_8 = 2a_5(XX \cdot o_x + YY \cdot o_y)$

$b_9 = XX \cdot x_p - YY \cdot y_p$

(2) $\overline{cc}_i = b_i + b_{i+5} \qquad (i = 0, \ldots, 4)$

(3) $do_1 = a_5 a_z - ZZ$

$do_2 = -2a_5 o_z$

$do_3 = -(a_5 a_z + ZZ)$

(4) $$\sum_{i=0}^{4} d_i\, t_i = \left(\sum_{i=0}^{3} do_i\, t^{i-1}\right)^2$$

d_i is computed from Subroutine CFSET. $(i = 0, \ldots, 4)$

(5) $$\sum_{i=0}^{8} e_i\, t^i = \left(\sum_{i=0}^{4} \overline{cc}_i\, t^i\right)^2$$

Coefficient e_i is computed from Subroutine CFSET.

(6) $f_0 = x_n^2 + y_n^2$

$f_1 = 4a_5(o_x\, x_n - o_y\, y_n)$

$f_2 = 2(x_n\, x_p + y_n\, y_p) + 4a_5^2(o_x^2 + o_y^2)$

$f_3 = 4a_5(o_x\, x_p - o_y\, y_p)$

$f_4 = x_p^2 + y_p^2$

(7) $\sum_{i=0}^{8} g_i\ t^i = \left(\sum_{i=0}^{4} d_i\ t^i\right)\left(\sum_{i=0}^{4} f_i\ t^i\right)$

Coefficient g_i is computed from Subroutine CFSET. $(i = 0, \ldots, 8)$

(8) $h_0 = f_0$

$h_1 = f_1$

$h_2 = f_2 + 2f_0$

$h_3 = f_3 + 2f_1$

$h_4 = f_0 + 2f_2 + f_4$

$h_5 = f_1 + 2f_3$

$h_6 = f_2 + 2f_4$

$h_7 = f_3$

$h_8 = f_4$

(9) $J_i = e_i + g_i - ah_i$ $\qquad (i = 0, \ldots, 8)$ $\quad (a = a_2{}^2 - a_{34}{}^2)$

(10) $I_0 = o_y\ y_n - o_x\ x_n$

$I_1 = -2a_5(o_x{}^2 + o_y{}^2) + 2(a_y\ y_n - a_x\ x_n)$

$I_2 = o_y\ y_p - o_x\ x_p + o_x\ x_n - o_y\ y_n - 4a_5(a_x\ o_x + a_y\ o_y)$

$I_3 = 2a_5(o_x{}^2 + o_y{}^2) + 2(a_y\ y_p - a_x\ x_p)$

$I_4 = o_x\ x_p - o_y\ y_p$

(11) $\left(\sum_{i=0}^{8} I_{2i}\ t^i\right) = \left(\sum_{i=0}^{4} I_i\ t^i\right)^2$

I_{2i} is computed from Subroutine CFSET. $(i = 0, \ldots, 8)$

(12) $\left(\sum_{i=0}^{6} II_i\ t^i\right) = \left(\sum_{i=0}^{4} I_i\ t^i\right)\left(\sum_{i=0}^{3} do_i\ t^{i-1}\right)$

II_i is computed from Subroutine CFSET. $(i = 0, \ldots, 6)$

(13) $K_0 = -o_z \cdot \overline{cc}_0$

$K_1 = -(o_z \cdot \overline{cc}_1 + 2a_z \cdot \overline{cc}_0)$

$K_2 = o_z \cdot \overline{cc}_0 - 2a_z \cdot \overline{cc}_1 - o_z \cdot \overline{cc}_2$

$K_3 = o_z \cdot \overline{cc}_1 - 2a_z \cdot \overline{cc}_2 - o_z \cdot \overline{cc}_3$

$K_4 = o_z \cdot \overline{cc}_2 - 2a_z \cdot \overline{cc}_3 - o_z \cdot \overline{cc}_4$

$K_5 = o_z \cdot \overline{cc}_3 - 2a_z \cdot \overline{cc}_4$

$K_6 = o_z \cdot \overline{cc}_4$

(14) $PI_i = K_i + II_i \qquad (i = 0, \ldots, 6)$

(15) $LL_0 = o_z^2$

$LL_1 = 4a_z \cdot o_z$

$LL_2 = 4a_z{}^2 - 2o_z{}^2$

$LL_3 = -4a_z \cdot o_z$

$LL_4 = o_z{}^2$

(16) $$\sum_{i=0}^{8} PL_i\, t^i = \left(\sum_{i=0}^{4} LL_i\, t^i\right)\left(\sum_{i=0}^{4} f_i\, t^i\right)$$

Coefficient PL_i is computed from Subroutine CFSET. $(i = 0, \ldots, 8)$

(17) $L_i = I_{2i} + PL_i \qquad (i = 0, \ldots, 8)$

(18) $$\left(\sum_{i=0}^{16} N_i\, t^i\right) = \left(\sum_{i=0}^{8} J_i\, t^i\right)^2$$

N_i is computed from Subroutine CFSET. $(i = 0, \ldots, 16)$

(19) $$\left(\sum_{i=0}^{12} m_i\, t^i\right) = \left(\sum_{i=0}^{6} PI_i\, t^i\right)^2 \times 4a_{34}{}^2$$

m_i is computed from Subroutine CFSET. $(i = 0, \ldots, 12)$

(20) $$\sum_{i=0}^{12} f_i\, t^i = (1+t^2)^2 \left(\sum_{i=0}^{4} f_i\, t^i\right)^2$$

f_i $(i = 0, \ldots, 12)$ is calculated from CFSET routine.

(21) $$\sum_{i=0}^{24} p_i\ t^i = \left(\sum_{i=0}^{12} m_i\ t^i\right)\left(\sum_{i=0}^{12} f_i\ t^i\right)$$

$$\sum_{i=0}^{24} q_i\ t^i = \left(\sum_{i=0}^{8} L_i\ t^i\right)\left(\sum_{i=0}^{16} N_i\ t^i\right)$$

P_i and q_i are computed from CFSET routine. $(i = 0, \ldots, 24)$

(22) $r_i = p_i - q_i \qquad (i = 0, \ldots, 24)$