JAERI-M
86-180

EXACT SOLUTION OF THE INVERSE PROBLEM FOR A SIX-LINK MANIPULATOR WITH A MECHANICAL OFFSET

December 1986

Shinobu SASAKI and Kunihiko NABESHIMA

日 本 原 子 力 研 究 所
Japan Atomic Energy Research Institute

JAERI-Mレポートは，日本原子力研究所が不定期に公刊している研究報告書です。
入手の問合わせは，日本原子力研究所技術情報部情報資料課（〒319-11 茨城県那珂郡東海村）あて，お申しこしください。なお，このほかに財団法人原子力弘済会資料センター（〒319-11 茨城県那珂郡東海村日本原子力研究所内）で複写による実費頒布をおこなっております。

JAERI-M reports are issued irregularly.
Inquiries about availability of the reports should be addressed to Information Division Department of Technical Information, Japan Atomic Energy Research Institute, Tokaimura, Naka-gun, Ibaraki-ken 319-11, Japan.

編集兼発行　日本原子力研究所
印　　刷　日青工業株式会社

Exact Solution of the Inverse Problem for a Six-Link Manipulator with a Mechanical Offset

Shinobu SASAKI and Kunihiko NABESHIMA

Department of Reactor Engineering,
Tokai Research Establishment
Japan Atomic Energy Research Institute
Tokai-mura, Naka-gun, Ibaraki-ken

(Received November 18, 1986)

In this paper, an algebraic approach to derive a joint solution for a six-link manipulator with a mechanical offset is proposed. Generally, derivation of exact joint solutions for such a multi-link manipulator is considered to be an extremely cumbersome problem and a practical approach might be limited to numerical approximation based on linearization and iteration. Compared with such a conventional method, the polynomial expression presented in this paper permits to determine all feasible solutions of the inverse kinematics of the manipulator.

An algorithm of the solution for a six-link manipulator was implemented in the computer code ARM3. Joint behaviors with the mechanical offset were obtained with sufficient accuracies.

Keywords: Six-Link Manipulator, Mechanical Offset, Inverse Kinematics, Computer Code ARM3

機械的オフセットをもつ6リンク・マニプレータに対する逆問題の厳密解

日本原子力研究所東海研究所原子炉工学部
佐々木　忍・鍋島　邦彦

（1986年11月18日受理）

本報告書は，機械的なオフセットをもった6リンク・マニプレータの関節解を誘導する代数的手法を提案している。一般に，このような多リンク構造のマニプレータに対する厳密な関節解の導出は極めて困難な問題と考えられ，実用的な方法としては，線形化反復法を中心とした数値近似に限定される。通常の手法に比べ，ここに示した多項式表示法は，マニプレータ逆運動学の可能解を全て与えることができる。

6リンク・マニプレータに対する解法のアルゴリズムは，計算コードARM 3としてプログラム化した。機械的なオフセットを伴った関節挙動は十分な精度で得られた。

東海研究所：〒319-11　茨城県那珂郡東海村白方白根2－4

Contents

目　　次

List of Figures

---

*) denotes analytical results without a mechanical offset structure.

1. Introduction

The study of kinematics of a robot manipulator is primarily concerned with the mathematical correspondence between the joint variables and Cartesian ones related to a spatial linkage mechanism.[(1)~(5)] As a rule, the transformation from the joint space into the Cartesian space, the so called " direct problem " is unique and easy to compute even for a six-link manipulator. On the other hand, though the solution of the inverse kinematics regarded as a non-linear mapping from the Cartesian into the joint co-ordinate space is one of the most important problem in the manipulator control, it is often intractable to derive except by numerical method based on linearization and iteration techniques (for instance, Jacobian method), which is currently a most common way to derive the individual joint solutions given the position and orientation at the end-point of a manipulator.

Admitting that the inverse Jacobian method may be certainly sufficient for practical purposes, however, this approach has intrinsic shortcom ngs in the sense that a solution obtained is highly dependent on the initial data (i.e. initial guess values) and a large deviation of the initial guess values from the exact ones is not allowed in obtaining a reliable joint solution due to the essential properties of linearization . Additionally, the problem of singularity may be a serious obstacle to the execution of the inverse Jacobian.[(6)]

Thus, generic and explicit analytic solutions for this problem seem impossible to obtain but for some special arm configurations, since kinematic equations are composed of transcendental functions of multi-variables.

Considering such circumstances, we have previously introduced how to solve the inverse problems from a different viewpoint apart from the traditional linearization techniques.[(6)] Its underlying notion was to transform the kinematic relationships into a non-linear polynomial with a single angular variable.

Since the roots of the equation are favorably correspondent to the solutions of the joint angle, they give a clue to extract at the same time all feasible solutions latent within kinematic relationships. In that sense, this approach is helpful to study every possible configuration of arm.
So far, the validity and usefulness of this method have been verified in terms of applying to a few different types of manipulators.[(7)~(9)]

The purpose of this paper is to extend the polynomial model so as to apply to a six-link manipulator having a mechanical offset and thereby to obtain relevant solutions of the inverse kinematics.

In the sections that follow, we present a schematic of its algorithm and several test calculations.

2. Mathematical Model Description

2.1 Representation of Kinematic Equations

On the basis of the orthogonal co-ordinates systems assigned at individual links of a manipulator, the kinematic relationships are first represented as a function of the joint angle variables using a simple matrix operation. By and large, the homogeneous matrix T_j, which specifies the position and orientation of the j th co-ordinate frame with reference to the supporting base (i.e. world or absolute co-ordinates system) is the chain product of successive co-ordinate transformation matrices of A_k, represented by:

$$T_j = \prod_{k=1}^{j} A_k \quad ; \quad \text{for } j = 1,2,\ldots,n. \qquad (1)$$

where A_k is a 4 by 4 tranformation matrix relating the k th homogeneous co-ordinate representation $p_k = (x_k, y_k, z_k, 1)^T$ to the k-1th one $p_{k-1} = (x_{k-1}, y_{k-1}, z_{k-1}, 1)^T$. Namely, $p_{k-1} = A_k\, p_k$.

For a manipulator with six links (j=6), the location and orientation of the end-point are determined in terms of six geometrical transformations matrices.

$$T_6 = A_1A_2A_3A_4A_5A_6 \qquad (2)$$

This matrix T_6 is employed so frequently in the analysis of a manipulator kinematics. Figure 1 shows a sketch of the manipulator to be studied here. Once the link co-ordinate systems are established for each link, kinematic equations in question can easily be developed using the A_k-matrix.

$a_1 = 700(\mathrm{mm})$

$a_2 = 500(\mathrm{mm})$

$a_3 + a_4 = 350(\mathrm{mm})$

$a_5 = 150(\mathrm{mm})$

$a_6 = 280(\mathrm{mm})$

$b = 115(\mathrm{mm})$

$-200 \leq \theta_1 \leq 200(\mathrm{deg})$

$-30 \leq \theta_2 \leq 120(\mathrm{deg})$

$-90 \leq \theta_3 \leq 90(\mathrm{deg})$

$-180 \leq \theta_4 \leq 180(\mathrm{deg})$

$-90 \leq \theta_5 \leq 90(\mathrm{deg})$

$-60 \leq \theta_6 \leq 60(\mathrm{deg})$

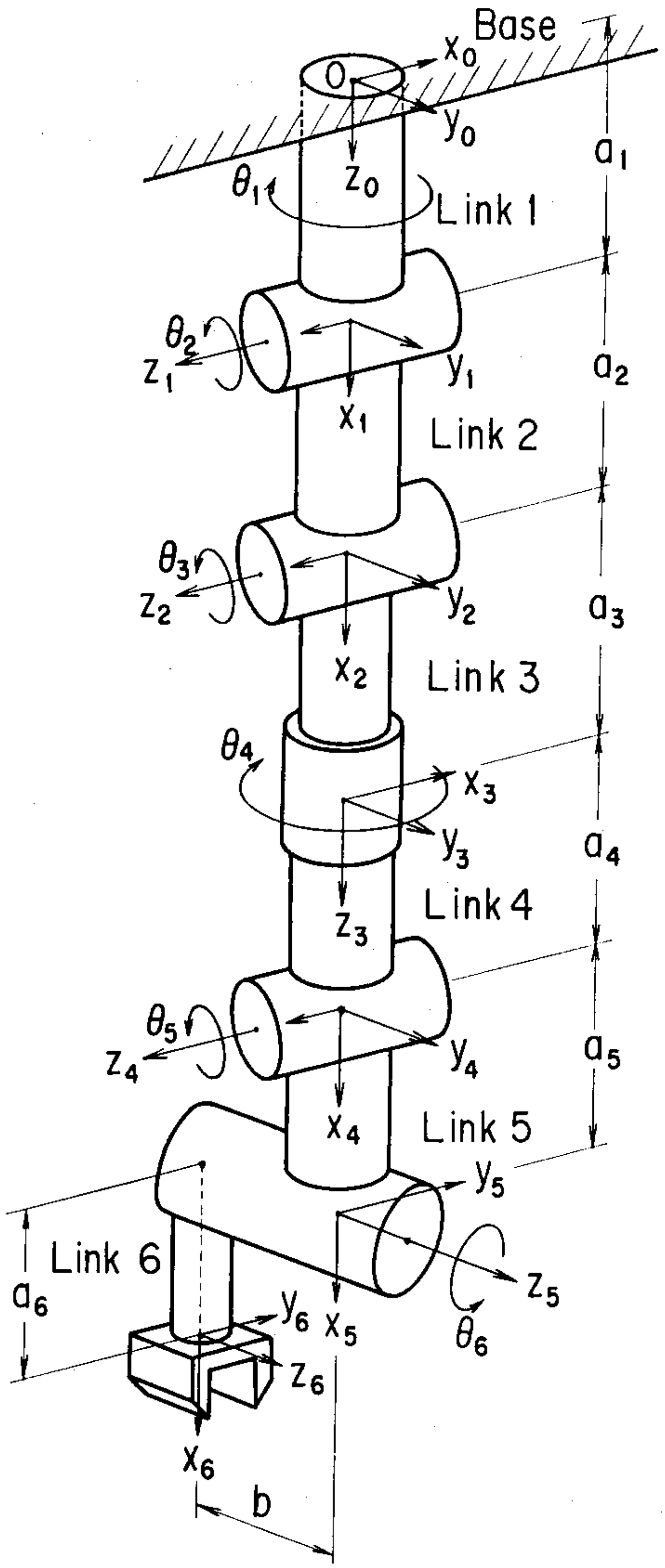

Fig.1 A Six-Link Manipulator and Link Co-ordinates Systems

Now we enter the matrix calculation between individual link co-ordinates systems. Throughout its derivation, the following assumptions and notations are made.

(1) The connection between links (joints) has only one degree of freedom, either rotational or revolute.

(2) The motion of revolute or rotational joints around the z-axis follows the " right screw rule " .

(3) A right handed system is used to represent the co-ordinate system of each link as shown in Fig.1.

(4) The symbols used throughout the derivation are :

(i) A transformation corresponding to rotations or revolutions around the x, y, z axes by an angle θ

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$Rot(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

(ii) A transformation representing translations by p, q, and r along the x, y, and z axes , respectively

$$Trans(p,q,r) = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

(iii) Abbreviations

$$c_i = \cos\theta_i; \quad s_i = \sin\theta_i; \quad c_{ij} = \cos(\theta_i + \theta_j); \quad s_{ij} = \sin(\theta_i + \theta_j) \tag{7}$$

Concerning the linkage relation of the manipulator depicted in Fig. 1, the geometrical structure of the joint 1 through joint 5 is the completely same as that reported in the previous analysis.(7) Hence, no explanation of their derivation on the individual co-ordinate transformation matrices from the base to the fifth joint is repeated and only the result of each matrix ($A_1 \sim A_4$) is shown. Since the focus in the present study is put in the offset mechanism between the link 5 and 6, the co-ordinate transformation matrices A_5 and A_6 are newly described.

$$A_1 = Rot(z_0,\theta_1)Trans(0,0,a_1)Rot(y_0,-\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -s_1 & -c_1 & 0 \\ 0 & c_1 & -s_1 & 0 \\ 1 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (8)$$

$$A_2 = Rot(z_1,\theta_2)Trans(a_2,0,0)$$

$$= \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (9)$$

$$A_3 = Rot(z_2,\theta_3)Trans(a_3,0,0)Rot(y_2,\tfrac{\pi}{2})$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -s_3 & c_3 & a_3c_3 \\ 0 & c_3 & s_3 & a_3s_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (10)$$

$$A_4 = Rot(z_3, \theta_4) Trans(0,0,a_4) Rot(y_3, -\frac{\pi}{2})$$

$$= \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -s_4 & -c_4 & 0 \\ 0 & c_4 & -s_4 & 0 \\ 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (11)$$

(1) A_5-matrix

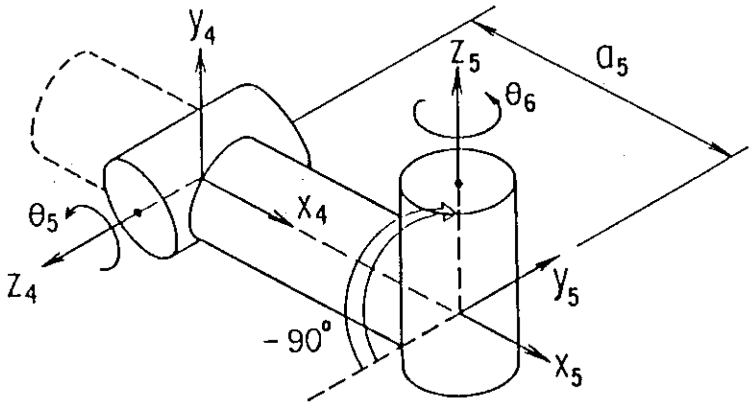

Fig. 2 Schematic of Link 4 to Link 5

From the illustration of Fig. 2, the upward and downward movement of the wrist parts, are realized by the combination of : a rotation around the z_4-axis by the angle θ_5, a translation by a_5 along the x_4-direction, and a rotation around the x_4-axis by - 90(deg). Hence, we obtain a new co-ordinate system (x_5, y_5, z_5) and A_5-matrix.

$$A_5 = Rot(z_4, \theta_5) Trans(a_5,0,0) Rot(x_4, -\frac{\pi}{2})$$

$$= \begin{pmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_5 & -s_5 & 0 & a_5c_5 \\ s_5 & c_5 & 0 & a_5s_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_5 & 0 & -s_5 & a_5c_5 \\ s_5 & 0 & c_5 & a_5s_5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (12)$$

(2) A_6-matrix

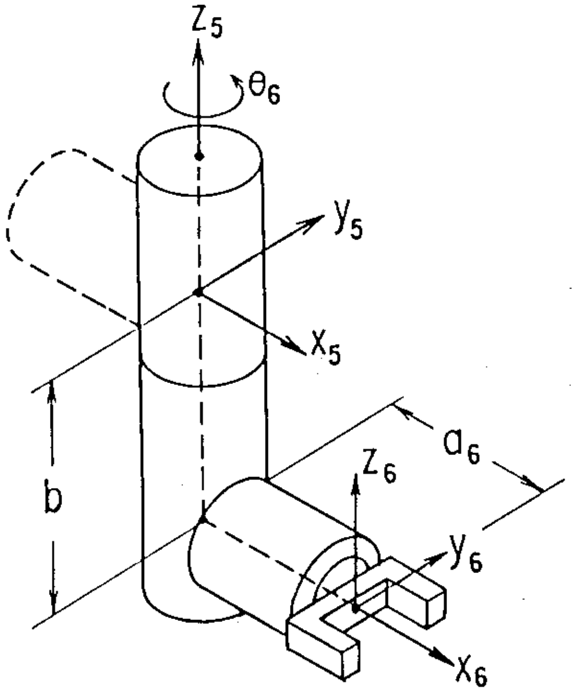

Fig. 3 Schematic of Link 5 to Final Link 6 with End-Effector

Referring to Fig. 3, a geometrical description includes : a rotation (i.e. left and right movement) of θ_6 in the final joint, a translation by b along the negative direction of the z_5-axis corresponding to the mechanical offset and furthermore, a translation by a_6 along the last link with an end-effector (or hand).

Thereby, the hand co-ordinate system (x_6, y_6, z_6) is assigned at the base of the end-effector.

$$A_6 = Rot(z_5,\theta_6)Trans(0,0,-b)Trans(a_6,0,0)$$

$$= \begin{pmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -b \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_6 & -s_6 & 0 & a_6c_6 \\ s_6 & c_6 & 0 & a_6s_6 \\ 0 & 0 & 1 & -b \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (13)$$

Now that we specified the individual A-matrices for a serial link manipulator, six chain products of these homogeneous transformations are postmultiplied successively. In other words,

$$A_1A_2 = \begin{pmatrix} 0 & -s_1 & -c_1 & 0 \\ 0 & c_1 & -s_1 & 0 \\ 1 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -s_1s_2 & -s_1c_2 & -c_1 & -a_2s_1s_2 \\ s_2c_1 & c_1c_2 & -s_1 & a_2c_1s_2 \\ c_2 & -s_2 & 0 & a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

$A_1A_2A_3$

$$= \begin{pmatrix} -s_1s_2 & -s_1c_2 & -c_1 & -a_2s_1s_2 \\ s_2c_1 & c_1c_2 & -s_1 & a_2c_1s_2 \\ 0 & -s_2 & 0 & a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -s_3 & c_3 & a_3c_3 \\ 0 & c_3 & s_3 & a_3c_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & s_1s_2s_3-s_1c_2c_3 & -s_1s_2c_3-s_1c_2s_3 & -a_3s_1s_2c_3-a_3s_1c_2s_3-a_2s_1s_2 \\ s_1 & -s_2s_3c_1+c_1c_2c_3 & s_2c_1c_3+c_1c_2s_3 & a_3s_2c_1c_3+a_3c_1c_2s_3+a_2c_1s_2 \\ 0 & -c_2s_3-s_2c_3 & c_2c_3-s_2s_3 & a_3c_2c_3-a_3s_2s_3+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & -s_1c_{23} & -s_1s_{23} & -a_3s_1s_{23}-a_2s_1s_2 \\ s_1 & c_1c_{23} & c_1s_{23} & a_3c_1s_{23}+a_2c_1s_2 \\ 0 & -s_{23} & c_{23} & a_3c_{23}+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

$A_1A_2A_3A_4$

$$= \begin{pmatrix} c_1 & -s_1c_{23} & -s_1s_{23} & -a_3s_1s_{23}-a_2s_1s_2 \\ s_1 & c_1c_{23} & c_1s_{23} & a_3c_1s_{23}+a_2c_1s_2 \\ 0 & -s_{23} & c_{23} & a_3c_{23}+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -s_4 & -c_4 & 0 \\ 0 & c_4 & -s_4 & 0 \\ 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -s_1s_{23} & -s_4c_1-s_1c_4c_{23} & -c_1c_4+s_1s_4c_{23} & -a_4s_1s_{23}-a_3s_1s_{23}-a_2s_1s_2 \\ c_1s_{23} & -s_1s_4+c_1c_4c_{23} & -s_1c_4-c_1c_{23}s_4 & a_4c_1s_{23}+a_3c_1s_{23}+a_2c_1s_2 \\ c_{23} & -c_4s_{23} & s_4s_{23} & a_4c_{23}+a_3c_{23}+a_2c_2+a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

$A_1A_2A_3A_4A_5$

$$
= \begin{pmatrix}
-S_1S_{23} & -S_4C_1-S_1C_4C_{23} & -C_1C_4+S_1S_4C_{23} & -a_4S_1S_{23}-a_3S_1S_{23}-a_2S_1S_2 \\
C_1S_{23} & -S_1S_4+C_1C_4C_{23} & -S_1C_4-C_1C_{23}S_4 & a_4C_1S_{23}+a_3C_1S_{23}+a_2C_1S_2 \\
C_{23} & -C_4S_{23} & S_4S_{23} & a_4C_{23}+a_3C_{23}+a_2C_2+a_1 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\times \begin{pmatrix}
C_5 & 0 & -S_5 & a_5C_5 \\
S_5 & 0 & C_5 & a_5S_5 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-S_1(S_{23}C_5+C_4C_{23}S_5)-S_4S_5C_1 & C_1C_4-S_1S_4C_{23} & S_1S_5S_{23}-S_4C_1C_5-S_1C_4C_{23}C_5 \\
C_1(C_5S_{23}+C_4S_5C_{23})-S_1S_4S_5 & S_1C_4+C_1C_{23}S_4 & -C_1S_{23}S_5-S_1S_4C_5+C_1C_4C_5C_{23} \\
C_{23}C_5-C_4S_{23}S_5 & -S_4S_{23} & -C_{23}S_5-C_4S_{23}C_5 \\
0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{matrix}
-a_5(C_5S_{23}+S_5C_4C_{23})S_1-a_5S_5S_4C_1-(a_3+a_4)S_1S_{23}-a_2S_1S_2 \\
a_5(C_5S_{23}+S_5C_4C_{23})C_1-a_5S_5S_4S_1+(a_3+a_4)C_1S_{23}+a_2C_1S_2 \\
a_5C_5C_{23}-a_5S_5C_4S_{23}+(a_3+a_4)C_{23}+a_2C_2+a_1 \\
1
\end{matrix} \qquad (17)
$$

$$
T_6 = A_1A_2A_3A_4A_5A_6
$$

$$
= \begin{pmatrix}
t_{11} & t_{12} & t_{13} & t_{14} \\
t_{21} & t_{22} & t_{23} & t_{24} \\
t_{31} & t_{32} & t_{33} & t_{34} \\
0 & 0 & 0 & 1
\end{pmatrix} \qquad (18\text{-a})
$$

where

$$
\begin{aligned}
t_{11} &= -s_1(s_{23}c_5c_6+c_{23}c_4c_6s_5+c_{23}s_4s_6)+c_1(c_4s_6-c_6s_4s_5) \\
t_{12} &= s_1(s_{23}s_6c_5+c_{23}c_4s_5s_6-c_{23}c_6s_4)+c_1(c_4c_6+s_4s_5s_6) \\
t_{13} &= s_1(s_{23}s_5-c_{23}c_4c_5)-c_1c_5s_4 \\
t_{14} &= -s_1[a_6(s_{23}c_5c_6+c_{23}c_4c_6s_5+c_{23}s_4s_6)+b(s_5s_{23}-c_4c_5c_{23}) \\
&\qquad +a_5(s_{23}c_5+c_{23}c_4s_5) + (a_3+a_4)s_{23}+a_2s_2] \\
&\qquad +c_1[a_6(c_4s_6-s_4s_5c_6)-a_5s_4s_5+bs_4c_5] \\
t_{21} &= s_1(c_4s_6-s_4s_5c_6)+c_1(s_{23}c_5c_6+c_{23}c_4c_6s_5+c_{23}s_4s_6) \\
t_{22} &= s_1(c_4c_6+s_4s_5s_6)-c_1(s_{23}s_6c_5+c_{23}c_4s_5s_6-c_{23}s_4c_6) \\
t_{23} &= -s_1s_4c_5-c_1(s_{23}s_5-c_{23}c_4c_5)
\end{aligned}
$$

$$t_{24} = s_1[a_6(c_4s_6-s_4s_5c_6)-a_5s_4s_5+bs_4c_5]+c_1[a_6(s_{23}c_5c_6$$
$$+c_{23}c_4c_6s_5+c_{23}s_4s_6)+a_5(s_{23}c_5+c_{23}c_4s_5)+b(s_{23}s_5-c_4c_5c_{23})$$
$$+(a_3+a_4)s_{23}+a_2s_2]$$

$$t_{31} = c_5c_6c_{23}-c_4c_6s_5s_{23}-s_4s_6s_{23}$$

$$t_{32} = -c_5c_{23}s_6+c_4s_5s_6s_{23}-s_4s_{23}c_6$$

$$t_{33} = -s_5c_{23}-c_4c_5s_{23}$$

$$t_{34} = a_6(c_5c_6c_{23}-s_5s_{23}c_4c_6-s_4s_6s_{23})+a_5(c_5c_{23}-c_4s_5s_{23})$$
$$+b(c_{23}s_5+c_4c_5s_{23})+(a_3+a_4)c_{23}+a_2c_2+a_1$$

From this representation, we can cast the position and orientation at the end-point of a manipulator with reference to the base co-ordinate system.

In passing, such kinematic relationships between links can elegantly be derived using another concept based on a spatial vector and rotation operator.[10]

2.2 How to Solve the Inverse Problem

This section presents an algebraic approach to derive a consistent joint angle solution given the matrix T_6 of a manipulator with mechanical offset. A transformation of kinematic expression into a non-linear algebraic equation, the solutions of this decision equation, and a determination of joint angle variables will explained in due order.

2.2.1 Transformation into a Polynomial

Consider the matrix T_6 to be of the form:

$$T_6 = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (18\text{-b})$$

In Eq.(18-b), the left upper 3 by 3 matrix is a rotation matrix made with orthogonal unit vectors - n, o, and a. As depicted in Fig. 3, n is the approach vector of the hand, which indicates the x_6-direction. O is the sliding vector of the hand, which indicates the direction (i.e. y_6-sense) of the finger motion as the finger opens and closes. On the other hand, a is the normal vector of the hand (i.e. in the z_6-direction). Assuming a parallel-jaw hand, it is perpendicular to the fingers of the manipulator. p is the position vector describing the location of the hand in the Cartesian space.

Equating the corresponding elements of the matrix T_6 in Eq.(18-a) and Eq.(18-b), we obtain kinematic expressions of the manipulator given in Fig. 1, where each component of T_6 in Eq.(18-b) is postulated to be known.

$$n_x = -C_6(S_1S_{23}C_5 + S_4S_5C_1 + S_1S_5C_4C_{23}) + S_6(C_1C_4 - S_1S_4C_{23}) \quad (19)$$

$$n_y = C_6(C_1C_5S_{23} - S_1S_4S_5 + C_1C_4C_{23}S_5) + S_6(S_1C_4 + C_1C_{23}S_4) \quad (20)$$

$$n_z = C_6(C_5C_{23} - C_4S_5S_{23}) - S_4S_{23}S_6 \quad (21)$$

$$o_x = C_6(C_1C_4 - S_1S_4C_{23}) + S_6(S_1S_{23}C_5 + S_4S_5C_1 + S_1S_5C_4C_{23}) \quad (22)$$

$$o_y = C_6(S_1C_4 + C_1C_{23}S_4) - S_6(C_1C_5S_{23} - S_1S_4S_5 + C_1C_4C_{23}S_5) \quad (23)$$

$$o_z = -C_6(S_4S_{23}) + S_6(C_4S_5S_{23} - C_5C_{23}) \quad (24)$$

$$a_x = S_1S_5S_{23} - S_1C_4C_5C_{23} - C_1C_5S_4 \quad (25)$$

$$a_y = C_1C_4C_5C_{23} - C_1S_5S_{23} - S_1S_4C_5 \quad (26)$$

$$a_z = -(C_4C_5S_{23} + C_{23}S_5) \quad (27)$$

$$\begin{aligned} P_x = & -a_6C_6(S_1S_{23}C_5 + S_4S_5C_1 + S_1S_5C_4C_{23}) + a_6S_6(C_1C_4 - S_1S_4C_{23}) \\ & -a_5(C_5S_1S_{23} + S_4S_5C_1 + S_1S_5C_4C_{23}) - (a_3 + a_4)S_1S_{23} \\ & -a_2S_1S_2 - b(S_1S_5S_{23} - S_4C_1C_5 - S_1C_4C_5C_{23}) \end{aligned} \quad (28)$$

$$\begin{aligned} P_y = & \ a_6C_6(C_1C_5S_{23} - S_1S_4S_5 + C_1C_4C_{23}S_5) + a_6S_6(S_1C_4 + C_1C_{23}S_4) \\ & +a_5(C_5C_1S_{23} - S_1S_4S_5 + C_1C_4C_{23}S_5) + (a_3 + a_4)C_1S_{23} \\ & +a_2S_2C_1 - b(C_1C_4C_5C_{23} - C_1S_5S_{23} - S_1S_4C_5) \end{aligned} \quad (29)$$

$$\begin{aligned} P_z = & \ a_6C_6(C_5C_{23} - C_4S_5S_{23}) + a_6(-S_4S_{23})S_6 + a_5(C_5C_{23} - S_5S_{23}C_4) \\ & +(a_3 + a_4)C_{23} + a_2C_2 + a_1 + b(C_{23}S_5 + C_4C_5S_{23}) \end{aligned} \quad (30)$$

Here, the last term in P_x, P_y and P_z is a portion caused by the offset structure. Eqs. (25) through (27) are uniquely determined from the orthonormal co-ordinate relation, that is $a = n \times O$. Now, we introduce the following parameters in view of the apparent features of the equations (19) through (24).

$$A = S_1S_{23}C_5 + S_4S_5C_1 + S_1S_5C_4C_{23} \quad (31)$$

$$B = C_1C_4 - S_1S_4C_{23} \quad (32)$$

$$C = C_1C_5S_{23} - S_1S_4S_5 + C_1C_4C_{23}S_5 \quad (33)$$

$$D = S_1C_4 + C_1C_{23}S_4 \quad (34)$$

$$E = C_5C_{23} - C_4S_5S_{23} \quad (35)$$

$$F = -S_4S_{23} \quad (36)$$

$$a_3 + a_4 = a_{34} \quad (37)$$

Consequently, the original equations (19) to (24) are simplified as follows.

$$-A\ c_6 + B\ s_6 = n_x \tag{38}$$
$$C\ c_6 + D\ s_6 = n_y \tag{39}$$
$$E\ c_6 + F\ s_6 = n_z \tag{40}$$
$$B\ c_6 + A\ s_6 = o_x \tag{41}$$
$$D\ c_6 - C\ s_6 = o_y \tag{42}$$
$$F\ c_6 - E\ s_6 = o_z \tag{43}$$

Similarly, the position vector in Eqs. (28), (29) and (30) is represented by:

$$P_x = -a_6 c_6 A + a_6 s_6 B - a_5 A - a_{34} s_1 s_{23} - a_2 s_1 s_2 - b a_x \tag{44}$$
$$P_y = a_6 c_6 C + a_6 s_6 D + a_5 C + a_{34} c_1 s_{23} + a_2 s_2 c_1 - b a_y \tag{45}$$
$$P_z = a_6 c_6 E + a_6 s_6 F + a_5 E + a_{34} c_{23} + a_2 c_2 + a_1 - b a_z \tag{46}$$

As can be noticed from Eqs. (38) through (43), each parameter defined in Eqs. (31) to (36) can be represented by a trigonometric function of only c_6 and s_6. Namely,

$$A = -n_x c_6 + o_x s_6 \tag{47}$$
$$B = n_x s_6 + o_x c_6 \tag{48}$$
$$C = n_y c_6 - o_y s_6 \tag{49}$$
$$D = n_y s_6 + o_y c_6 \tag{50}$$
$$E = n_z c_6 - o_z s_6 \tag{51}$$
$$F = o_z c_6 + n_z s_6 \tag{52}$$

From the equations (44) and (45), we obtain

$$p_x c_1 + p_y s_1 = a_6 (n_x c_1 + n_y s_1) + a_5(-A c_1 + C s_1) - b(a_x c_1 + a_y s_1) \tag{53}$$

On simplification, we have

$$(p_x - a_6 n_x + a_5 A + b a_x) c_1 = (a_6 n_y + a_5 C - p_y - b a_y) s_1\ , \tag{54}$$

that is, $$\tan\theta_1 = \frac{s_1}{c_1} = \frac{p_x - a_6 n_x + a_5 A + b a_x}{a_6 n_y + a_5 C - p_y - b a_y} = \frac{XX + a_5 A}{a_5 C - YY}$$

$$= \frac{a_5 o_x s_6 - a_5 n_x c_6 + XX}{a_5 n_y c_6 - a_5 o_y s_6 - YY} \tag{55}$$

where
$$\begin{cases} XX = p_x - a_6 n_x + b a_x \\ YY = p_y - a_6 n_y + b a_y \end{cases} \tag{56}$$

Let $\tan\frac{\theta_6}{2} = t$. Then $c_6 = \frac{1 - t^2}{1 + t^2}$, $s_6 = \frac{2t}{1 + t^2}$ and $\tan\theta_6 = \frac{2t}{1 - t^2}$

By substituting them into Eq. (55), it holds so that

$$\tan\theta_1 = \frac{a_5 o_x \left(\frac{2t}{1 + t^2}\right) - a_5 n_x \left(\frac{1 - t^2}{1 + t^2}\right) + XX}{a_5 n_y \left(\frac{1 - t^2}{1 + t^2}\right) - a_5 o_y \left(\frac{2t}{1 + t^2}\right) - YY}$$

$$= \frac{x_n + 2 a_5 o_x t + x_p t^2}{y_n - 2 a_5 o_y t + y_p t^2} \tag{57}$$

where
$$\left.\begin{aligned} x_p &= a_5 n_x + XX \\ x_n &= -a_5 n_x + XX \\ y_p &= -(a_5 n_y + YY) \\ y_n &= a_5 n_y - YY \end{aligned}\right\} \tag{58}$$

From Eq. (46) we have

$$\left.\begin{aligned} P_z - a_6 n_z - a_1 + b a_z &= a_5 E + a_{34} c_{23} + a_2 c_2 \\ &= a_5 (n_z c_6 - o_z s_6) + a_{34} c_{23} + a_2 c_2 \\ a_5 c_6 (n_z - o_z \tan\theta_6) - zz + a_{34} c_{23} &= -a_2 c_2 . \\ \text{where} \quad zz = p_z - a_6 n_z - a_1 + b a_z & \end{aligned}\right\} \tag{59}$$

And also Eq. (44) × $(-s_1)$ + Eq. (45) × c_1 reduces to

$$a_5c_6\{(n_yc_1 - n_xs_1) + (s_1o_x - o_yc_1)\tan\theta_6\}$$
$$-\{-p_xs_1 + p_yc_1 + a_6(n_xs_1 - n_yc_1) - b(a_xs_1 - a_yc_1)\} + a_{34}s_{23} = -a_2s_2 \quad (60)$$

As described above, we have obtained two principal equations (59) and (60) from kinematic equations.

From here on, we will make further simplification so that these equations might be unified into one single mathematical relationship. To keep the two expressions short, at first, let us introduce the following notations ψ and η.

$$\psi = a_5c_6(n_z - o_z\tan\theta_6) - zz ,$$

and (61)

$$\eta = a_5c_6\{n_yc_1 - n_xs_1 + (s_1o_x - o_yc_1)\tan\theta_6\}$$
$$-\{-p_xs_1 + p_yc_1 + a_6(n_xs_1 - n_yc_1) - b(a_xs_1 - a_yc_1)\} \quad (62)$$

That is, Eqs. (59) and (60) are rewritten as:

$$\psi + a_{34}c_{23} = -a_2c_2 \quad (63)$$

$$\eta + a_{34}s_{23} = -a_2s_2 \quad (64)$$

Adding the square of Eq. (63) and Eq. (64), we obtain the following simple form.

$$\left.\begin{array}{l} \psi^2 + \eta^2 - a = -2a_{34}(\psi c_{23} + \eta s_{23}) \\ \text{where} \quad a = a_2{}^2 - a_{34}{}^2 \end{array}\right\} \quad (65)$$

After this, expressions of ψ and η in terms of θ_6 are similar to that of the reference (7).

From the marked similarity in subsequent derivation, we come to a conclusion that manipulator kinematic relationships having an offset mechanism can be represented by means of a single polynomial as in the previous model. That is the key point in the present paper. In order to compute real roots of this algebraic equation as exactly as possible, the Bairstow's numerical technique[11] was used in the computer code. (see Appendix 1)

2.2.2 Determination of Articulated Variables

(1) Calculation of θ_6

Once the desired solutions t are found from the algebraic equation a joint angle θ_6 can easily be calculated.

That is, $\tan\frac{\theta_6}{2} = t$

thus, we have

$$\theta_6 = 2\tan^{-1} t \, . \tag{66}$$

Corresponding trigonometric function is

$$s_6 = \sin\theta_6$$

$$c_6 = \cos\theta_6 \quad .$$

Using these values, we can determine the parameters A to F given by Eqs. (47) through (52).

(2) Calculation of θ_1

Let

$$X_1 = XX + a_5 A$$

and $$Y_1 = a_5 C - YY \, ,$$

then, Eq. (55) reduces to

$$X_1 \cdot c_1 = Y_1 \cdot s_1 \quad \text{(See Eq. (56) for XX and YY)}$$

Thus, $$\theta_1 = \tan^{-1}\left(\frac{X_1}{Y_1}\right) \tag{67}$$

$$s_1 = \sin\theta_1$$

$$c_1 = \cos\theta_1$$

(3) Calculation of θ_{23}

From Eq. (65), we have

$$\psi^2 + \eta^2 - a = -2a_{34}(\psi c_{23} + \eta s_{23})$$

$$= -2a_{34}\sqrt{\psi^2 + \eta^2}\,\sin(\theta_{23} + \varepsilon) \, .$$

where $$\varepsilon = \tan^{-1}(\psi/\eta), \quad a = a_2{}^2 - a_{34}{}^2 \tag{68}$$

Thus, we have

$$\sin(\theta_{23}+\varepsilon) = (\psi^2+\eta^2-a)/-2a_{34}\sqrt{\psi^2+\eta^2}$$

$$\cos(\theta_{23}+\varepsilon) = \pm\sqrt{1-\sin^2(\theta_{23}+\varepsilon)}\ .$$

Hence,

$$\tan(\theta_{23}+\varepsilon) = \frac{\sin(\theta_{23}+\varepsilon)}{\cos(\theta_{23}+\varepsilon)}$$

or

$$\theta_{23} = \tan^{-1}\left(\frac{\pm(\psi^2+\eta^2-a)/(-2a_{34}\sqrt{\psi^2+\eta^2})}{\sqrt{1-\{(\psi^2+\eta^2-a)/(-2a_{34}\sqrt{\psi^2+\eta^2})\}^2}}\right) - \tan^{-1}\left(\frac{\psi}{\eta}\right) \qquad (69)$$

$$s_{23} = \sin\theta_{23}$$

$$c_{23} = \cos\theta_{23}$$

(4) Calculation of θ_4

Making use of Eqs. (32) and (34), c_4 and s_4 are described as follows.

$$c_4 = B\ c_1 + D\ s_1$$

$$s_4 = (-B\ s_1 + D\ c_1)/c_{23} \quad (c_{23} \neq 0)$$

if $c_{23} = 0$, we use Eq. (36) to determine s_4.

or $\quad s_4 = -F/s_{23}$

Thus, we have

$$\theta_4 = \tan^{-1}\left(\frac{s_4}{c_4}\right)\ . \qquad (70)$$

(5) Calculation of θ_5

By Eq. (31) × c_1 - Eq. (33) × s_1, we obtain

$$s_5 = (Ac_1 - Cs_1)/s_4$$
$$= \{c_1(-n_x c_6 + o_x s_6) - s_1(n_y c_6 - o_y s_6)\}/s_4 \quad (s_4 \neq 0) \qquad (71)$$

On the other hand, Eq. (31) × s_1 + Eq. (33) × c_1 leads to

$$c_5 s_{23} + s_5 c_4 c_{23} = A\ s_1 + C\ c_1\ . \qquad (72)$$

In the case of $s_4 = 0$, Eq. (95) $\times$ c_{23} - Eq. (35) $\times$ s_{23} results in

$$s_5 = \{c_{23}(A\ s_1 + C\ c_1) - E\ s_{23}\}/c_4\ . \tag{73}$$

By making Eq. (73) $\times$ s_{23} + Eq. (35) $\times$ c_{23}, it holds that

$$c_5 = (A\ s_1 + C\ c_1)s_{23} + E\ c_{23}\ . \tag{74}$$

Hence,

$$\theta_5 = \tan^{-1}(\frac{s_5}{c_5}) \tag{75}$$

(6) <u>Calculation of θ_2</u>

Eqs. (59) and (60) give:

$$c_2 = \{p_z - a_6 n_z - a_1 + b a_z - a_5(n_z c_6 - o_z s_6) - a_{34} c_{23}\}/a_2$$

$$s_2 = \{-p_x s_1 + p_y c_1 + a_6(n_x s_1 - n_y c_1) - b(a_x s_1 - a_y c_1) - a_5(n_y c_1 - n_x s_1)c_6 - a_5(s_1 o_x - o_y c_1)s_6 - a_{34}\ s_{23}\}/a_2$$

Therefore, we have

$$\theta_2 = \tan^{-1}(\frac{s_2}{c_2})\ . \tag{76}$$

(7) <u>Calculation of θ_3</u>

From the calculation of θ_{23} and θ_2, we can obtain

$$\theta_3 = \theta_{23} - \theta_2 \tag{77}$$

3. Test Calculations

As the embodiment of the algorithm mentioned in the preceding section, the computer code ARM3 was developed in order to leave much of the computational drudgery to a digital computer. In this section, the validity of its mathematical model or approach is verified by means of several test runs.

Now, in giving the motion schemes of the finger tip of a manipulator, the spatial trajectory is considered such that it follows a straight line or a circle or some other specific curves. As the simulation of a simple manipulator task, it may be sufficient to consider the first two specifications - straight line path and circle. Here we investigate behaviors of joint angles corresponding to these two motion schemes.

At the outset, one approximation of a simple straight line with the initial point A and the terminal point B is made in terms of points with n equi-distant spaces on this path segment. They start at the initial point ($N = 0$) and end at the $n+1$ th point ($N = n$). The position and orientation at each point along the trajectory is determined by means of a linear interpolation technique, respectively. In addition to this straight line trajectory, we are now in a position to reproduce individual joint angles corresponding to the circular motion of the finger tip. For convenience sake, the circle is assumed to be in any of xy-plane, yz-plane or zx-plane.

As with a line approximation, the circumference is approximated in terms of n points with equal intervals. For a circle with a radius r, centered on M (X_c, Y_c, Z_c), the equation can be written as follows :

$$x = X_c + r\cos$$
$$y = Y_c + r\sin \qquad \text{(a circle on the xy-plane,}$$
$$z = Z_c \qquad \text{located at } z = Z_c \text{)}$$

$x = X_c$
$y = Y_c + r\cos\theta$ (a circle on the yz-plane,
$z = Z_c + r\sin\theta$ located at $x = X_c$)

$x = X_c + r\cos\theta$
$y = Y_c$ (a circle on the zx-plane,
$z = Z_c + r\sin\theta$ located at $y = Y_c$)

where $\theta = (2\pi/n)k$ ($k = 1,\ldots,n$)

n : the number of points present on the circumference

In every case tried here, the orientation of the finger tip was kept unvaried during a movement. The calculated result of each joint is represented as the angular displacement from the home position (θ_i ,$i = 1, \ldots, 6$) of the manipulator shown in Fig. 1.

(I) Sample Problem 1

(a) Horizontal movement of a downward directed end-effector (see Fig. 4)

(b) position co-ordinate of the initial point A

(-0.10,0.35,1.63) (m in unit)

(c) position co-ordinate of the terminal point B

(0.10,0.35,1.63) (m in unit)

(d) the number of points (position numbers) = 41

(e) direction cosines

$n_x = 0.0$; $n_y = 0.0$; $n_z = 1.0$

$o_x = 1.0$; $o_y = 0.0$; $o_z = 0.0$

(f) convergence condition

EPS = 10^{-4}, EPS1 = 10^{-3}

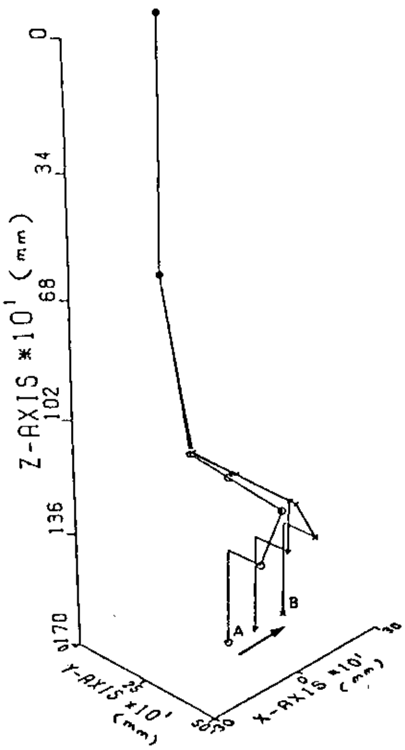

Fig. 4 Motion Scheme for Sample Problem 1

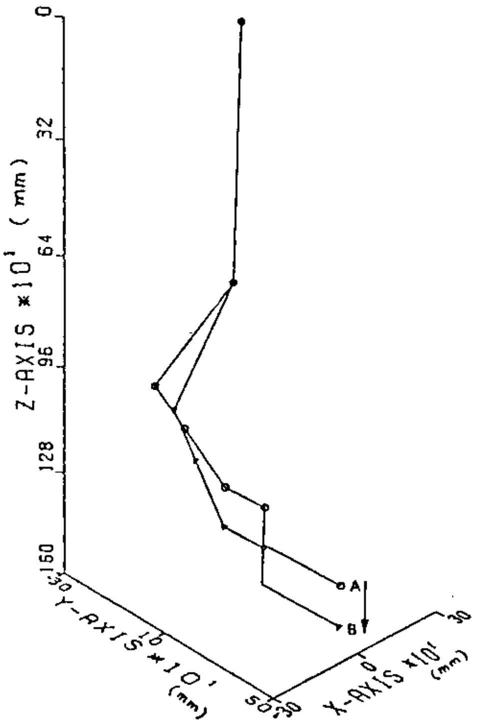

Fig. 5 Motion Scheme for Sample Problem 2

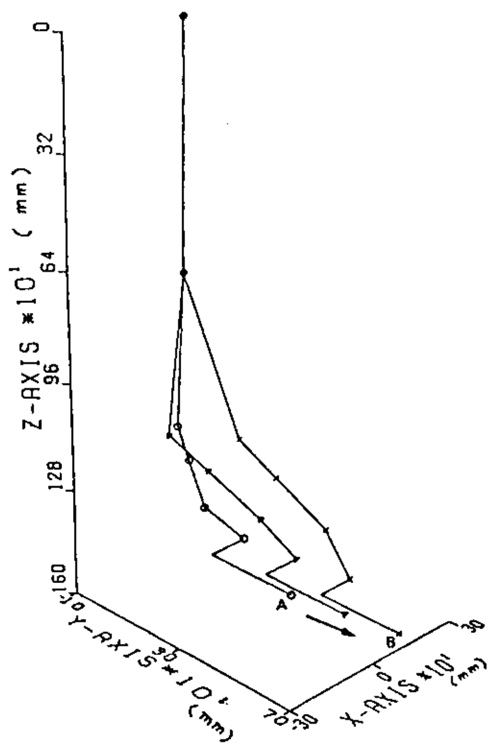

Fig. 6 Motion Scheme for Sample Problem 3

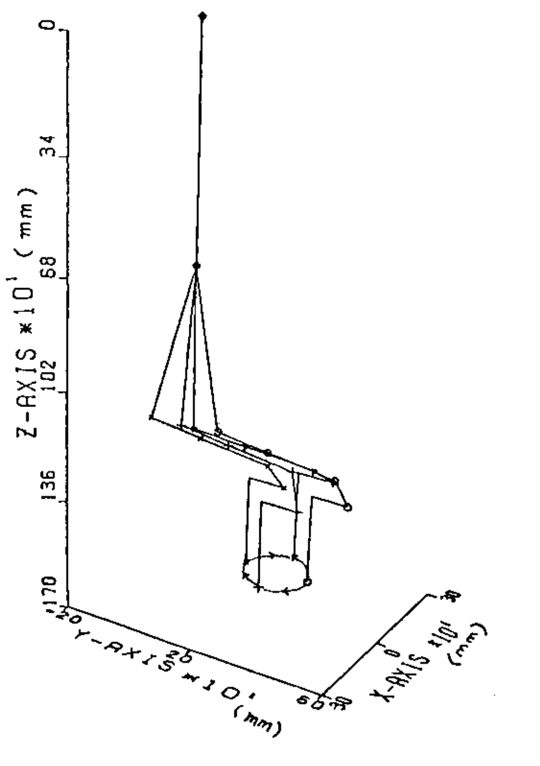

Fig. 7 Motion Scheme for Sample Problem 4

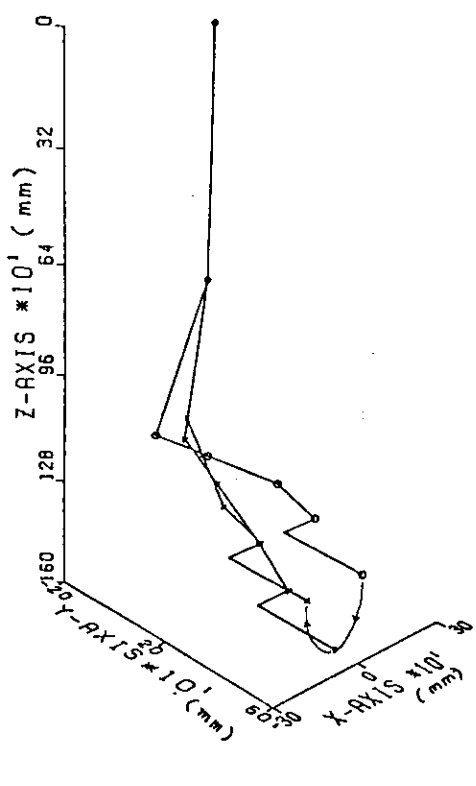

Fig. 8 Motion Scheme for Sample Problem 5

As can be noticed in Fig. 4, each configuration of the manipulator is identified using a special symbolic mark such as circle, cross and triangle. The positions described with the same symbol are indicative of the respective origins (i.e. the spatial positions of individual joints in the six-link manipulator) of seven link co-ordinates frames given in Fig. 1.

Now, the calculated joint angles under the above motion scheme are depicted in Figs. 4.1.1 through 4.6.1. In the first example applied, fourty one intervals (4.88 mm each) were assigned between the initial and terminal points. For the two kinds of solutions of θ_6 in Fig. 4.6.1, the remaining joint angle solutions had six possible combinations as designated by the symbols G_1 through G_6. In these curves, groups G_2 and G_4 indicate the complete solution curves corresponding to the line motion of the finger tip from the starting point to the end, where the variation width of each joint angle is tabulated as follows.

	G_2	G_4
θ_1 (deg):	-3.14 ~ 3.14	-12.22 ~ 12.22
θ_2 (deg):	13.09 ~ 13.33	72.54 (nearly const.)
θ_3 (deg):	71.20 ~ 74.78	-74.78 ~ -73.07
θ_4 (deg):	-29.96 ~ 29.96	-12.22 ~ 12.22
θ_5 (deg):	-87.87 ~ -83.69	0.51 ~ 2.27
θ_6 (deg):	-29.81 ~ 29.81	-0.11 ~ 0.11

Speaking of characteristics between the two, we notice that the variation of the finger tip (θ_6) in G_2 was significant, reflected in the value of the angle θ_4.

In the calculations of the angle θ_1, a phase difference of $\pm$180(deg) relating to the group G_2 has brought another groups G_1 and G_3, both of which terminated at the position number 21 due to the mechanical upper limit of θ_4 as can be demonstrated in Fig. 4.4.1, however.

After passing this point, groups G_5 and G_6 showed up as new curves to meet the lower limit of θ_4 in place of G_1 and G_3.

In order to investigate the features of angular behaviors between a mechanical offset and non-offset structure, comparisons were attempted by adding Figs. 4.1.2 through 4.6.2, which are indicative of analytical results without the offset mechanism. For the identical spatial movement of the finger tip, the number of real roots of a polynomial (i.e. θ_6) was the same between the two cases. And the entire joint response had almost the same trends, although their differences were identified to some extent in magnitude.

(II) Sample Problem 2

(a) vertical movement of the finger tip (see Fig. 5)

(b) position co-ordinate of the initial point A

(0.00,0.43,1.35) (m in unit)

(c) position co-ordinate of the terminal point B

(0,00,0.43,1.55) (m in unit)

(d) the number of points (position numbers) = 40

(e) direction cosines

$n_x = 0.0$; $n_y = 1.0$; $n_z = 0.0$

$o_x = 1.0$; $o_y = 0.0$; $o_z = 0.0$

(f) convergence condition

EPS = 10^{-4}, EPS1 = 10^{-3}

Secondly, vertical movement of the finger tip with a constant orientation is treated. The results of the present test runs are shown in Figs. 5.1.1 through 5.6.1. As seen, no desired joint solutions were found at the first portion of the interval AB due to the prescribed mechanical constraints of θ_3. After a position advancement of 7.5 cm in the z-direction, there existed feasible solutions to the end.

For a single solution of $\theta_6 = 0$, the remaining joint solutions possess three curves G_1, G_2 and G_3, which were determined from the possibility of θ_1 within the operation range. Notice that the graph G_3 was unavailable before the condition of the lower limit value -30(deg) of θ_2 was satisfied.

In a similar manner, results without the present offset model are demonstrated in Figs. 5.1.2 through 5.6.2. In this case, there existed complete solution sets G_1 and G_2 to meet the specified conditions of individual joints. Including the partial solutions, the absolute values of solutions curves G_1, G_2 and G_3 were small through the available intervals, as compared as those of the offset model.

(III) Sample Problem 3

(a) movement along any line trajectory (see Fig. 6)

(b) position co-ordinate of the initial point A

(-0.25,0.65,1.30) (m in unit)

(c) position co-ordinate of the terminal point B

(0.10,0.70,1.55) (m in unit)

(d) the number of points (position numbers) = 50

(e) direction cosines

$n_x = 0.0$; $n_y = 1.0$; $n_z = 0.0$

$o_x = 0.0$; $o_y = 0.0$; $o_z = 1.0$

(f) convergence condition

EPS = 10^{-4}, EPS1 = 10^{-3}

In the successive computation, the orientation of the finger tip is given so that joint 1 or joint 4 may revolve by -90(deg) around the z-axis. With this sense maintained, the finger-tip of the manipulator moves at intervals of 8.6 (mm) between the two points (a distance of about 43 cm) in the work space.

Calculated results of the ARM3 are shown in Figs. 6.1 through 6.6. A complete solution curve G_2 was obtained as a simulation for the above motion scheme. The remaining two solution graphs G_1 and G_3 are the partial solutions feasible found in the calculation process of θ_1.

(IV) Sample Problem 4

(a) circle movement on the xy-plane (see Fig. 7)

(b) position co-ordinate of the center point M

(0.15,0.20,1.68) (m in unit)

(c) a radius of circle r = 0.10 (m)

(d) the number of points = 60

(e) direction cosines

$n_x = 0.0$; $n_y = 0.0$; $n_z = 1.0$

$o_x = 1.0$; $o_y = 0.0$; $o_z = 0.0$

(f) convergence condition

EPS = 10^{-4}, EPS1 = 10^{-3}

Figures 7.1 through 7.6 show behaviors of individual joint angles while the finger-tip of the manipulator moves along a circumference on the xy-plane. Needless to say, a determination of reasonable solutions with a constant orientation is quite dependent on the input data associated with center point M or radius. In some cases there exist a number of feasible angular solutions and in other cases inconsistent solutions. Shown is only one example of them.

At the present sample calculation, a circumference was divided into 60 segments and we have obtained three feasible solution groups - G_1, G_2 and G_3. Seeing the computed results of θ_5 and θ_6, plotting curves G_1 and G_2 were in complete agreement. (see Figs. 7.5 and 7.6)

In addition to these three complete solution sets, the calculation predicted two partial solutions. Behaviors of the partial solutions were omitted to avoid the complicatedness.

(V) Sample Problem 5

(a) circle movement on the zx-plane (see Fig. 8)

(b) position co-ordinate of the center point M

(-0.10,0.60,1.40) (m in unit)

(c) a radius of circle $r = 0.12$ (m)

(d) the number of points = 60

(e) direction cosines

$n_x = 0.0$; $n_y = 1.0$; $n_z = 0.0$

$o_x = 0.0$; $o_y = 0.0$; $o_z = 1.0$

(f) convergence condition

EPS = 10^{-4}, EPS1 = 10^{-3}

Finally, computed results of individual joint angles for the sample problem 5 are demonstrated in Figs. 8.1 through 8.6, which are solutions of the inverse kinematics corresponding to a circular trajectory drawn in the zx-plane. As exemplified in figures, any group of angular solutions was not sufficient for reproducing the entire circular motion. On inquiry of solution behaviors, we found that there were no relevant solutions between the two points (-0.19, 0.60, 1.33) and (0.00,0.60, 1.32) along the circumference. This is merely responsible for the restraints of the angle θ_3. Provided that this angle was set to any value larger than the prescribed one 90 (deg) of the mechanical upper limit, groups G_2 and G_5 would have completed a single feasible curve. Further, it is possible to unify groups G_1 and G_4 into a continuous curve seemingly by widening the specified range of both θ_1 and θ_3, with the consequence that we will obtain another profile of joint angles ($\theta_1 \sim \theta_6$).

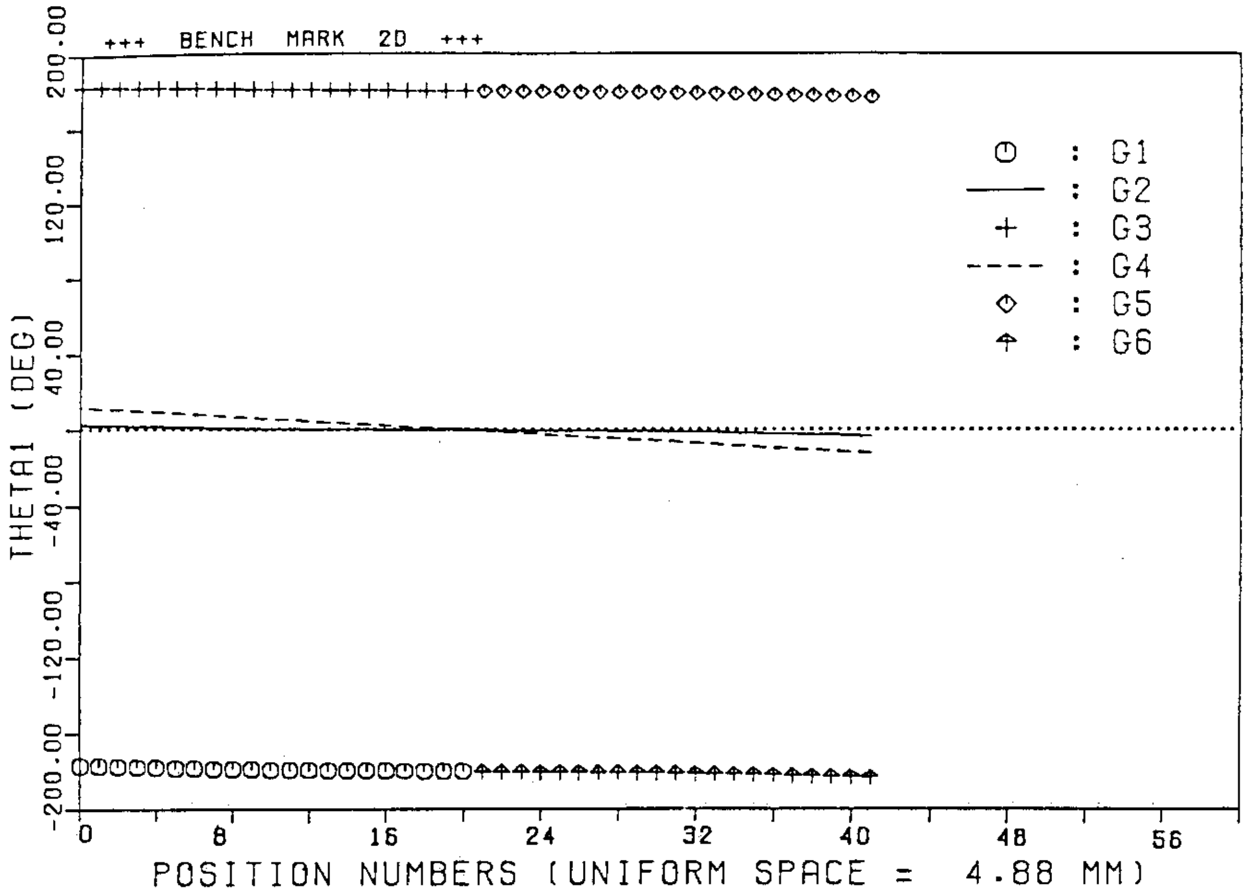

Fig.4.1.1 Behavior of Joint Angle θ_1 (Sample Problem 1)

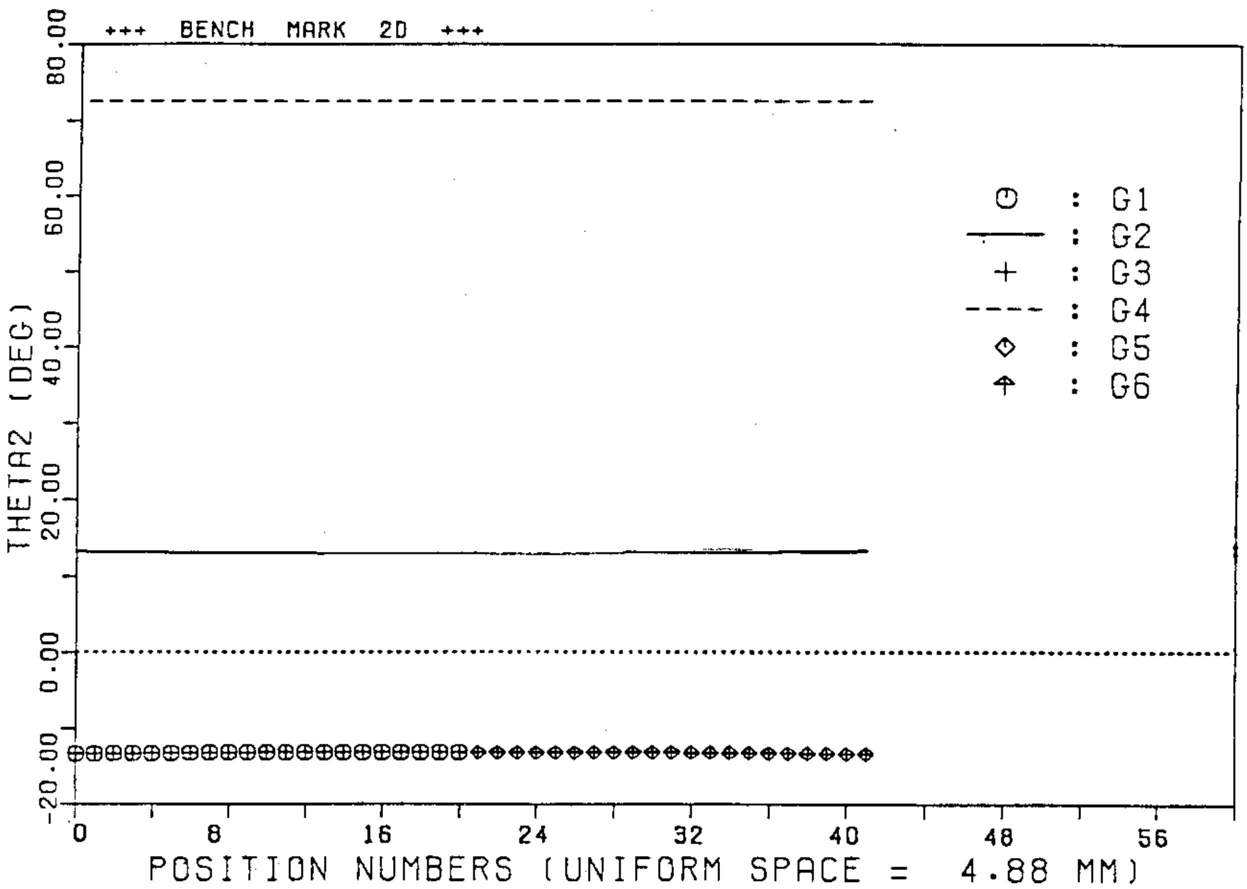

Fig.4.2.1 Behavior of Joint Angle θ_2 (Sample Problem 1)

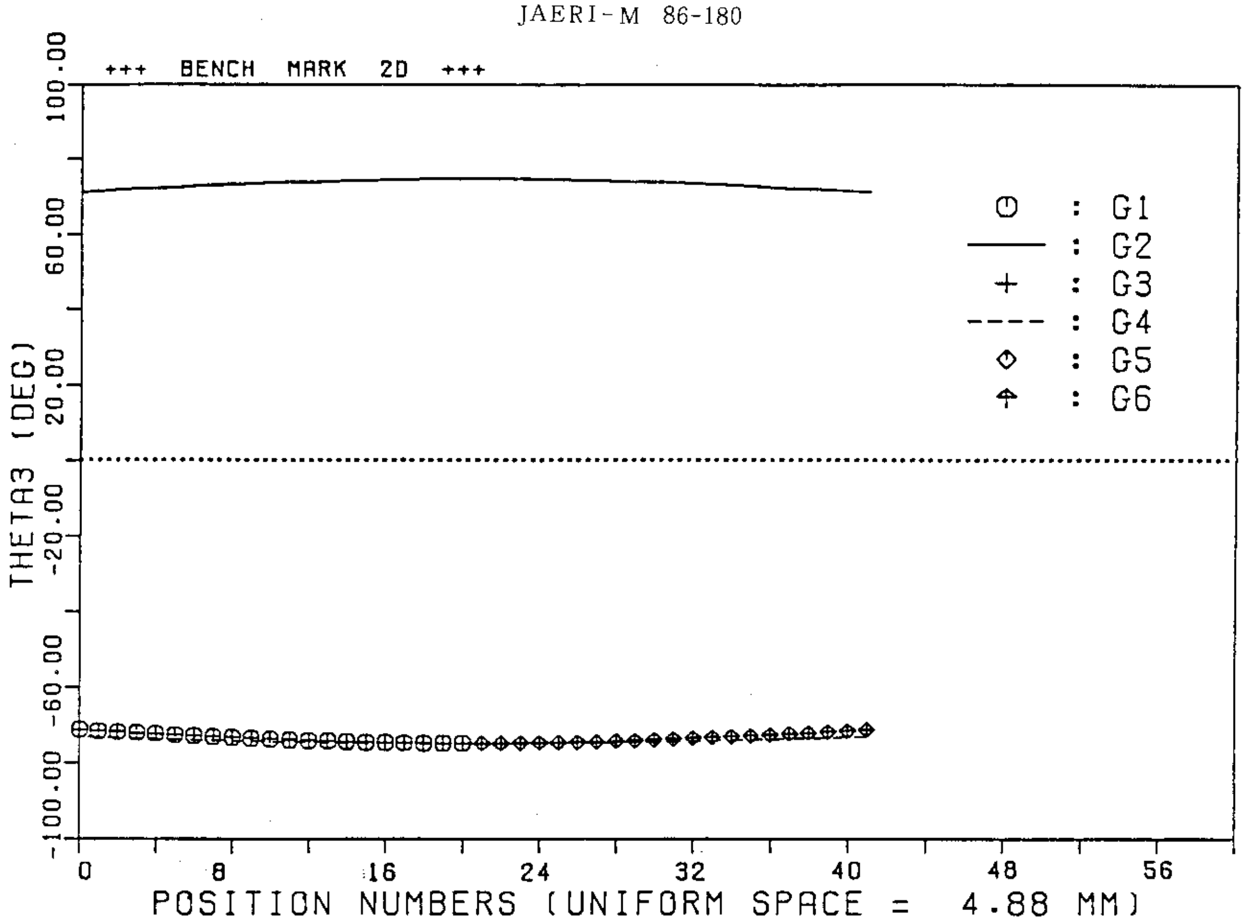

Fig.4.3.1 Behavior of Joint Angle θ_3 (Sample Problem 1)

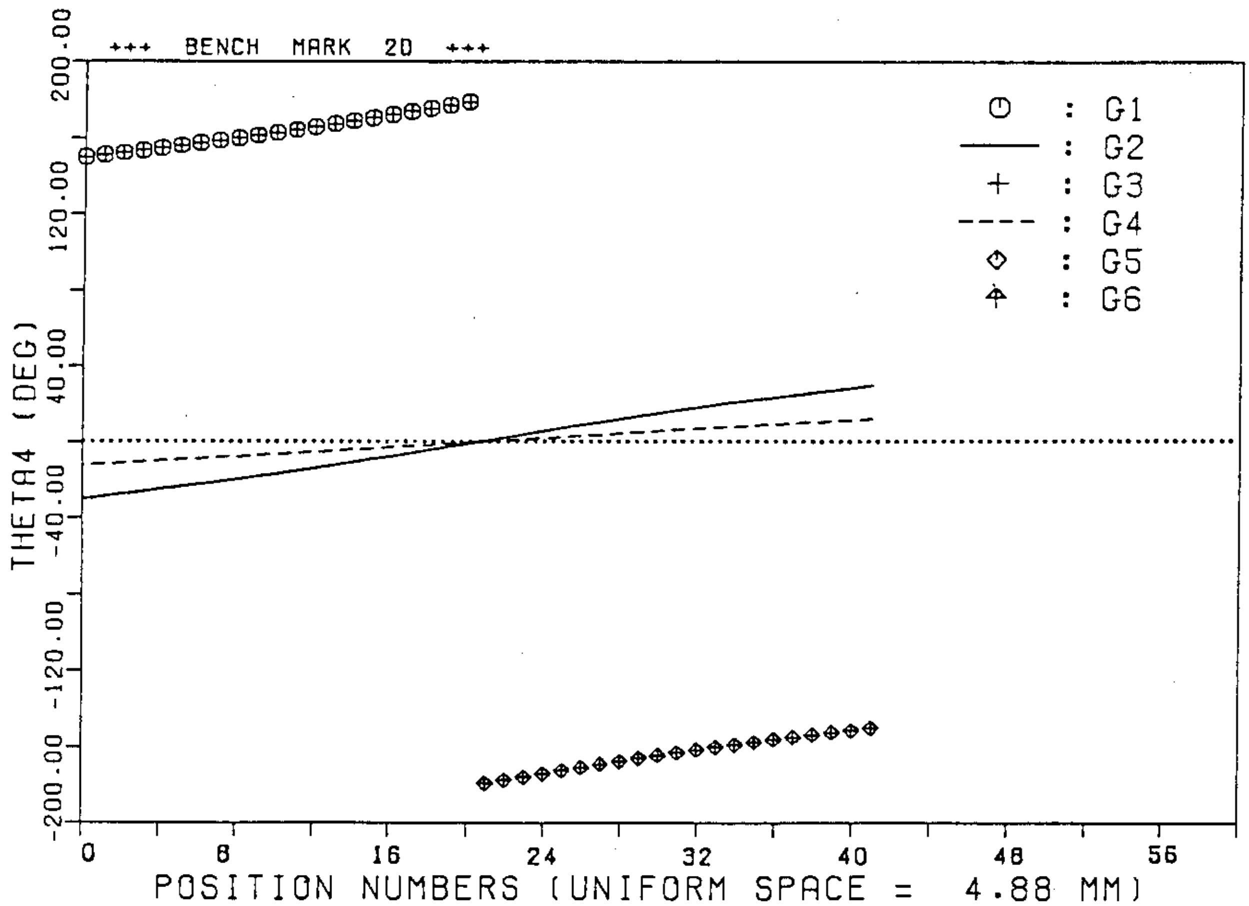

Fig.4.4.1 Behavior of Joint Angle θ_4 (Sample Problem 1)

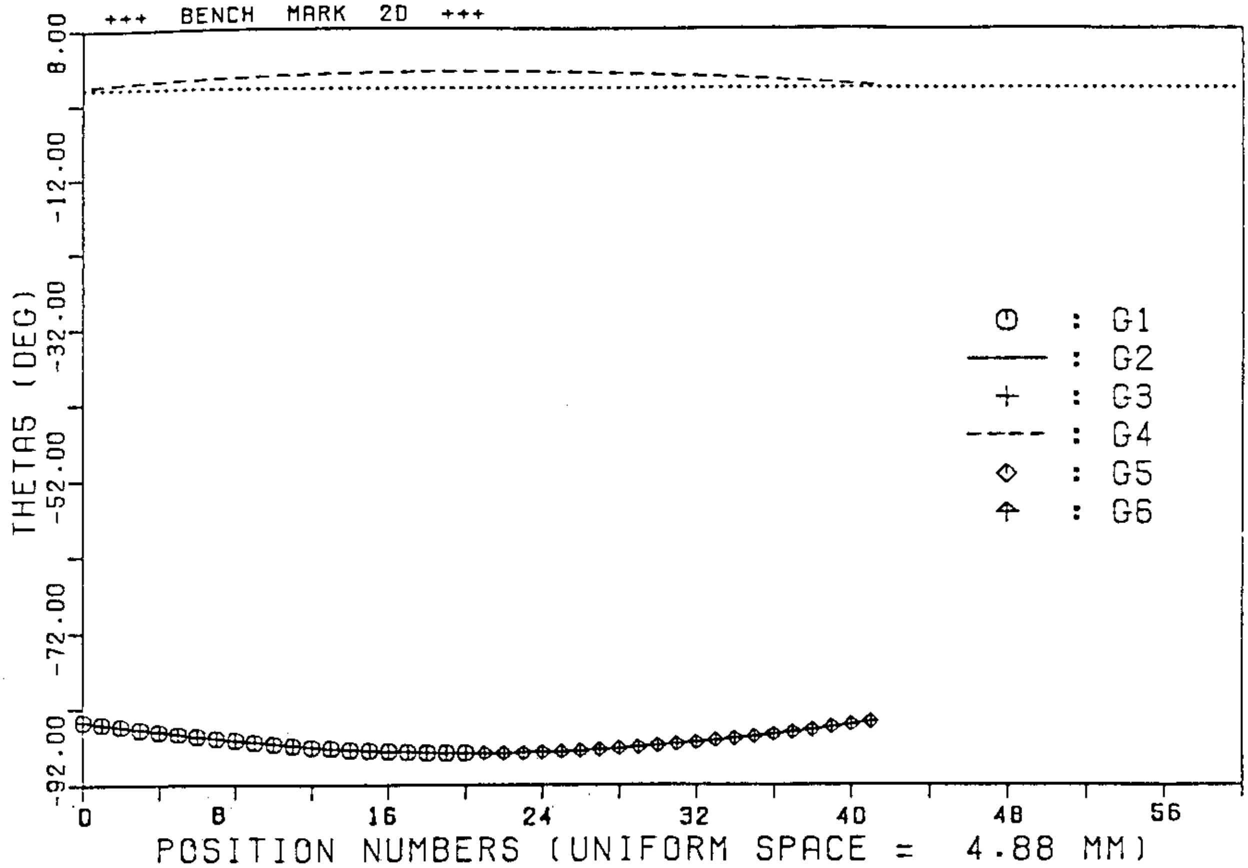

Fig.4.5.1 Behavior of Joint Angle θ_5 (Sample Problem 1)

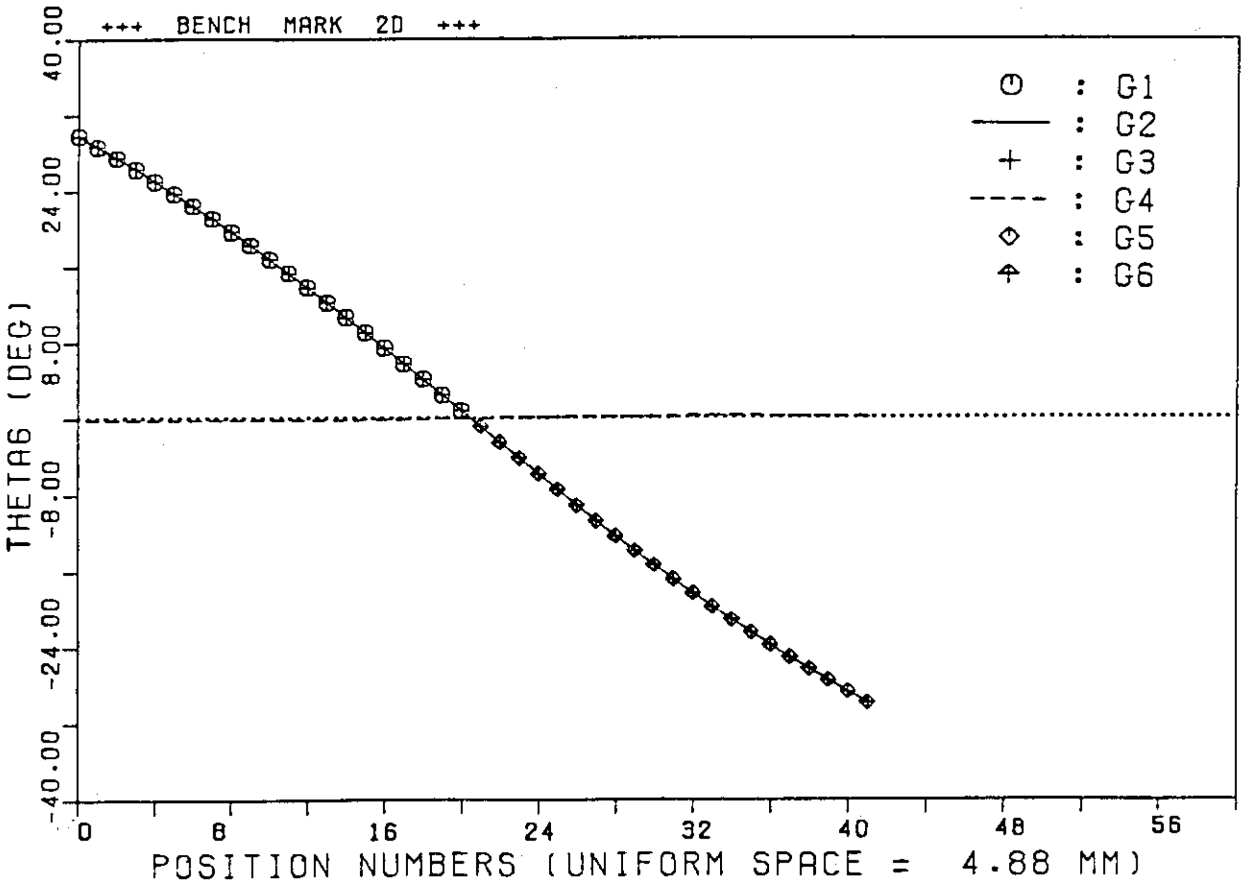

Fig.4.6.1 Behavior of Joint Angle θ_6 (Sample Problem 1)

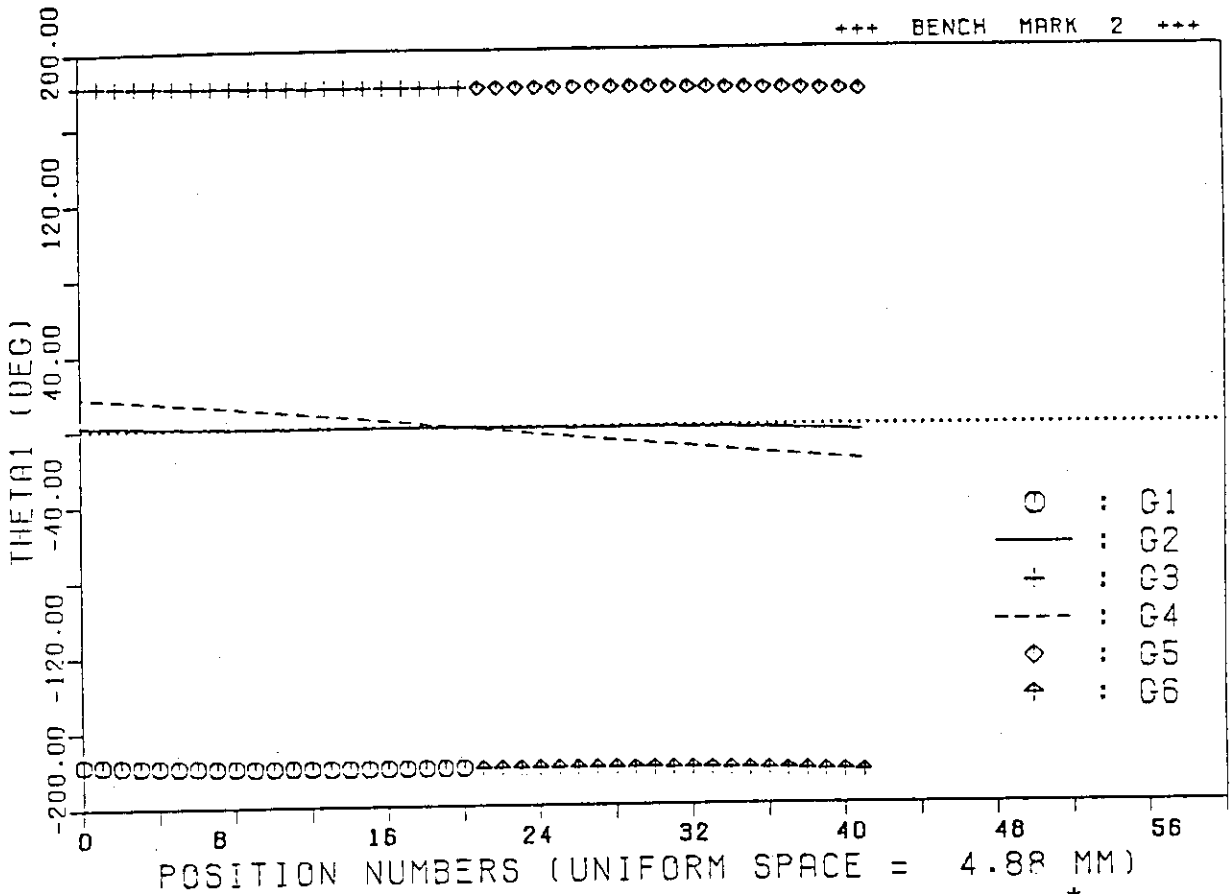

Fig.4.1.2 Behavior of Joint Angle θ_1 (Sample Problem 1)*

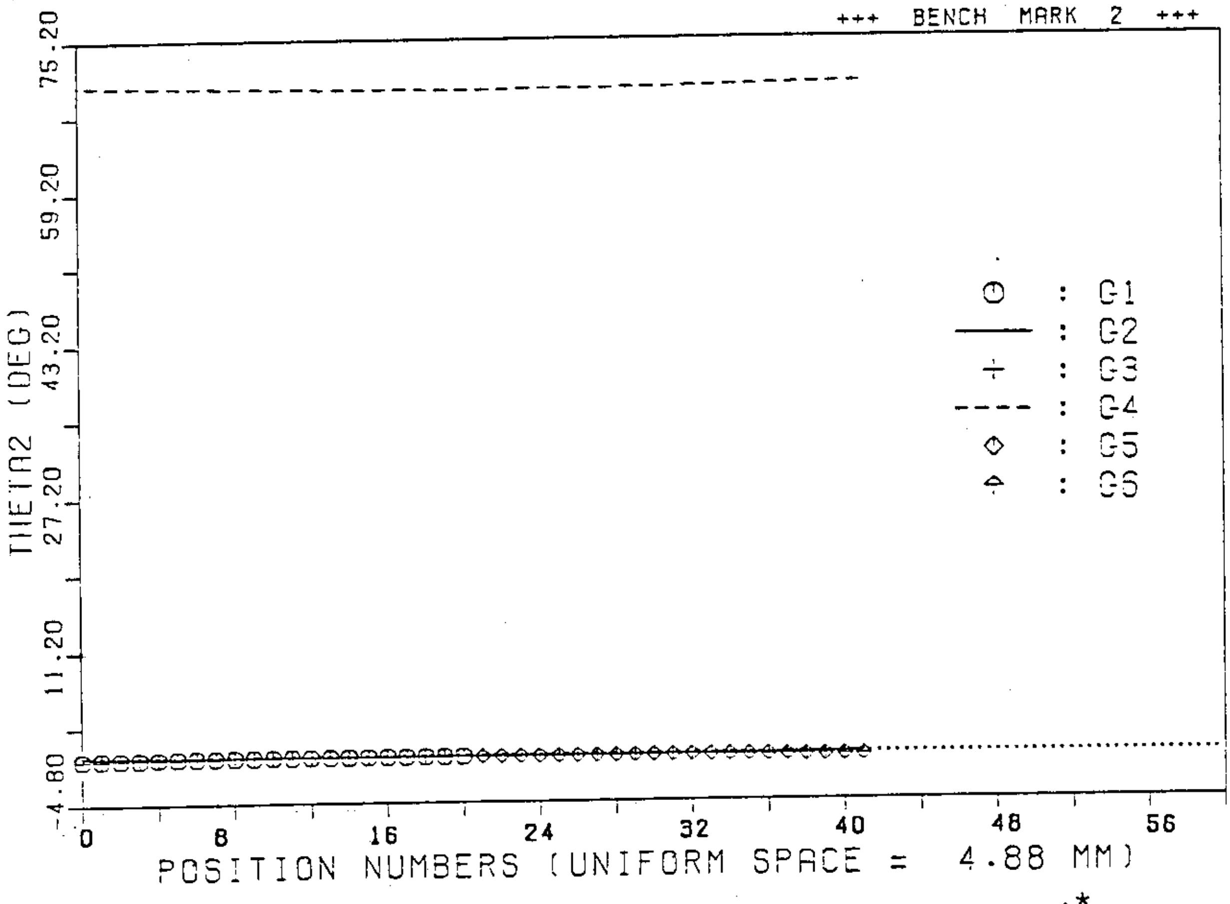

Fig.4.2.2 Behavior of Joint Angle θ_2 (Sample Problem 1)*

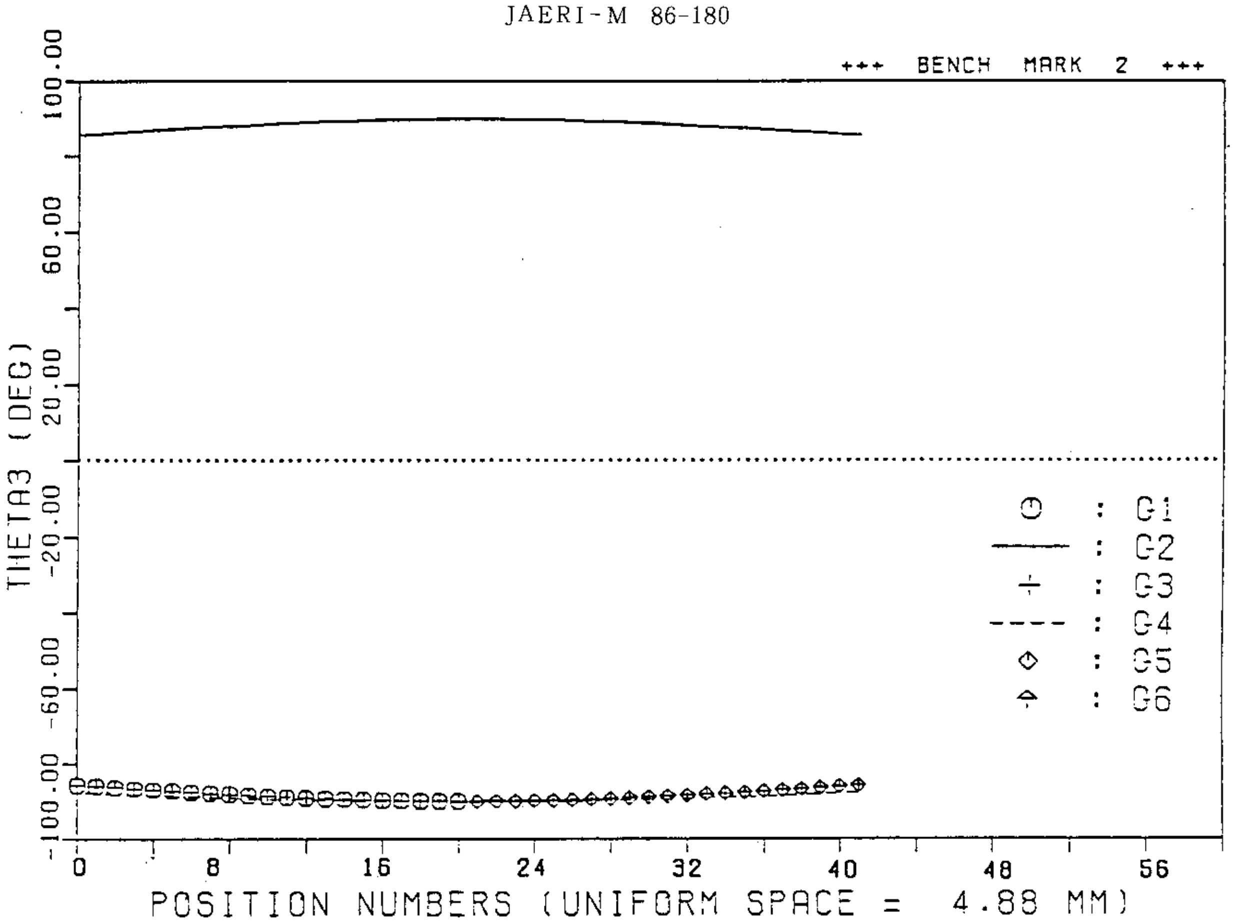

Fig.4.3.2 Behavior of Joint Angle θ_3 (Sample Problem 1)*

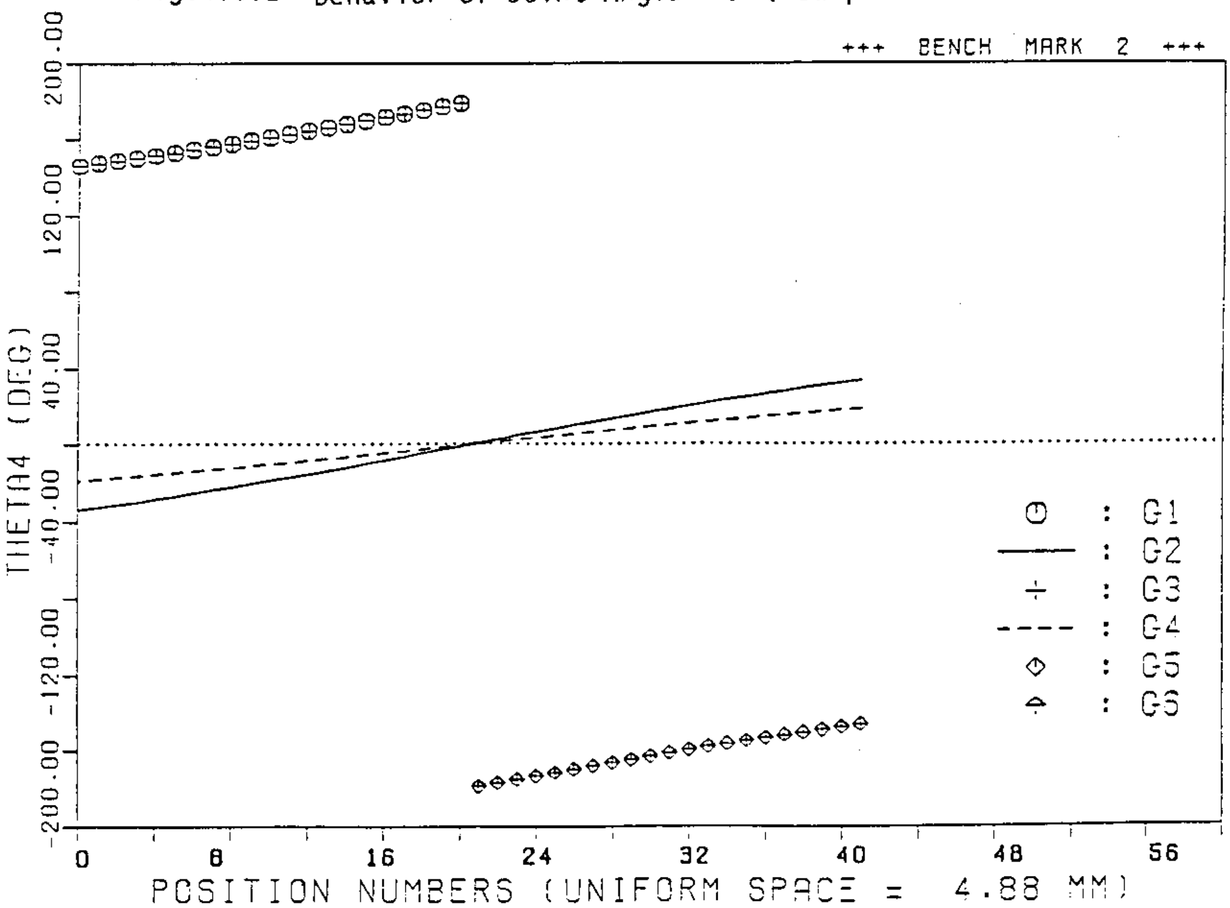

Fig.4.4.2 Behavior of Joint Angle θ_4 (Sample Problem 1)*

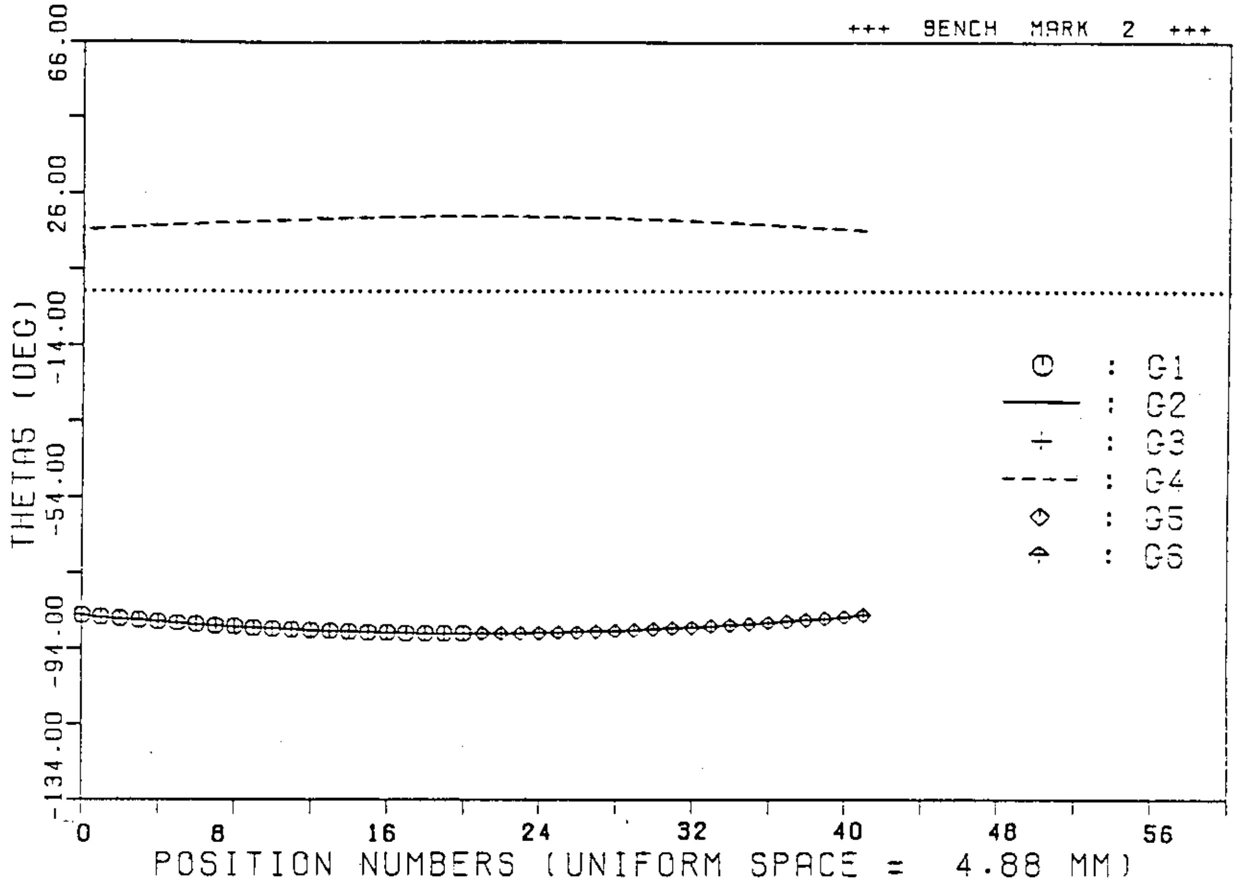

Fig.4.5.2 Behavior of Joint Angle θ_5 (Sample Problem 1)*

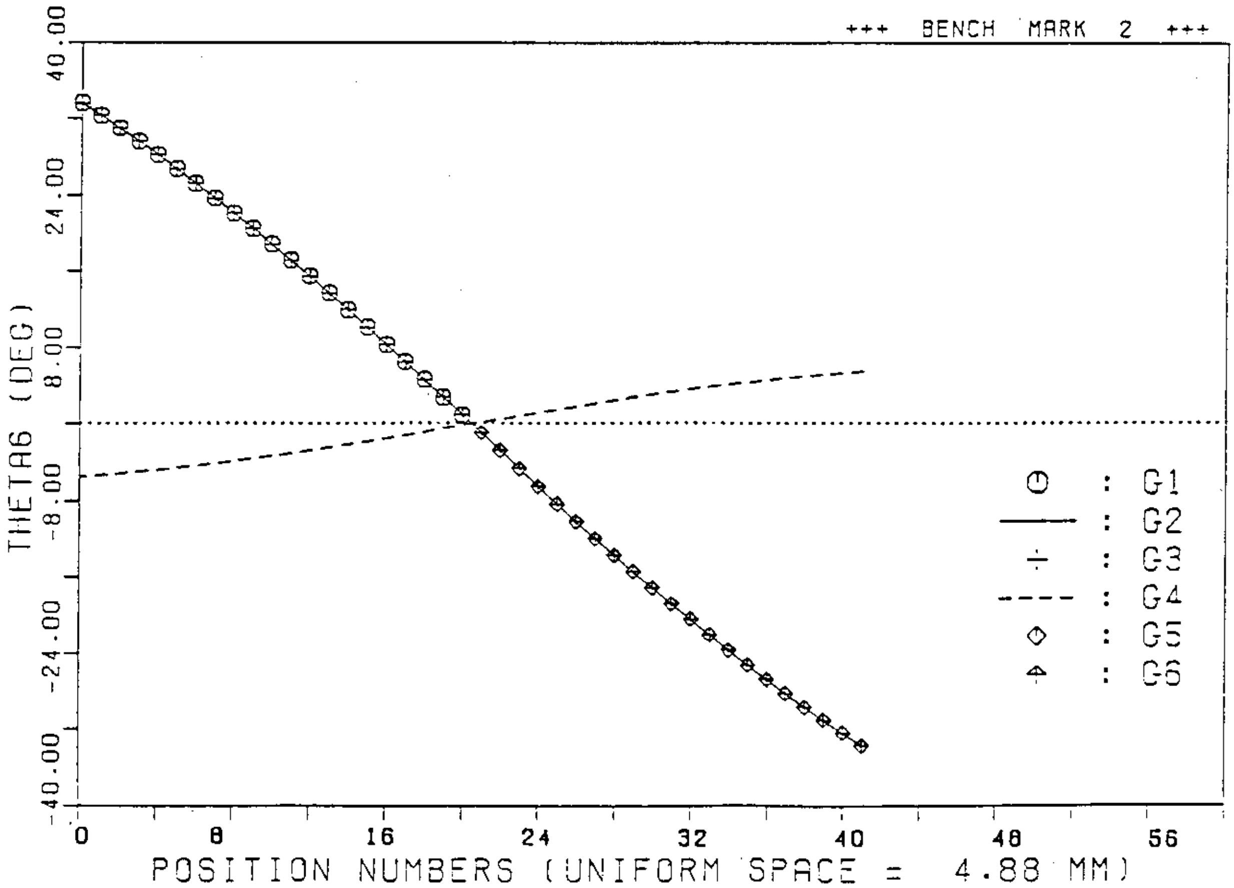

Fig.4.6.2 Behavior of Joint Angle θ_6 (Sample Problem 1)*

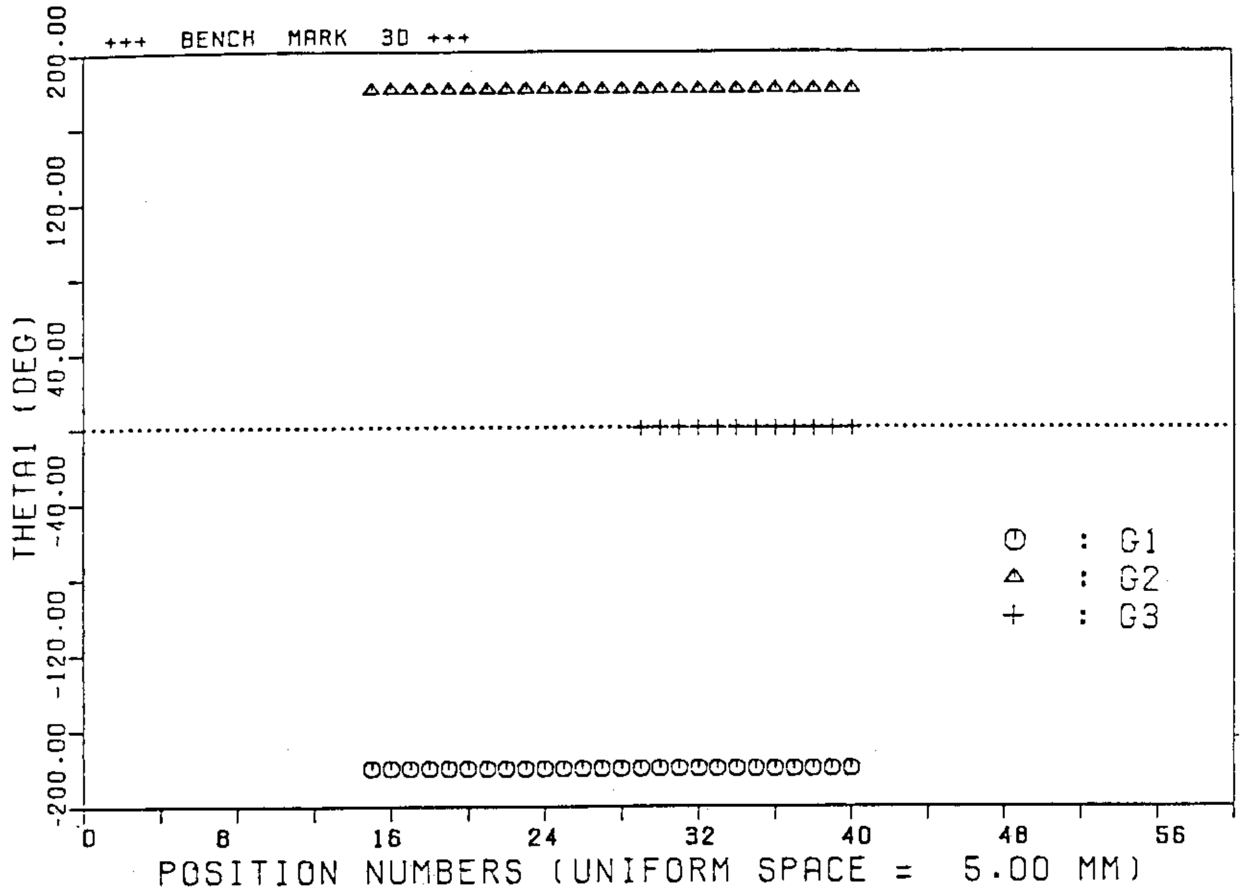

Fig.5.1.1 Behavior of Joint Angle θ_1 (Sample Problem 2)

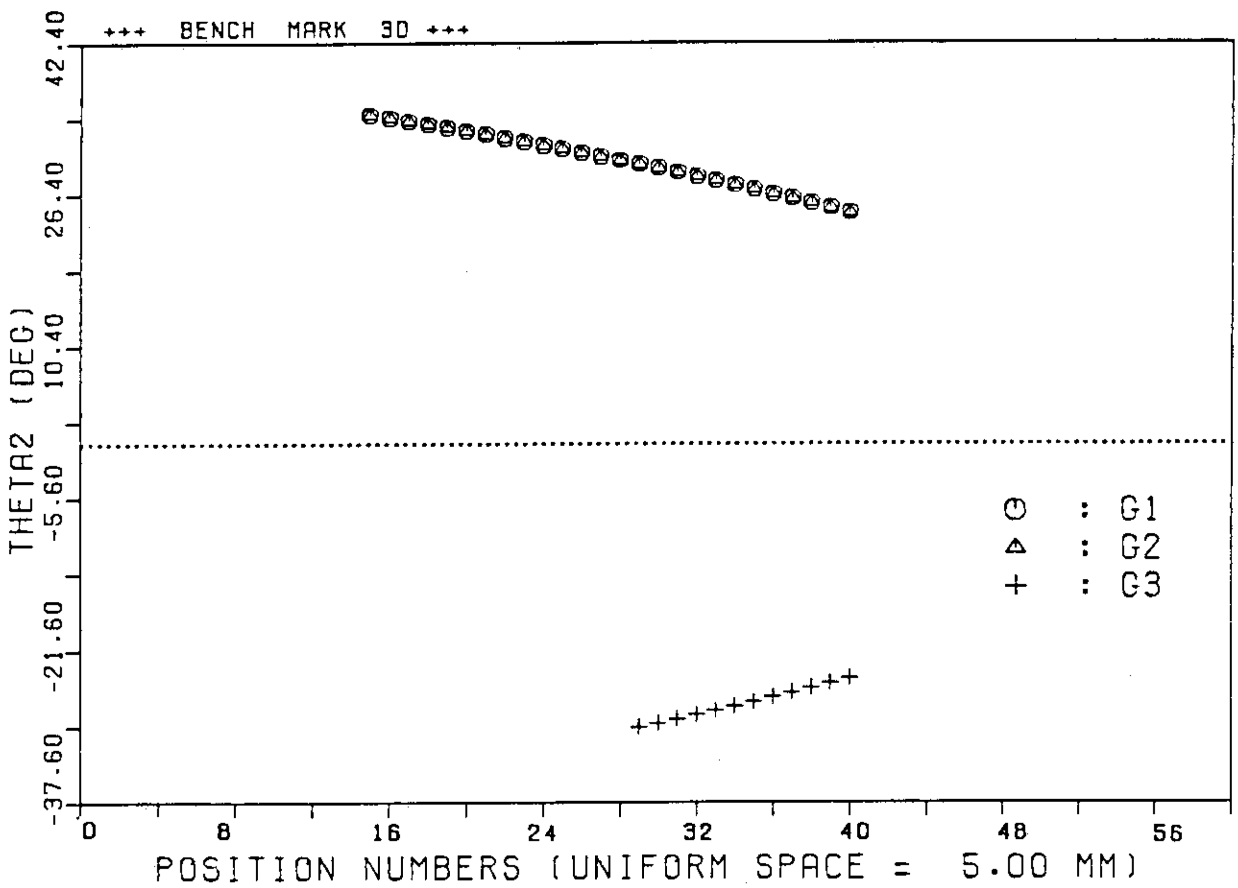

Fig.5.2.1 Behavior of Joint Angle θ_2 (Sample Problem 2)

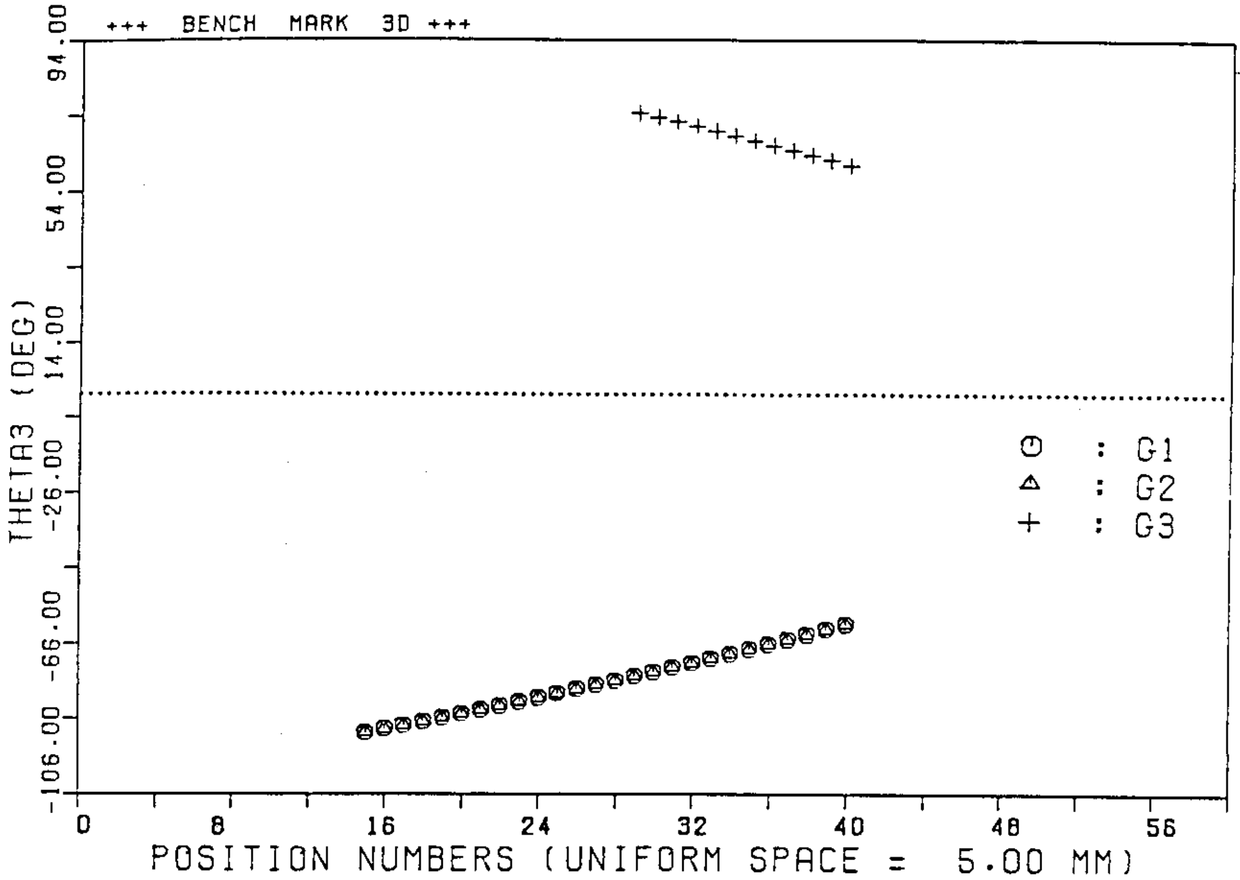

Fig.5.3.1 Behavior of Joint Angle θ_3 (Sample Problem 2)

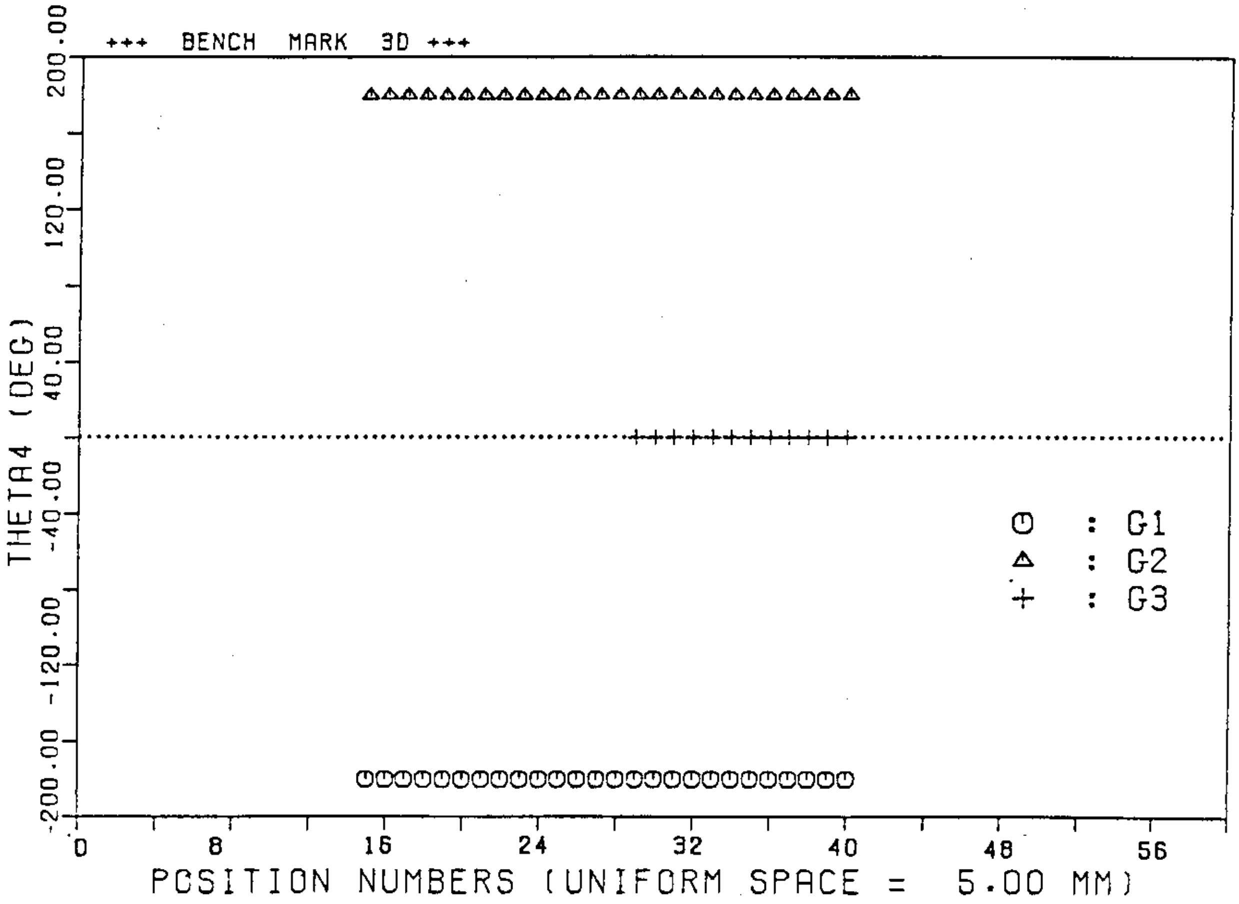

Fig.5.4.1 Behavior of Joint Angle θ_4 (Sample Problem 2)

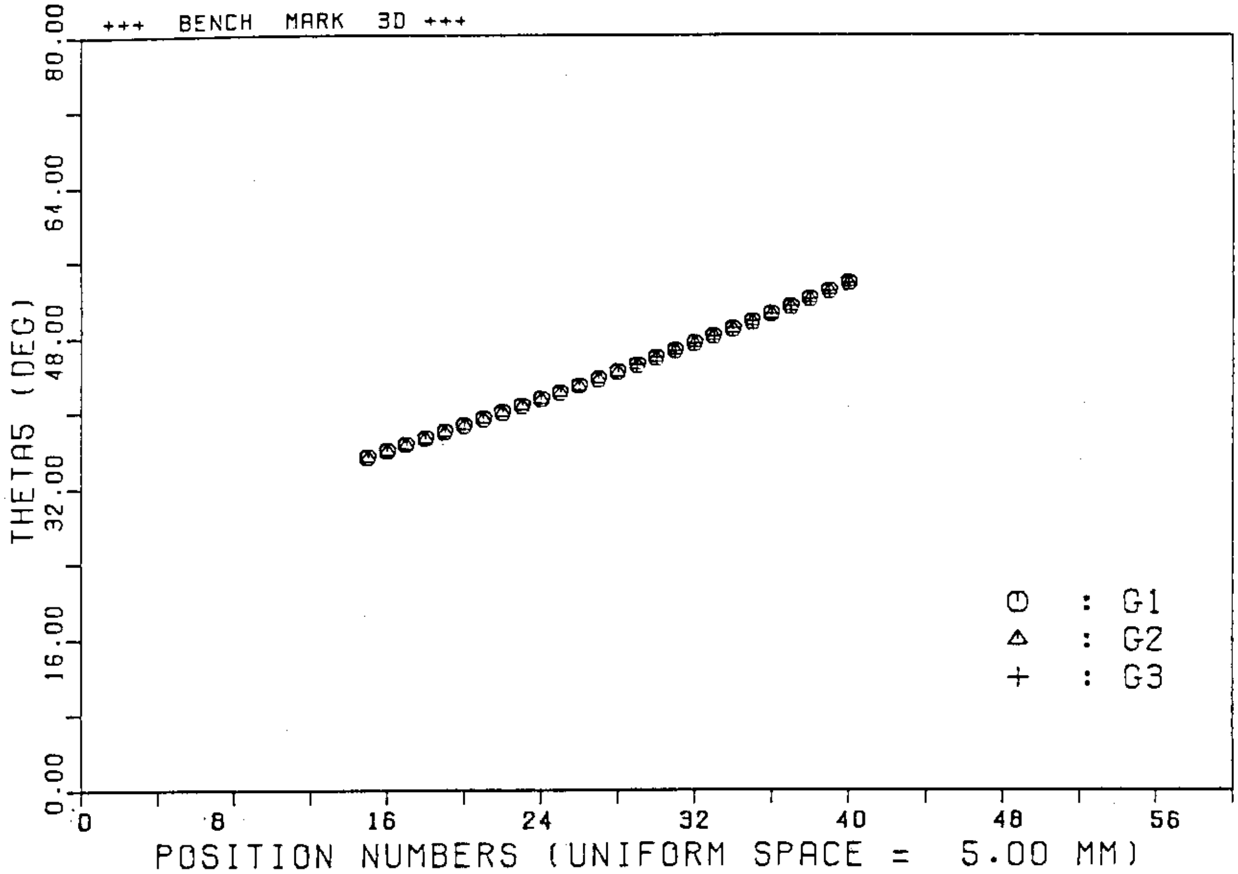

Fig.5.5.1 Behavior of Joint Angle θ_5 (Sample Problem 2)

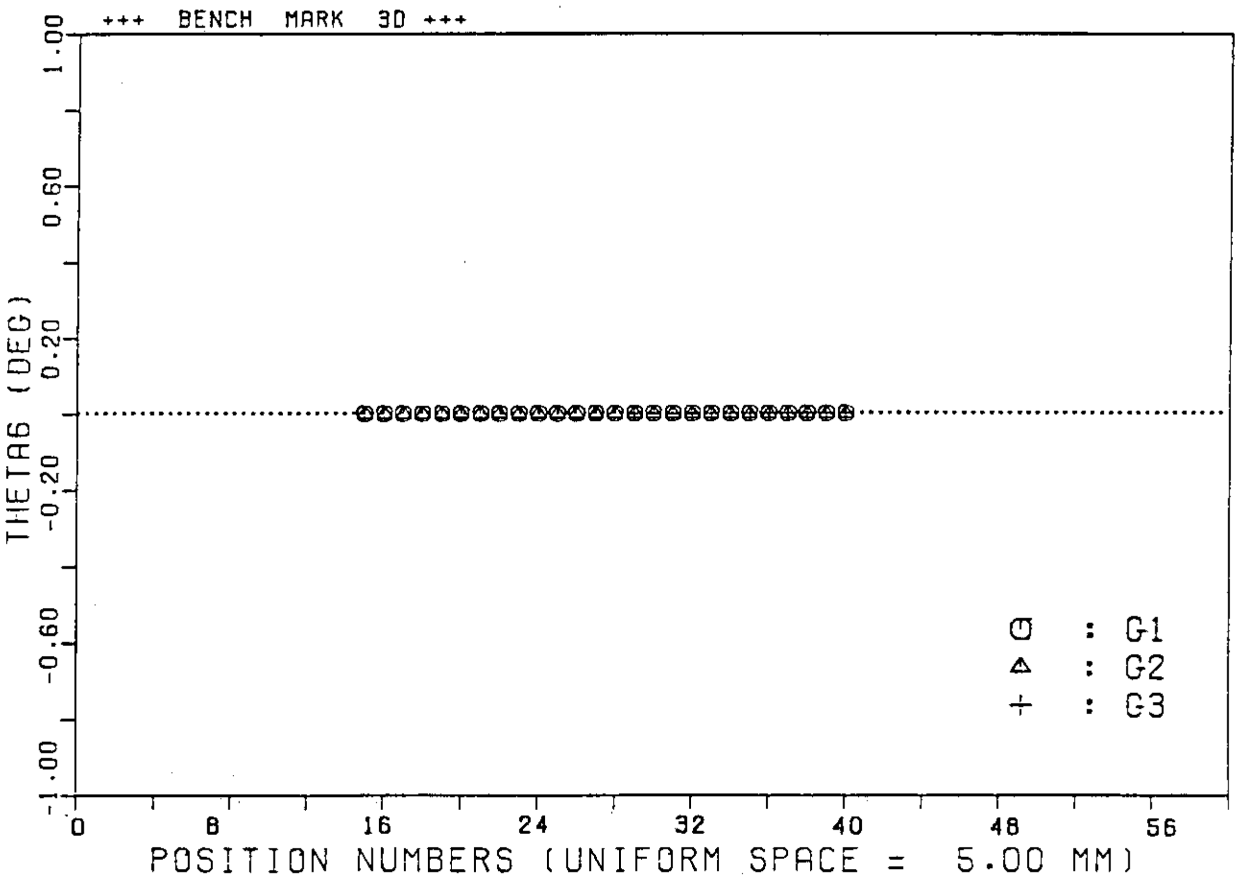

Fig.5.6.1 Behavior of Joint Angle θ_6 (Sample Problem 2)

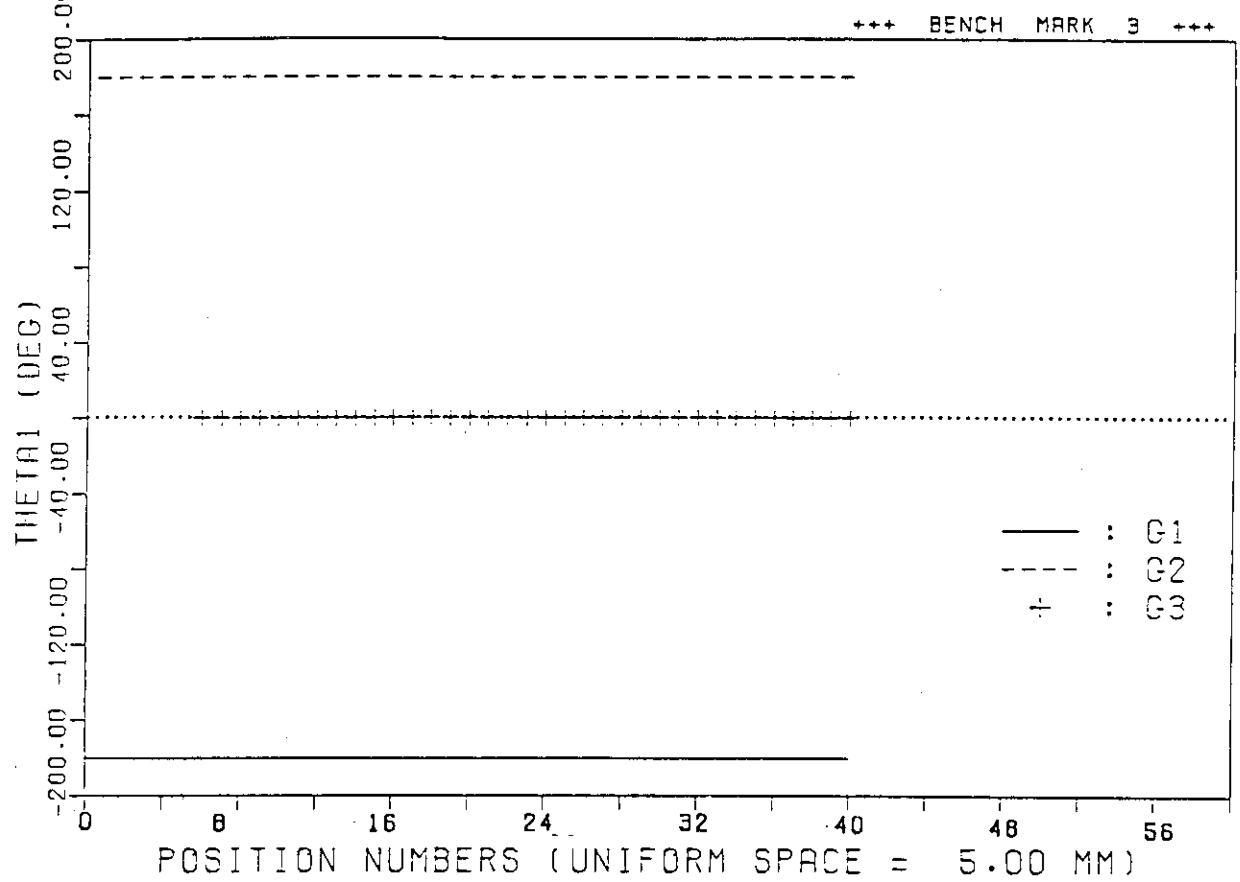

Fig.5.1.2 Behavior of Joint Angle θ_1 (Sample Problem 2)*

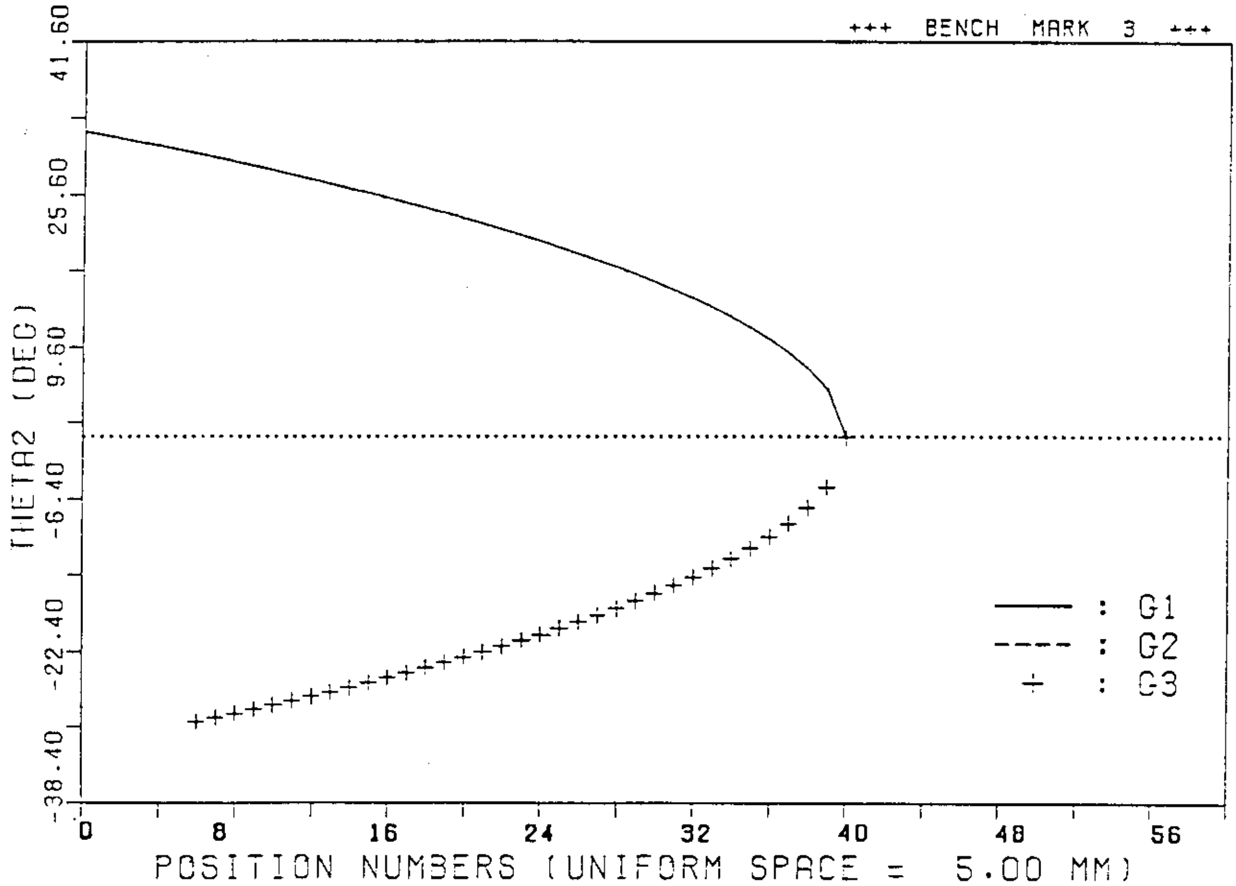

Fig.5.2.2 Behavior of Joint Angle θ_2 (Sample Problem 2)*

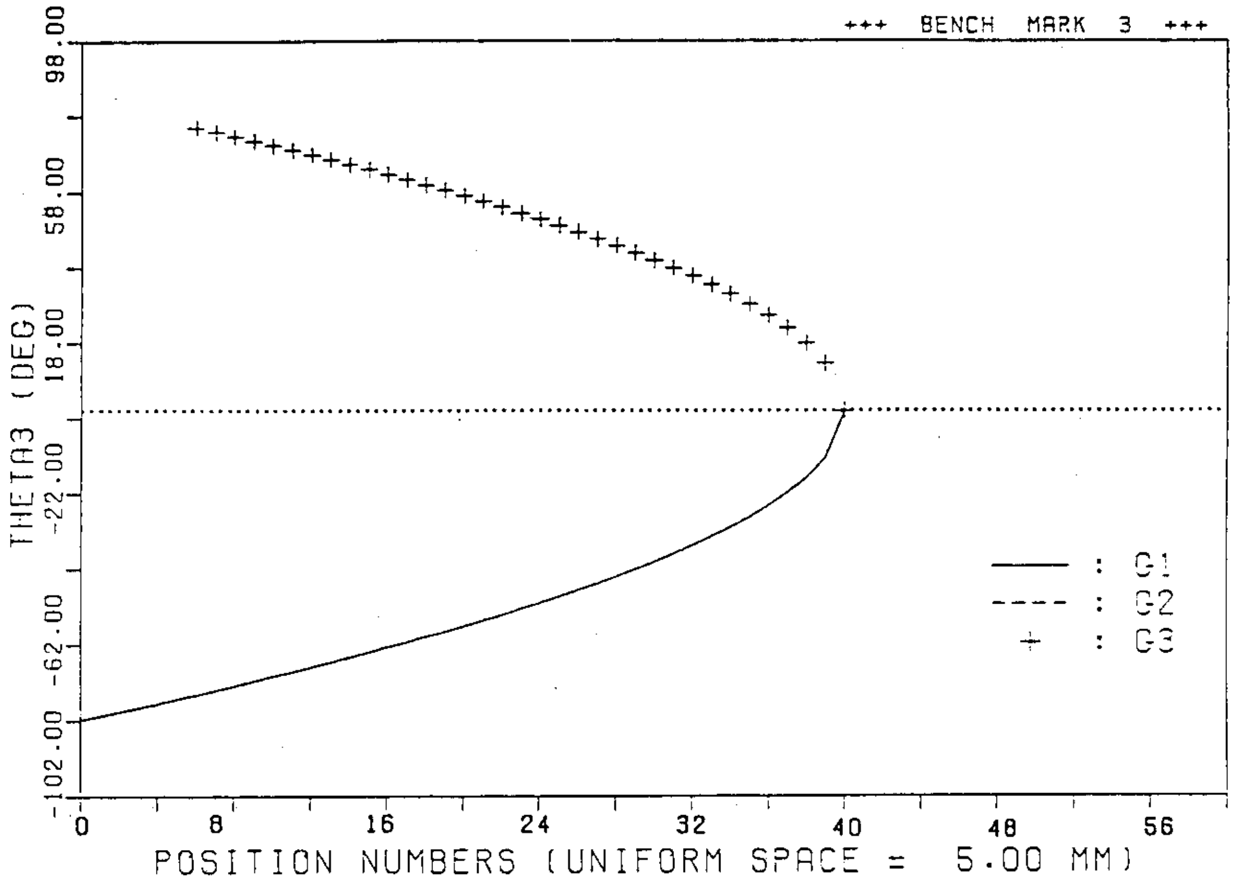

Fig.5.3.2 Behavior of Joint Angle θ_3 (Sample Problem 2)*

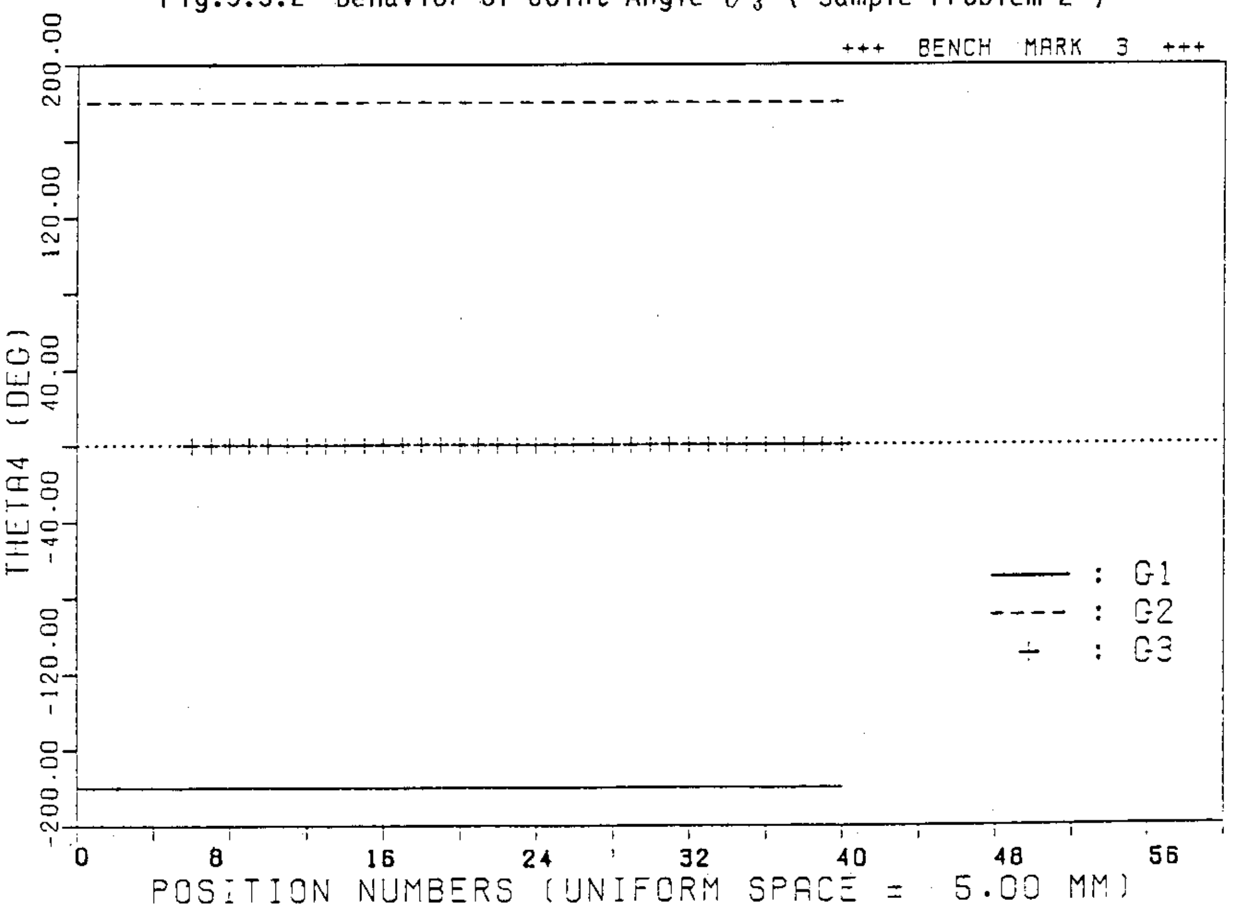

Fig.5.4.2 Behavior of Joint Angle θ_4 (Sample Problem 2)*

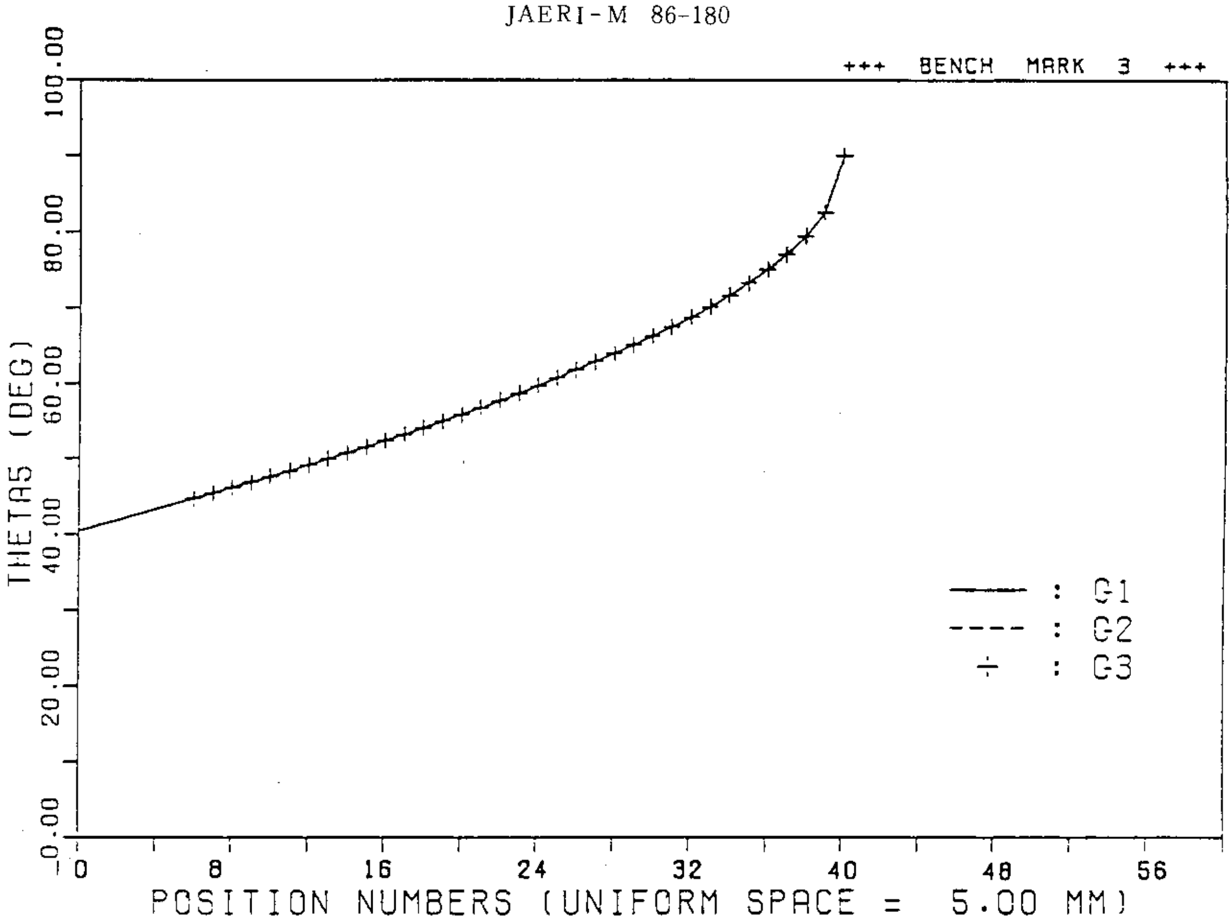

Fig.5.5.2 Behavior of Joint Angle θ_5 (Sample Problem 2)*

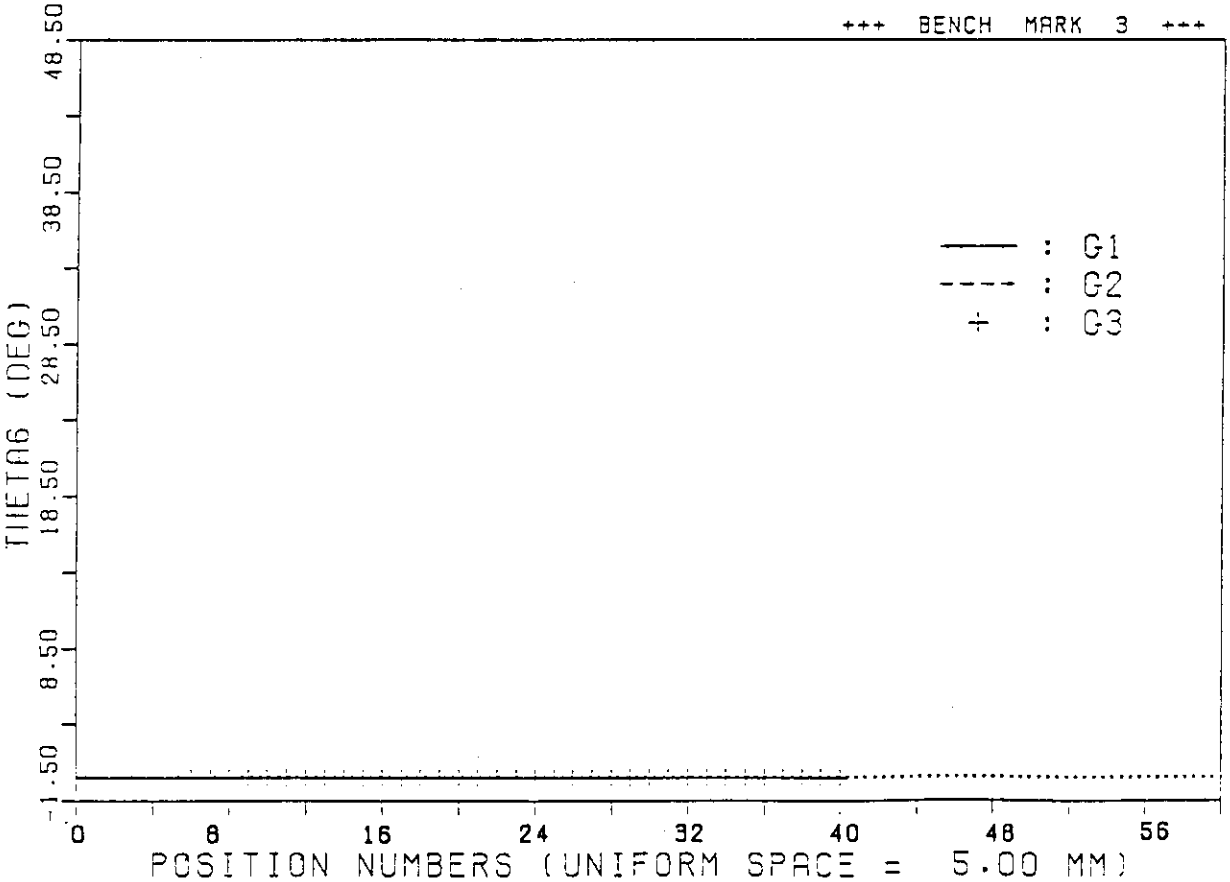

Fig.5.6.2 Behavior of Joint Angle θ_6 (Sample Problem 2)*

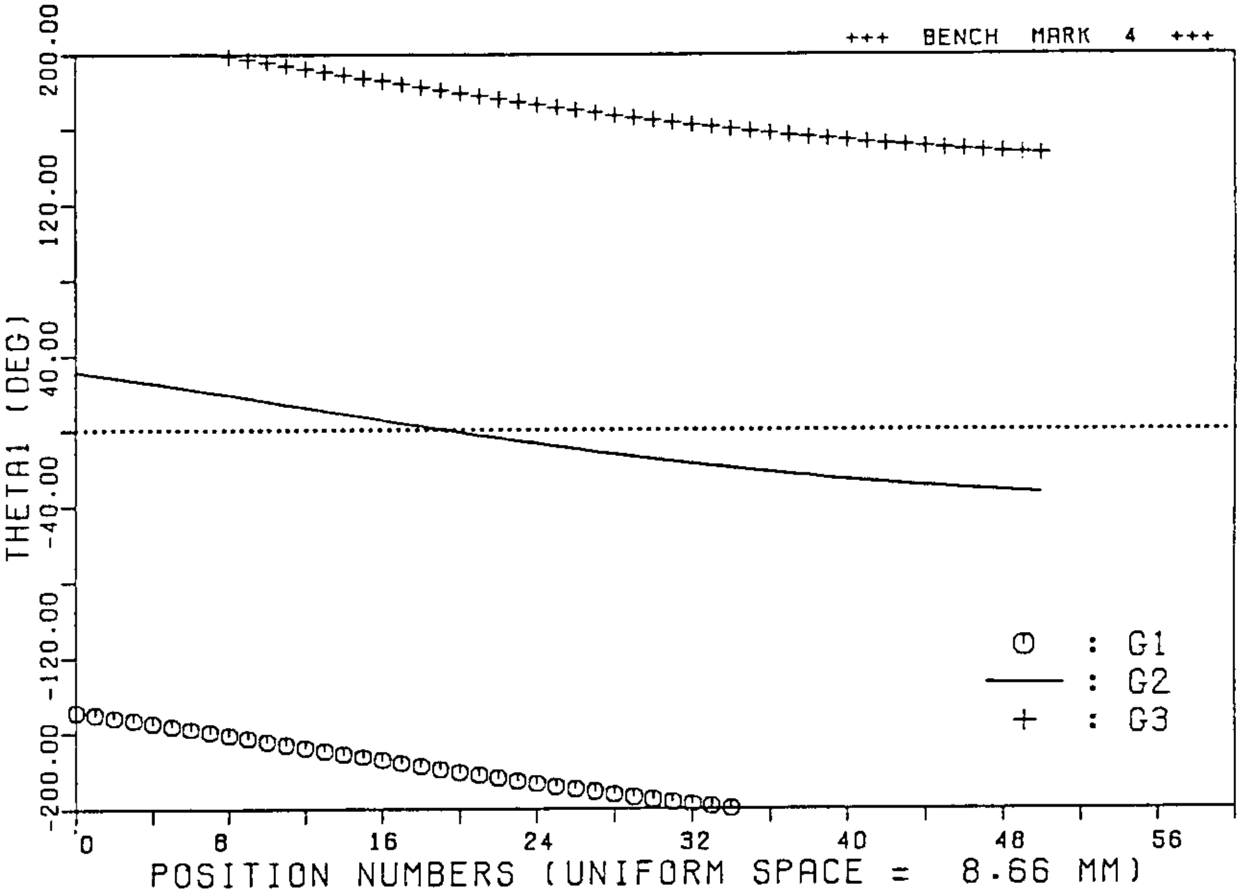

Fig. 6.1 Behavior of Joint Angle θ_1 (Sample Problem 3)

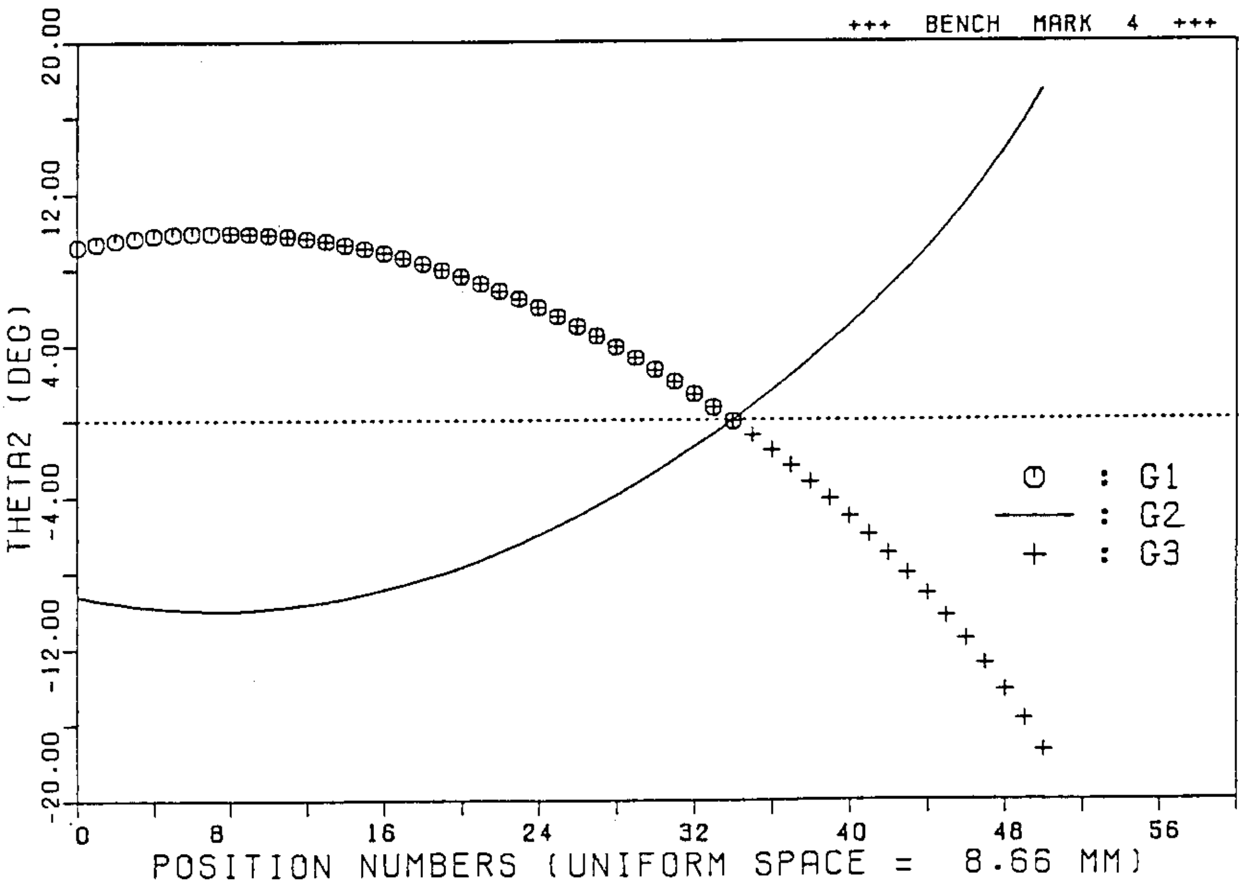

Fig. 6.2 Behavior of Joint Angle θ_2 (Sample Problem 3)

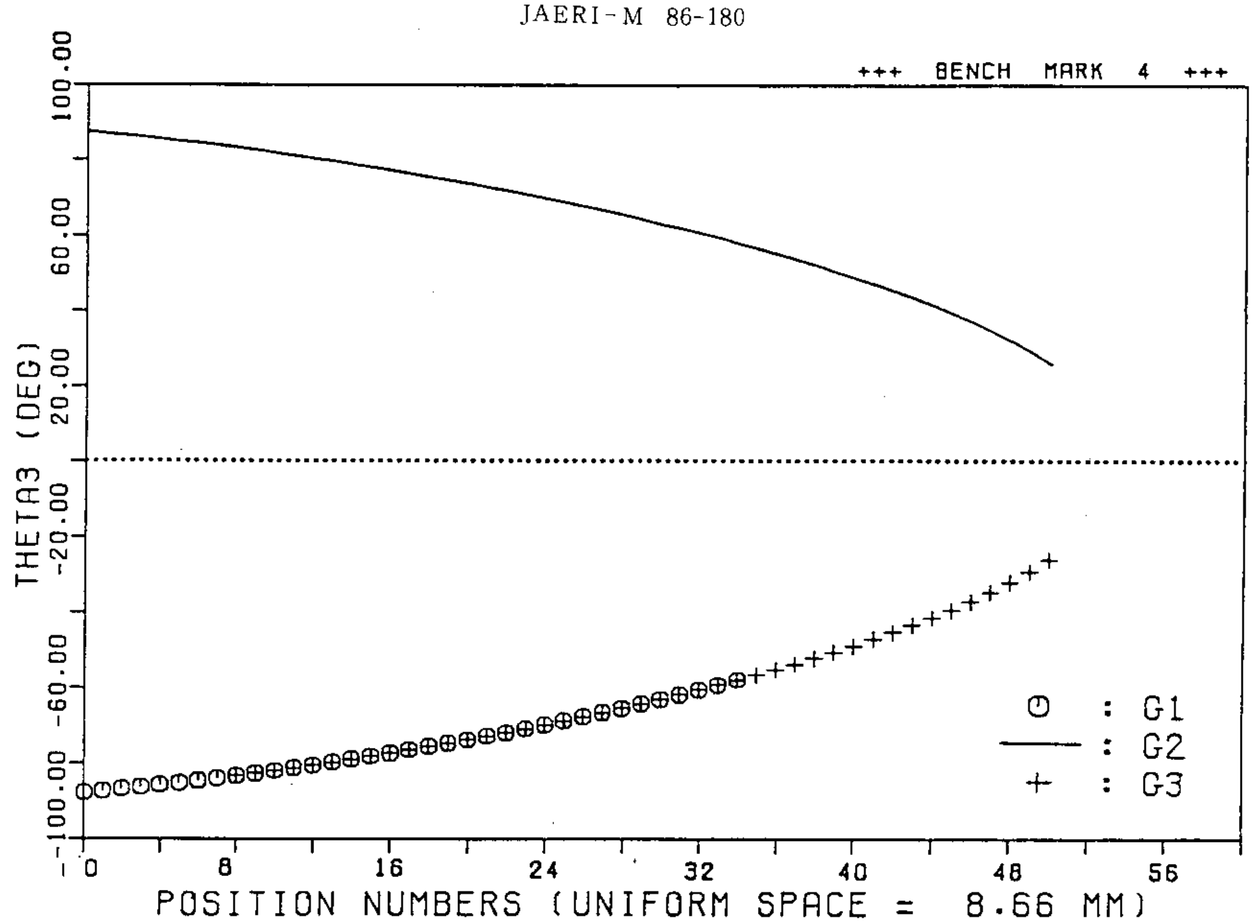

Fig. 6.3 Behavior of Joint Angle θ_3 (Sample Problem 3)

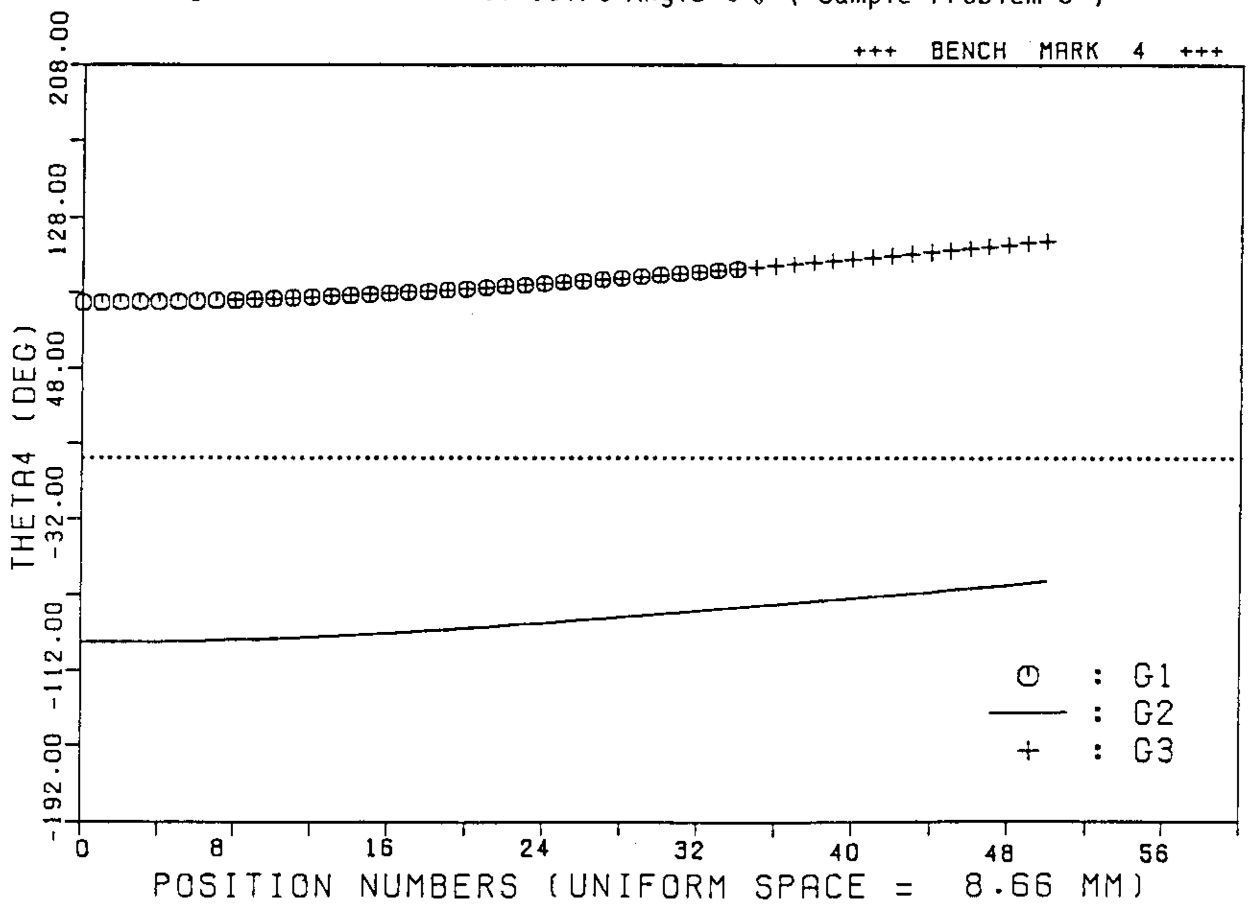

Fig. 6.4 Behavior of Joint Angle θ_4 (Sample Problem 3)

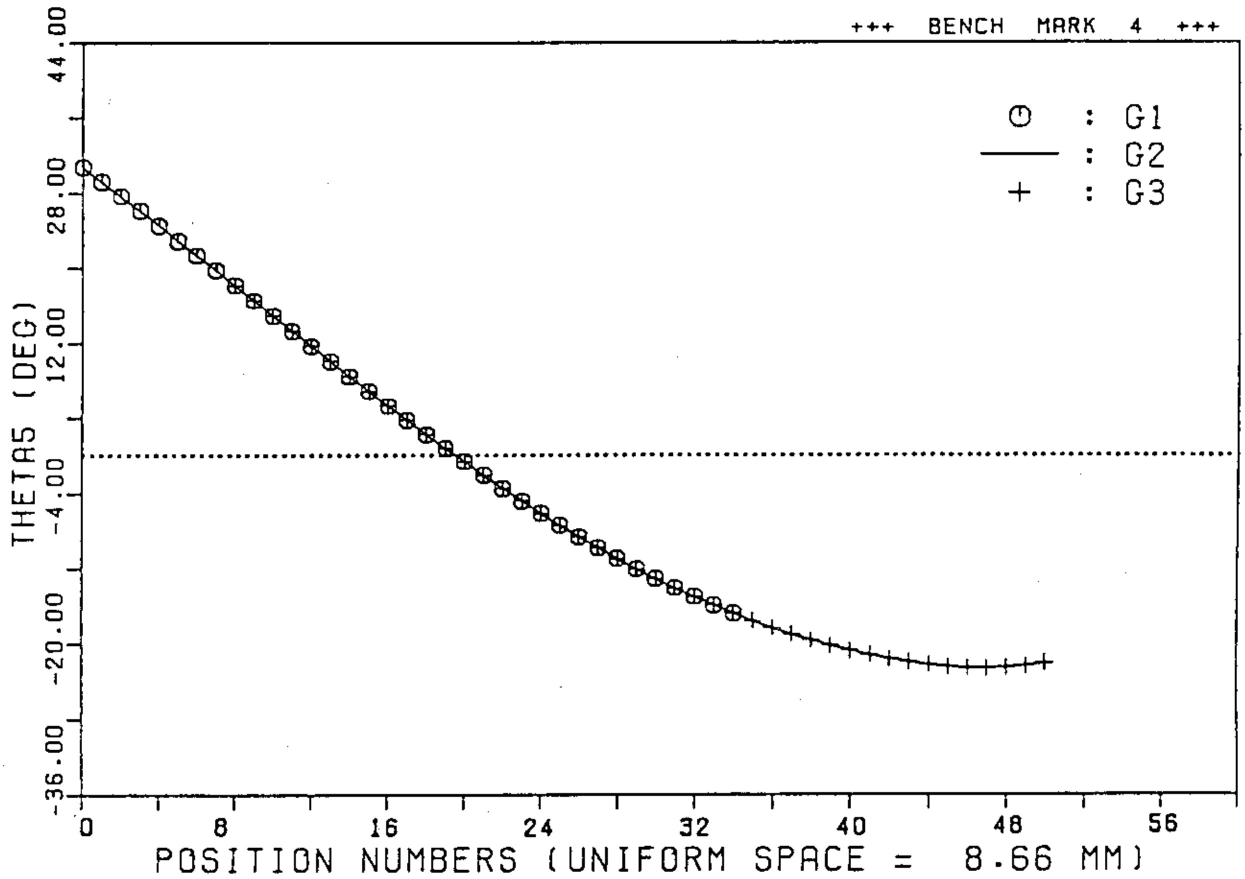

Fig. 6.5 Behavior of Joint Angle θ_5 (Sample Problem 3)

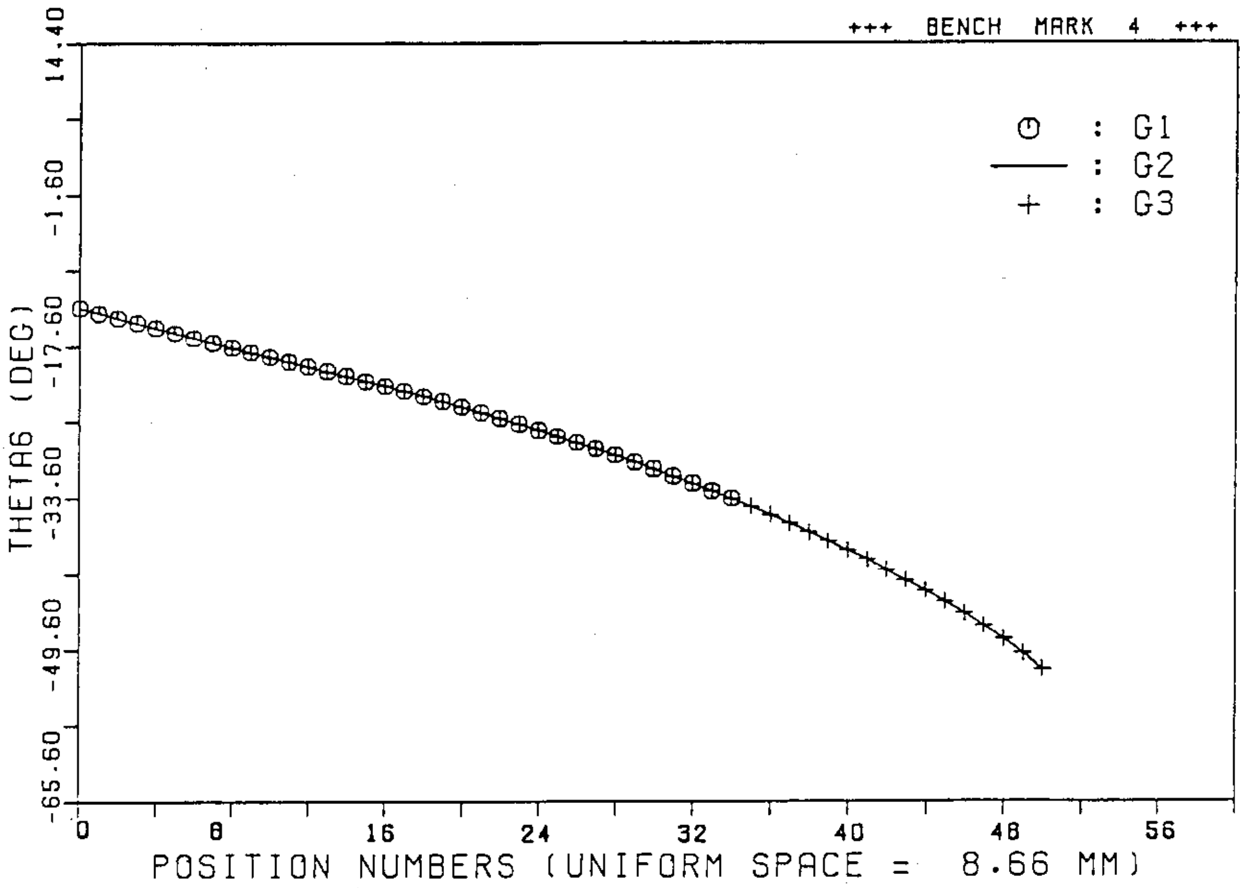

Fig. 6.6 Behavior of Joint Angle θ_6 (Sample Problem 3)

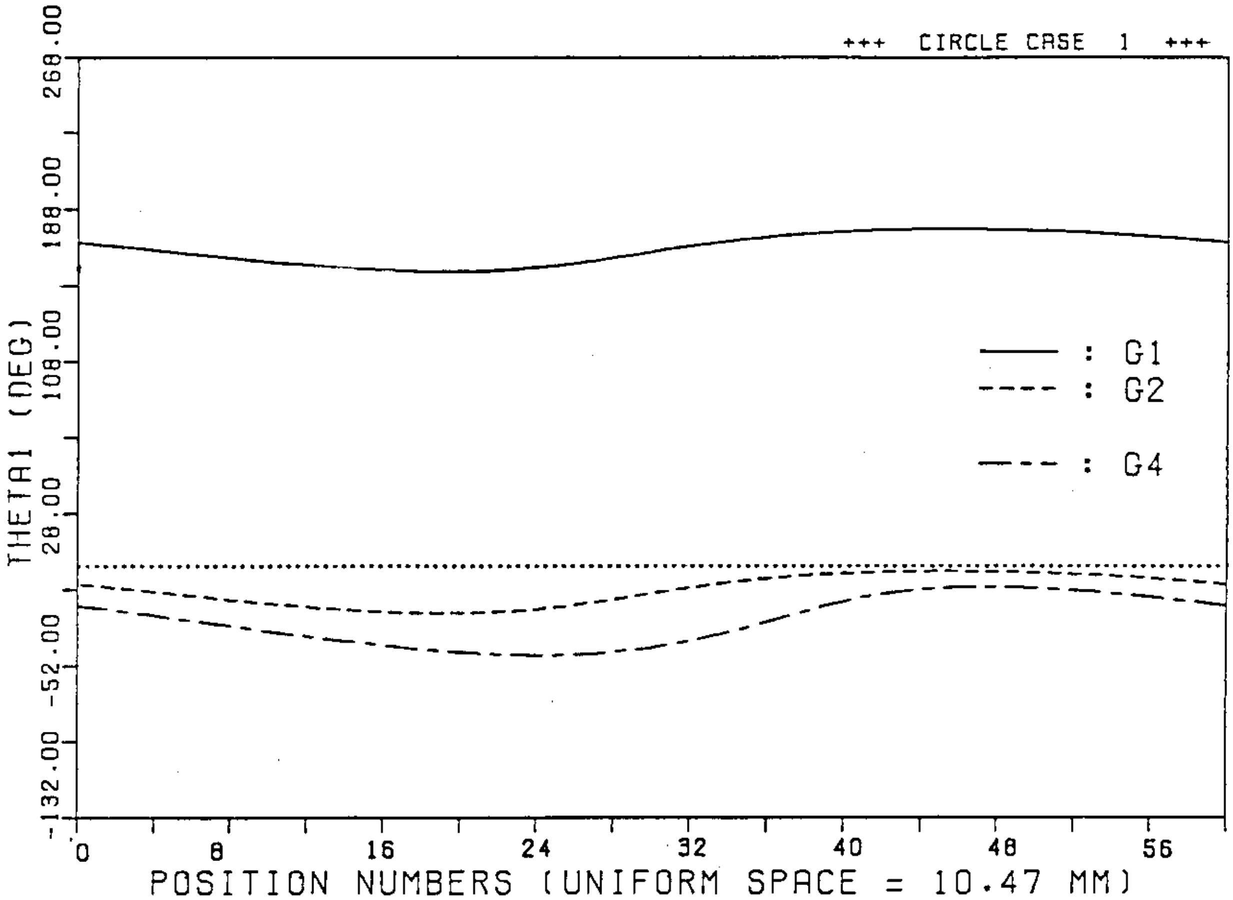

Fig. 7.1 Behavior of Joint Angle θ_1 (Sample Problem 4)

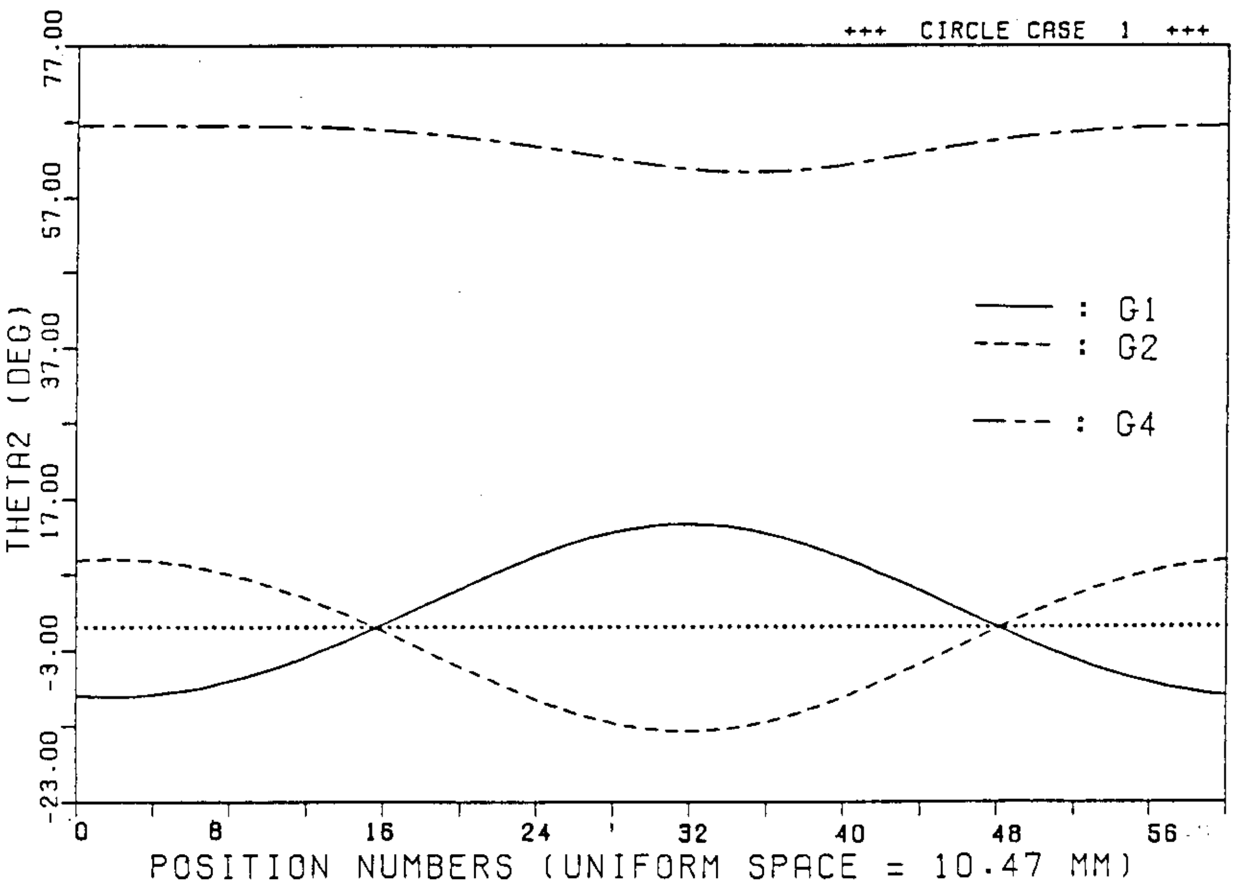

Fig. 7.2 Behavior of Joint Angle θ_2 (Sample Problem 4)

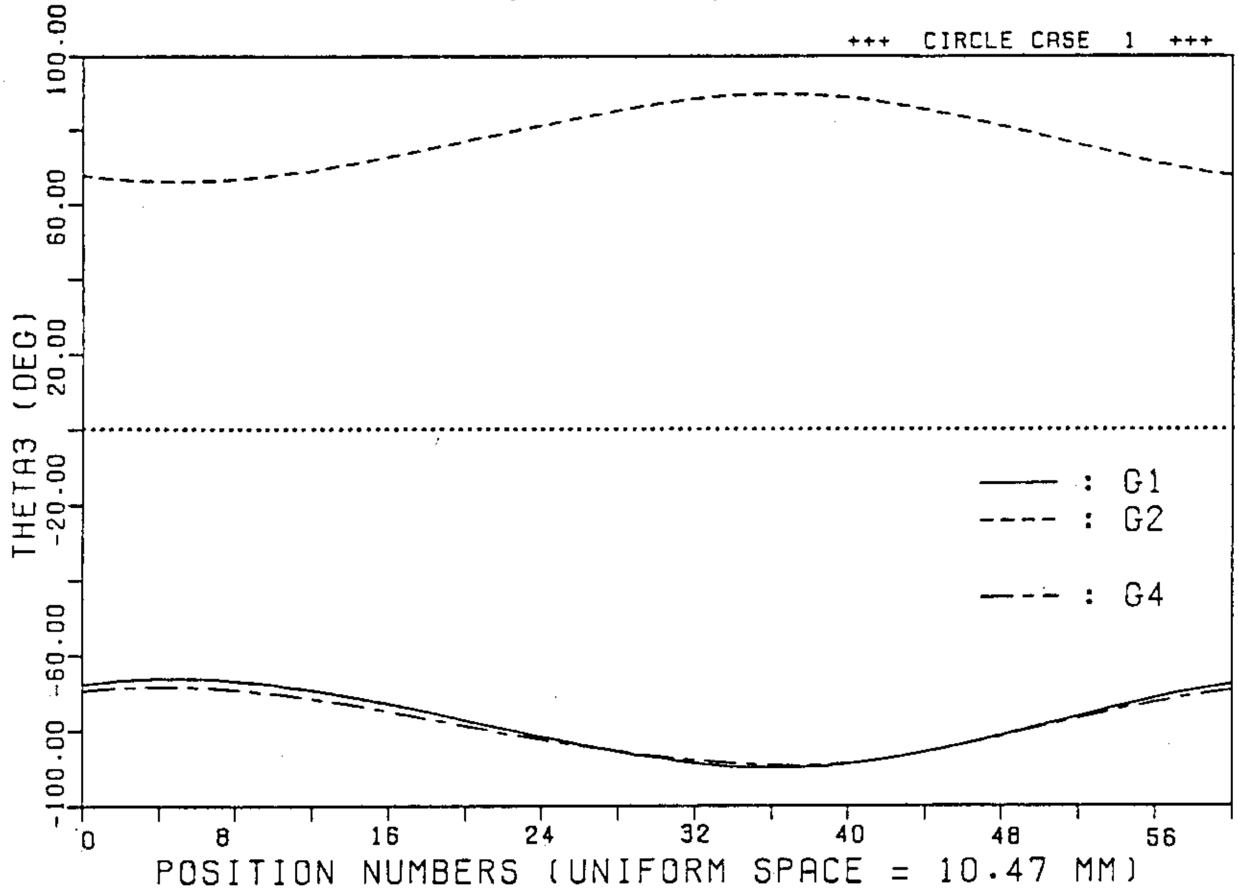

Fig. 7.3 Behavior of Joint Angle θ_3 (Sample Problem 4)

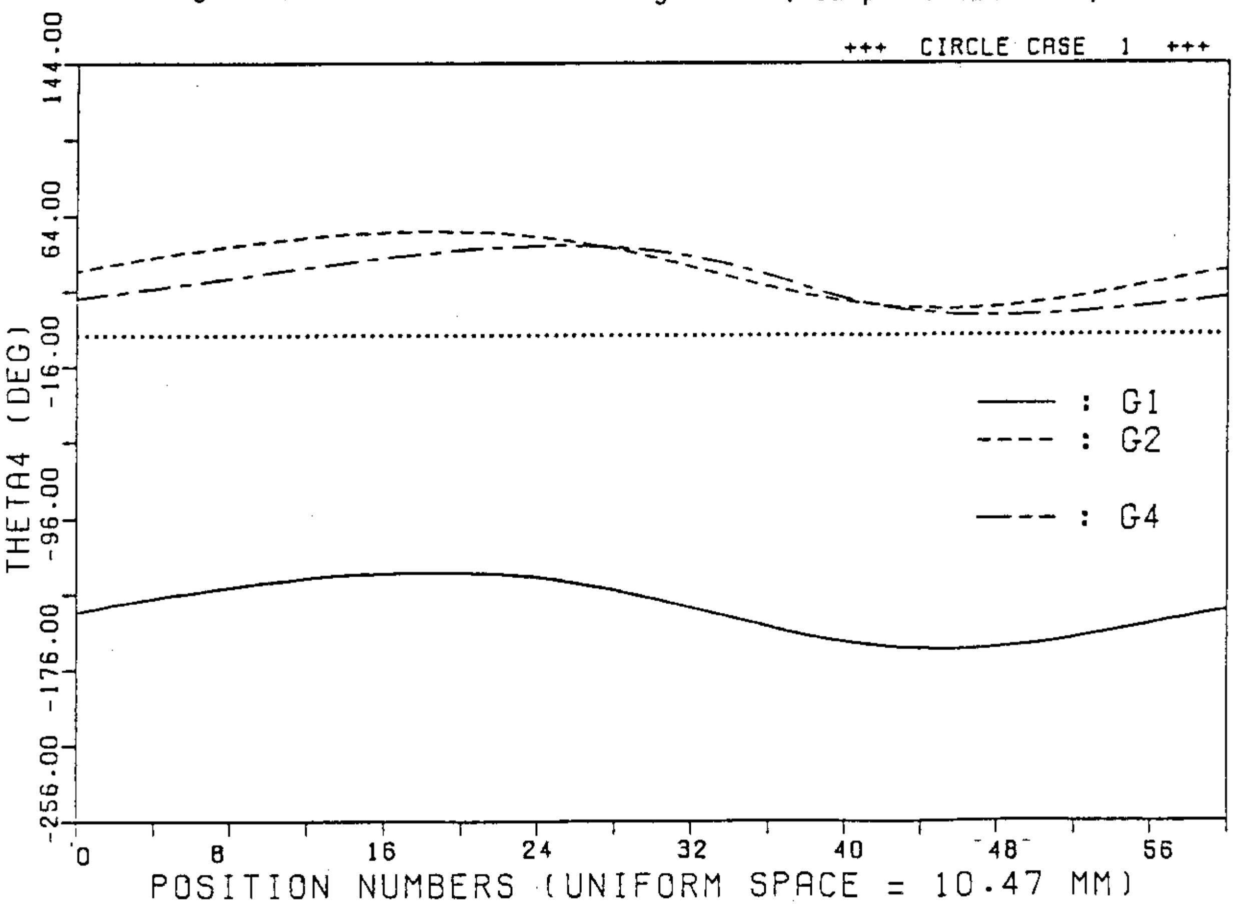

Fig. 7.4 Behavior of Joint Angle θ_4 (Sample Problem 4)

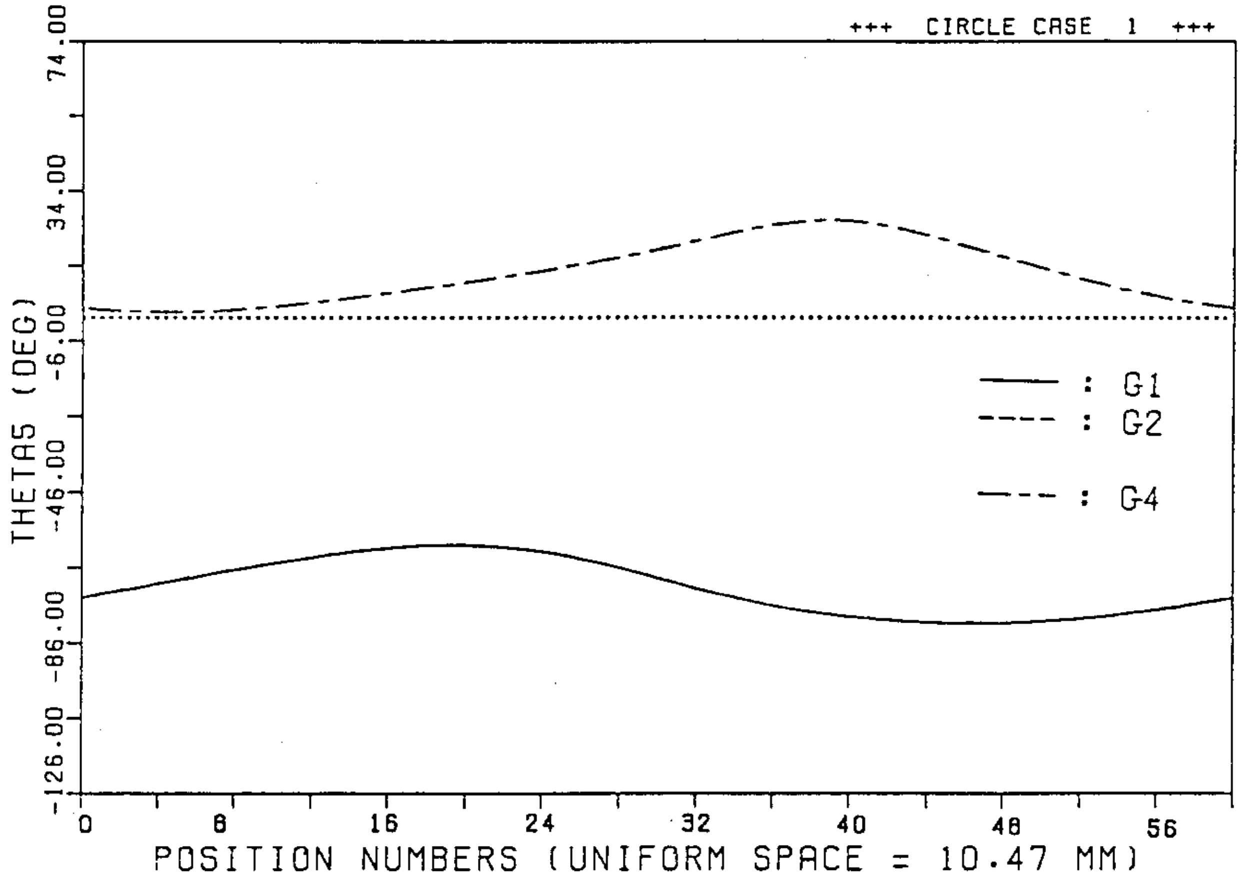

Fig. 7.5 Behavior of Joint Angle θ_5 (Sample Problem 4)

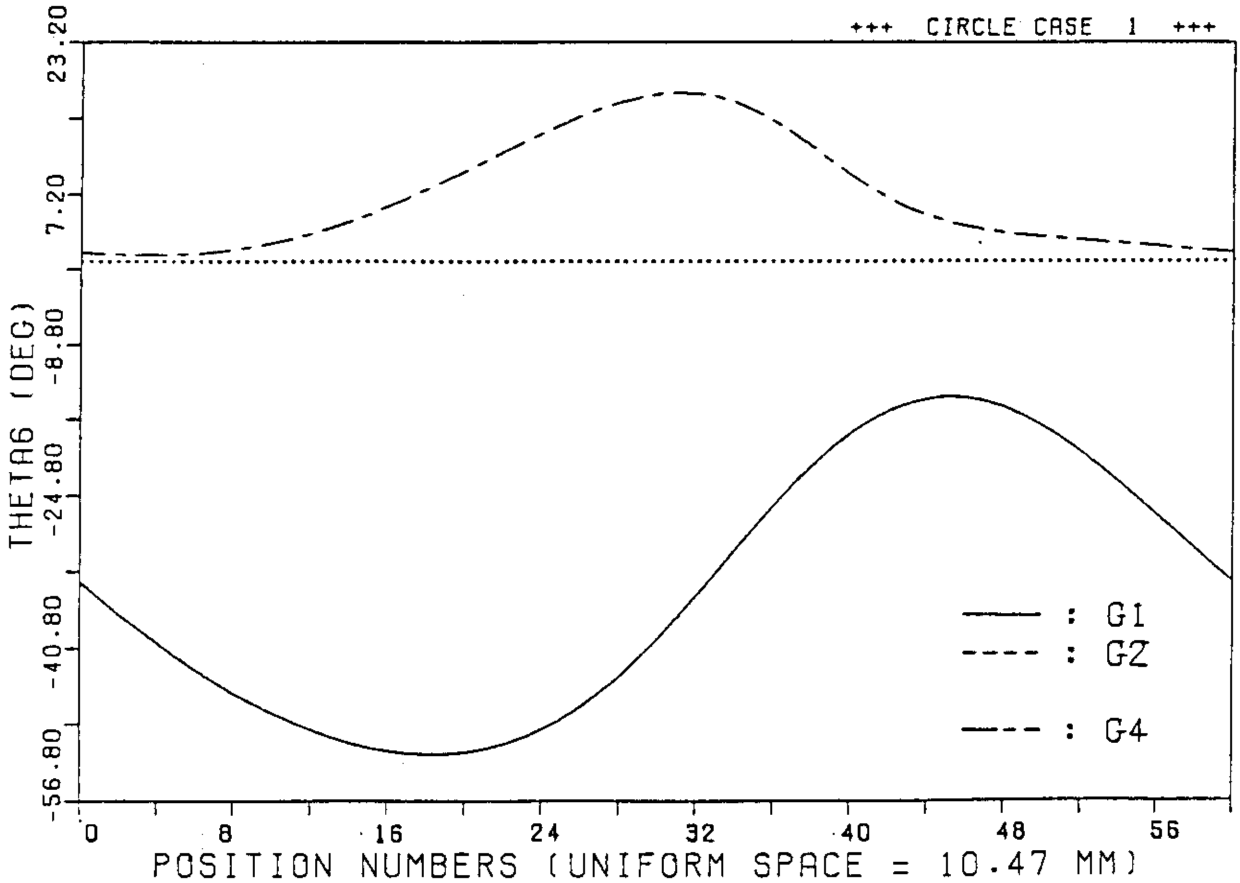

Fig. 7.6 Behavior of Joint Angle θ_6 (Sample Problem 4)

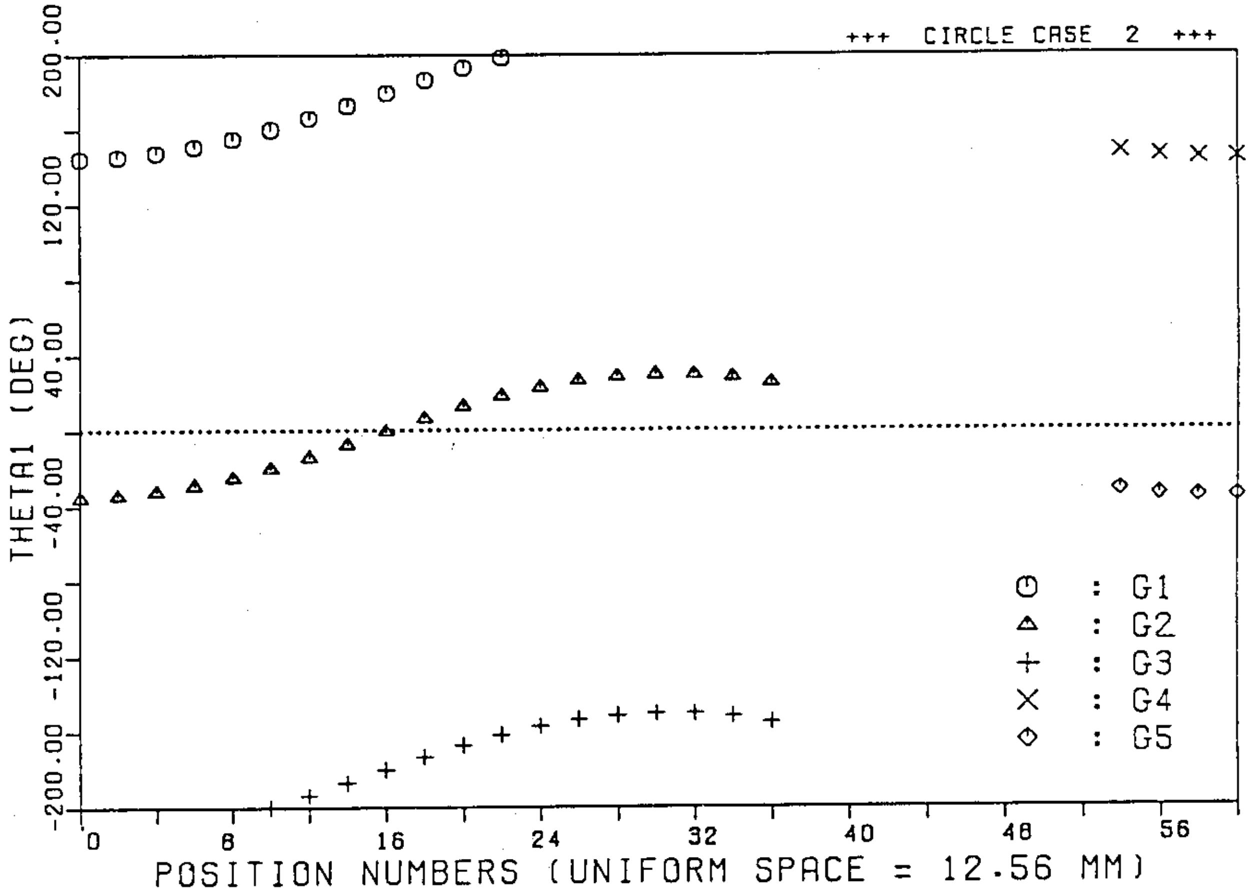

Fig. 8.1 Behavior of Joint Angle θ_1 (Sample Problem 5)

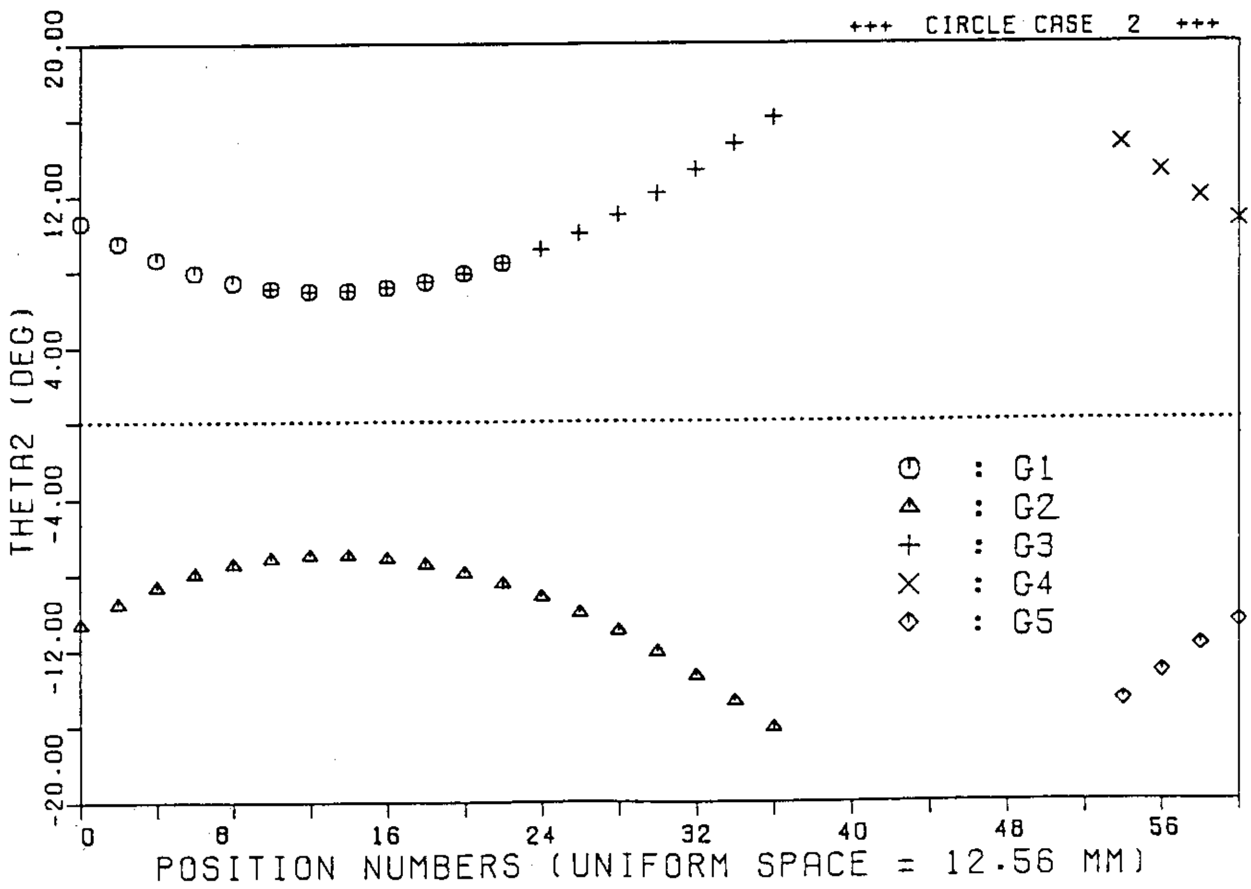

Fig. 8.2 Behavior of Joint Angle θ_2 (Sample Problem 5)

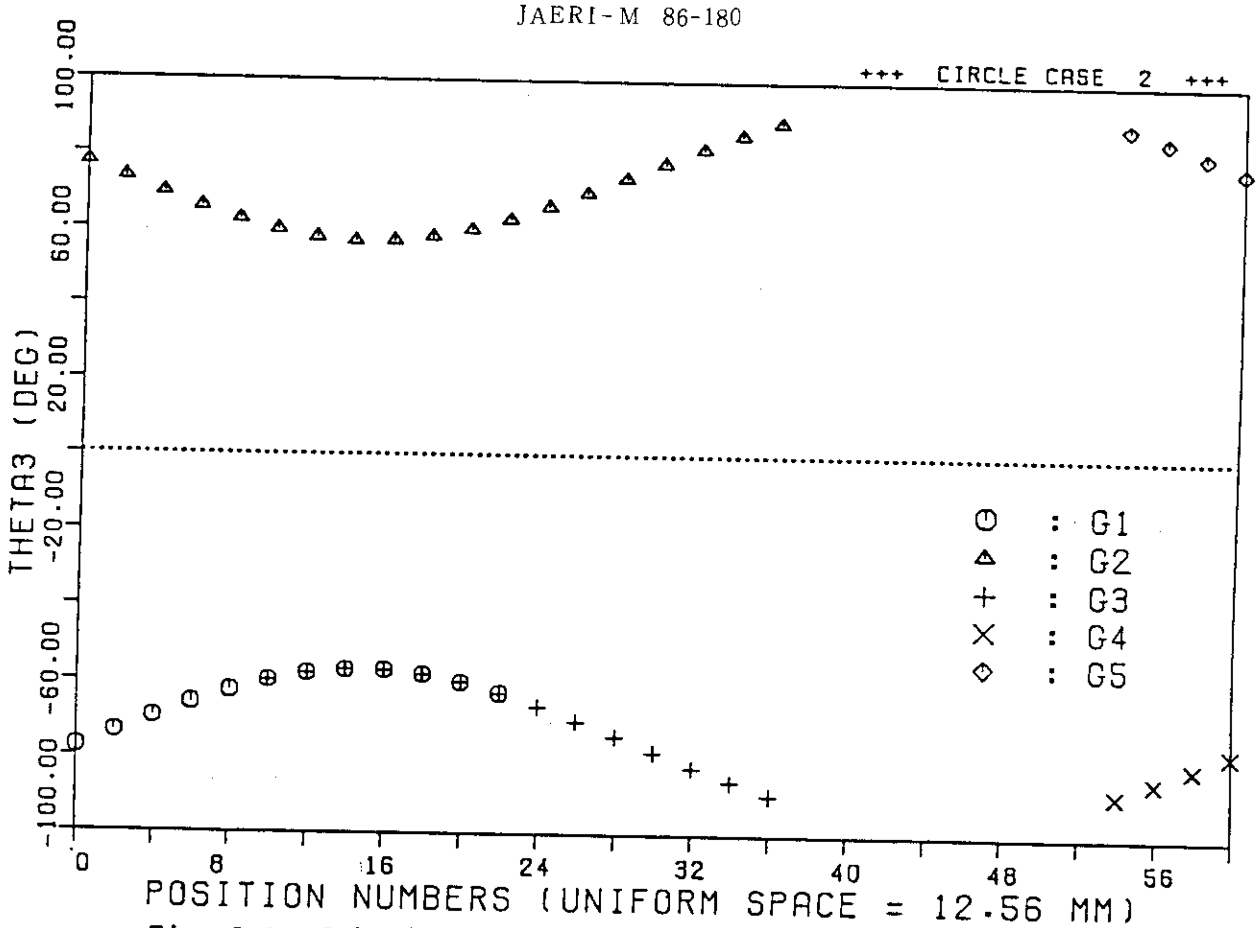

Fig. 8.3 Behavior of Joint Angle θ_3 (Sample Problem 5)

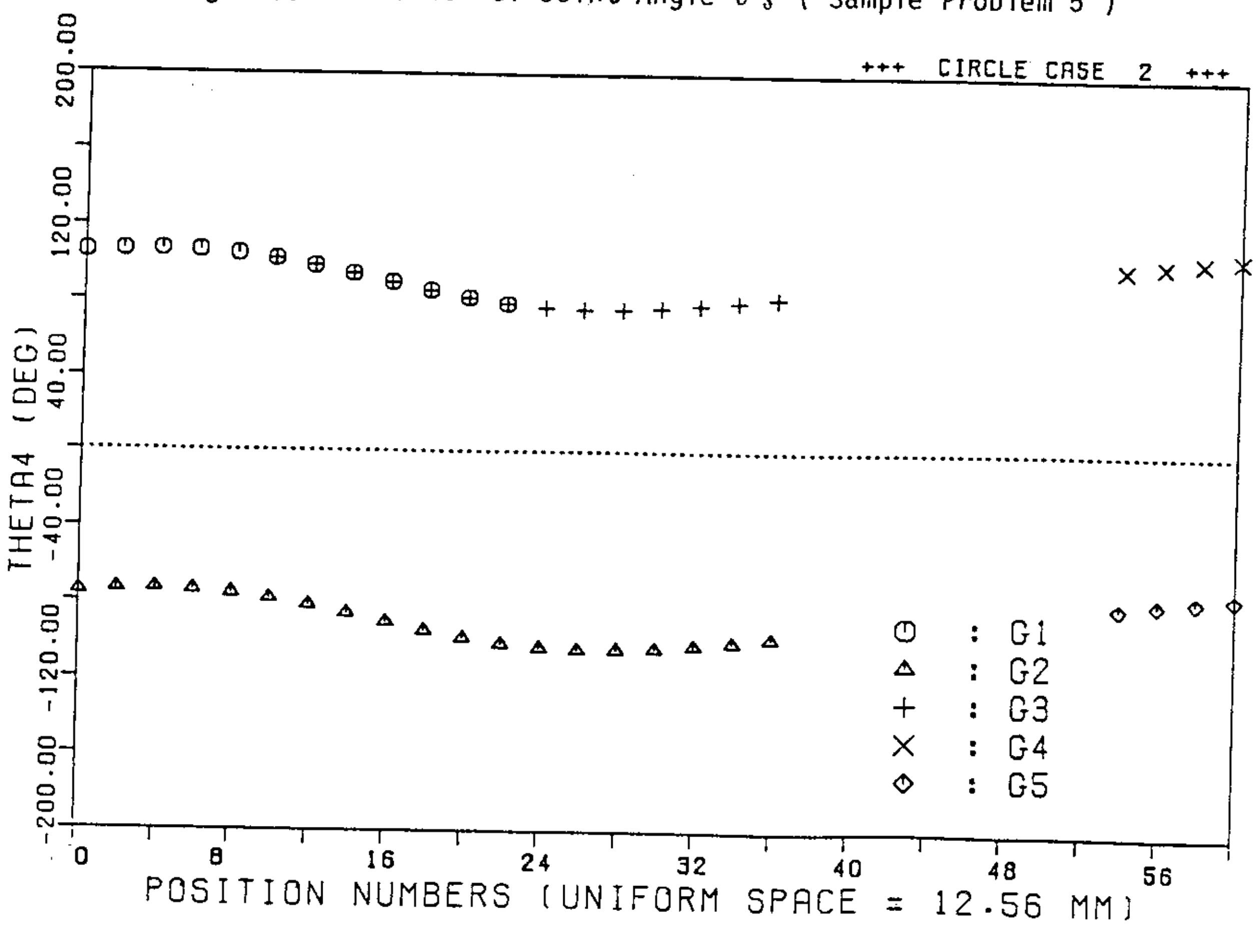

Fig. 8.4 Behavior of Joint Angle θ_4 (Sample Problem 5)

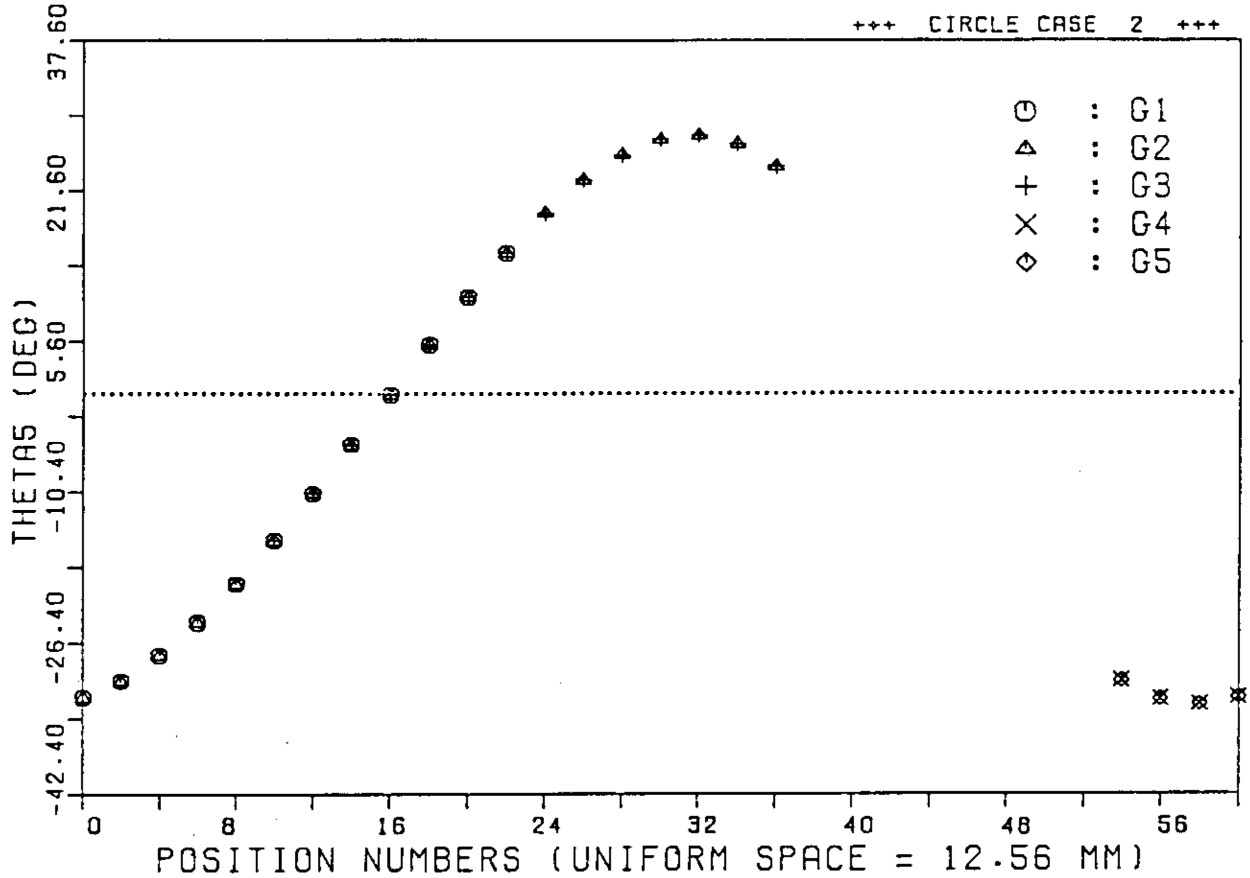

Fig. 8.5 Behavior of Joint Angle θ_5 (Sample Problem 5)

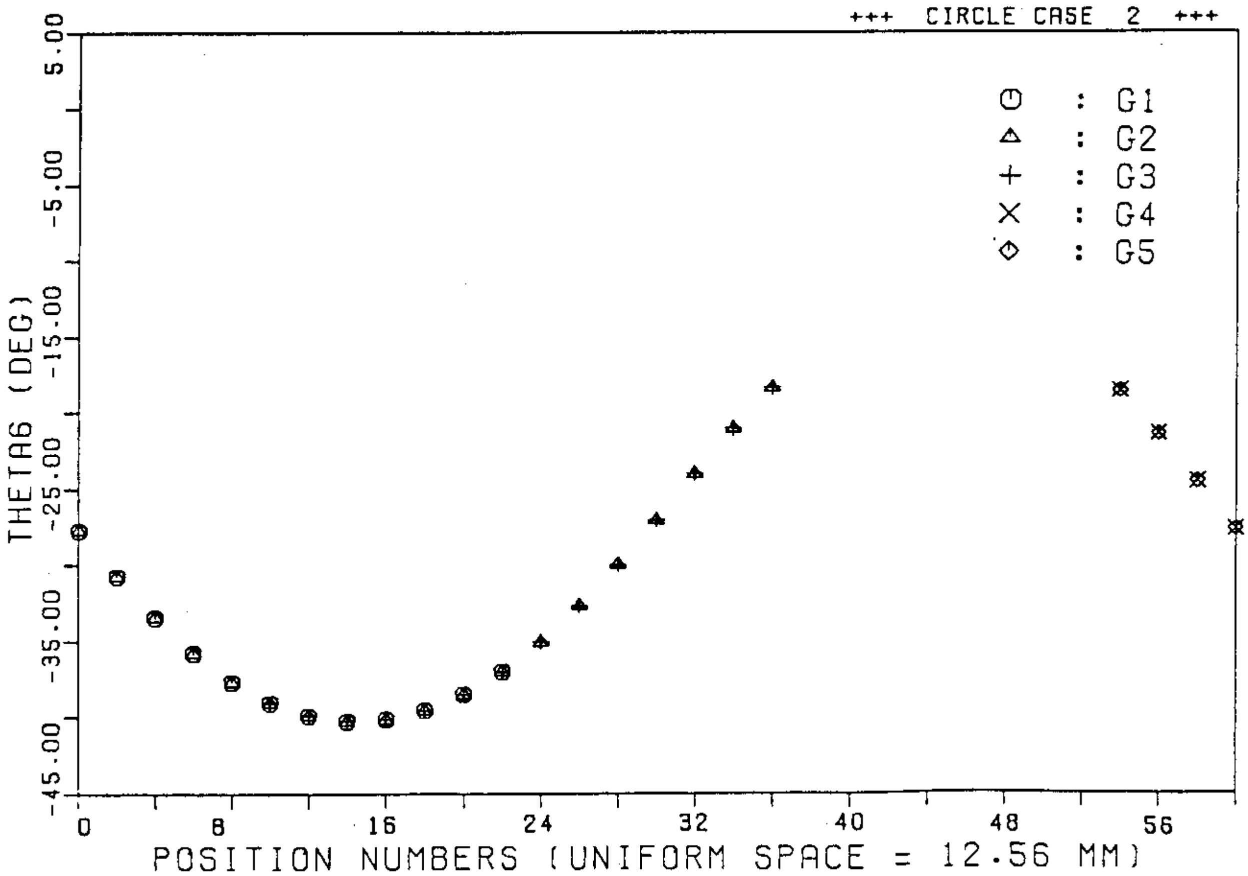

Fig. 8.6 Behavior of Joint Angle θ_6 (Sample Problem 5)

4. Concluding Remarks

Using the present algorithm, the inverse problem of a manipulator with a mechanical offset was solved successfully for most cases given and complete angular behaviors were obtained with sufficient accuracy.
Throughout the previous and present results, we have been broadening the domain of application for the method. Its advantage lies in the capability to obtain exact solution behaviors of the non-linear system composed of joint variables. In particular, as exemplified in the test calculations, the present polynomial method is useful in determining all feasible joint configurations corresponding to the same position and orientation of the finger tip in the actual work space. Since the proposed method has been developed yet for general cases, generalization of the concept remains as a future problem to be studied.

Acknowledgements

Thanks are due to Mr. Shinohara, Chief of Reactor Control Laboratory, for encouraging this work. We also thank our colleagues for their technical support.

4. Concluding Remarks

Using the present algorithm, the inverse problem of a manipulator with a mechanical offset was solved successfully for most cases given and complete angular behaviors were obtained with sufficient accuracy.
Throughout the previous and present results, we have been broadening the domain of application for the method. Its advantage lies in the capability to obtain exact solution behaviors of the non-linear system composed of joint variables. In particular, as exemplified in the test calculations, the present polynomial method is useful in determining all feasible joint configurations corresponding to the same position and orientation of the finger tip in the actual work space. Since the proposed method has been developed yet for general cases, generalization of the concept remains as a future problem to be studied.

Acknowledgements

Thanks are due to Mr. Shinohara, Chief of Reactor Control Laboratory, for encouraging this work. We also thank our colleagues for their technical support.

References

(1) Denavit, J. and Hartenberg, R.S. : " A Kinematic Notation for Low-pair Mechanism Based on Matrices ", Trans. ASME, J. of Applied Mechanics, June 1955, pp 215/221.

(2) Paul, R.P. : " Robot Manipulators ; Mathematics, Programming and Control ", The MIT Press, 1981.

(3) Coiffet, P. : " Robot Technology - Modelling and Control ", Vol. 1, Prentice-Hall, 1983.

(4) Brady, M., Hollerbach, J.M., Johnson,T.L., Lozano-Perez,T. and Mason, M.T. : " Robot Motion ; Planning and Control ", The MIT Press, 1983.

(5) Nakano, E. : " Introduction to Robotics ", Ohmu-sha, 1983.

(6) Uchiyama, M. :" Mechanism and Analysis of Robot-Arm ", J. of the Society of Instrument and Control Engineers (Japanese), 25-1, 7/14, 1986.

(7) Sasaki, S. :" A Method of Solving the Inverse Kinematics of a Manipulator Arm ", JAERI-M 86-018, Feb. 1986.

(8) Sasaki, S. :" Computer Code ARM1 for Solving the Inverse Kinematics of a Six-link Manipulator Arm ", JAERI-M 86-059, Mar. 1986.

(9) Sasaki, S. :" A Rigorous Algorithm for Solving the Inverse Kinematics of a Manipulator Arm ", JAERI-M 86-108, July 1986.

(10) Sasaki, S. :" Derivation of Manipulator Kinematics Based on a Vector Formulation ", JAERI-M 86-122, July 1986.

(11) Isoda, K. and Ohno, Y. :" Handbook of Numerical Calculations ", Ohmusha, 1971.

Appendix 1. Outline of Bairstow's method

We use Bairstow's iterative method to find an approximation to a quadratic factor of a given polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = 0 \tag{1}$$

Let $Q(x) = x^2 + px + q$ be any quadratic polynomial with real coefficients. Then

$$\left.\begin{array}{l} f(x) = (x^2 + px + q)Q_1(x) + Rx + S \\ \text{where a quotient } Q_1(x) = b_0x^{n-2} + b_1x^{n-3} + \dots b_{n-2} \,. \end{array}\right\} \tag{2}$$

$$\left.\begin{array}{l} \text{Additionally, assume } Q\ (x) = (x^2 + px + q)Q_2(x) + \bar{R}x + \bar{S}, \\ \text{where} \quad Q_2(x) = c_0x^{n-4} + c_1x^{n-5} + \dots + c_{n-4} \end{array}\right\} \tag{3}$$

Thus, we get:

$$\left.\begin{array}{l} b_0 = a_0,\ b_1 = a_1 - pb_0 \\ b_k = a_k - pb_{k-1} - qb_{k-2} \quad (k = 2,3, \dots, n) \\ c_0 = b_0,\ c_1 = b_1 - pb_0 \\ c_k = b_k - pc_{k-1} - qc_{k-2} \quad (k = 2,3, \dots, n-1) \end{array}\right\} \tag{4}$$

Using the coefficient relations in Eqs.(1) to (4), we have

$$R = b_{n-1}, \quad S = b_n + pb_{n-1} \tag{5}$$

$$\bar{R} = C_{n-3}, \quad \bar{S} = C_{n-2} + pC_{n-3} \,. \tag{6}$$

Suppose that $XQ_1(x)$ divided by $(x^2 + px + q)$ gives $R^*x + S^*$ as the remainder.

or

$$
\begin{aligned}
xQ_1(x) &= Q_3(x)(x^2 + px + q)\ R^*x + S^* \\
&= (x^2 + px + q)xQ_2(x) + \bar{R}x^2 + \bar{S}x \\
&= (x^2 + px + q)xQ_2(x) + c_{n-3}x^2 + (c_{n-2} + pc_{n-3})x \\
&= (x^2 + px + q)(xQ_2(x) + c_{n-3}) + c_{n-2}x - qc_{n-3} \qquad (7)
\end{aligned}
$$

Hence, we obtain

$$
\left.\begin{aligned}
R^* &= c_{n-2} \\
S^* &= -c_{n-3}\,q = c_{n-1} - b_{n-1} + p\,c_{n-2}
\end{aligned}\right\} \qquad (8)
$$

On the other hand, if we differentiate Eq.(2) with respect to p and q, we have

$$
\left.\begin{aligned}
xQ_1(x) &= -(x^2 + px + q)\frac{\partial Q_1}{\partial p} - x\frac{\partial R}{\partial p} - \frac{\partial S}{\partial p} \\
Q_1(x) &= -(x^2 + px + q)\frac{\partial Q_1}{\partial q} - x\frac{\partial R}{\partial q} - \frac{\partial S}{\partial q}
\end{aligned}\right\} \qquad (9)
$$

From these Eqs.(3), (6), (7) and (8),

$$
\left.\begin{aligned}
\frac{\partial R}{\partial p} &= -R^* = -c_{n-2} \\
\frac{\partial S}{\partial p} &= -S^* = -(c_{n-1} - b_{n-1} + p\,c_{n-2}) \\
\frac{\partial R}{\partial q} &= -\bar{R} = -c_{n-3} \\
\frac{\partial S}{\partial q} &= -\bar{S} = -(c_{n-2} + p\,c_{n-3})
\end{aligned}\right\} \qquad (10)
$$

If $Q(x) = Q'(x) = (x^2 + p'x + q')$ is a factor of $f(x)$, R and S in Eq.(2) must be zero.

i.e.,
$$
\left.\begin{aligned}
R(p,q) &= 0 \\
S(p,q) &= 0
\end{aligned}\right\} \qquad (11)
$$

Suppose that $Q(x)$ is an approximation to $Q'(x)$; then, p and q are assumed to have approximations to p' and q', respectively.

or $p' = p + \Delta p$

$q' = q + \Delta q$

Therefore,

$$\left.\begin{array}{l} R(p+\Delta p,\ q+\Delta q) = 0 \\ S(p+\Delta p,\ q+\Delta q) = 0 \end{array}\right\} \qquad (12)$$

Expanding in Taylor series and truncating after the first-order terms, we get:

$$\left.\begin{array}{l} R(p,q) + \dfrac{\partial R}{\partial p}\Delta p + \dfrac{\partial R}{\partial q}\Delta q = 0 \\ S(p,q) + \dfrac{\partial S}{\partial R}\Delta p + \dfrac{\partial S}{\partial q}\Delta q = 0 \end{array}\right\} \qquad (13)$$

Solving Eq.(13), we find that:

$$\left.\begin{array}{l} \Delta p = \dfrac{R\cdot c_{n-2} - S\cdot c_{n-3}}{c_{n-2}^2 - \bar{c}_{n-1}\cdot c_{n-3}} = \dfrac{b_{n-1}c_{n-2} - b_n c_{n-3}}{c_{n-2}^2 - \bar{c}_{n-1}\cdot c_{n-3}} \\ \Delta q = \dfrac{S\cdot c_{n-2} - R\cdot c_{n-1}}{c_{n-2}^2 - c_{n-1}\cdot c_{n-3}} = \dfrac{b_n c_{n-2} - b_{n-1}\bar{c}_{n-1}}{c_{n-2}^2 - \bar{c}_{n-1}\cdot c_{n-3}} \end{array}\right\} \qquad (14)$$

,

where $c_{n-2}^2 - \bar{c}_{n-1}c_{n-3} \neq 0$

$$\bar{c}_{n-1} = c_{n-1} - b_{n-1} \qquad (15)$$

Now we can replace p by $(p + \Delta p)$, and q by $(q + \Delta q)$, and repeat the above procedure.

Appendix 2. Input Data Formats

The current code written in FORTRAN 77 is operational on the FACOM-380 computer. In this section, the input data requirements are presented in the form necessary for computer execution from TSS terminal.

Record 1	TITLE (A50)*	=	Title of problem
Record 2	NDEL(I*4) (NDEL > 0)	=	Position numbers between initial and terminal position. If NDEL = 0, Record NO.7 through NO.9 are neglected, because a point interpolation is not used.
Record 3	EPS(R*8)	=	Convergence condition related to the determination of a quadratic factor ($x^2 + px + q$)
	EPS1(R*8)	=	Check of the validity of the calculated joint angles. Joint angles solutions are substituted into the components of the T_6 matrix and compared with the given data.
	MC(I*4)	=	Index of the trajectory 0: simulation of straight line 1: simulation of circle on the xy-plane 2: simulation of circle on the yz-plane 3: simulation of circle on the zx-plane

*) ( ) denotes the type of variable.

Record 4∿Record 6 ---- Initial point data or center of circle ----

Record 4 DATA(1,4,1) = P_x(R*8) = Initial position of x-direction of the manipulator hand (m) or x-coordinate at the center of circle trajectory (m)

DATA(2,4,1) = P_y(R*8) = Initial position of y-direction of the manipulator hand (m) or y-coordinate at the center of circle trajectory (m)

DATA(3,4,1) = P_z(R*8) = Initial position of z-direction of the manipulator hand (m) or z-coordinate at the center of circle trajectory (m)

Record 5 GET('A1') = Option of orientation calculation

= '0' : user specified

= '1' : Roll-Pitch-Yaw transformation

= 'z' : Return to the initial stage Record 1 (re-trial of input data)

= excepting the above letter (default) : Euler transformation

Notice that this data is regarded as to be dummy in case of circle.

Only the next data correponding to GET='0' of Record 6 is required.

Record 6

(i) If GET = '0', then

DATA(1,1,1) = NX(R*8) = x-component of approach vector n

DATA(2,1,1) = NY(R*8) = y-component of approach vector n

DATA(3,1,1) = NZ(R*8) = z-component of approach vector n

DATA(1,2,1) = OX(R*8) = x-component of sliding vector o

DATA(2,2,1) = OY(R*8) = y-component of sliding vector o

DATA(3,2,1) = OZ(R*8) = z-component of sliding vector o

(ii) If GET = '1', then

w_1(R*8) = Rotation angle about the z-axis (deg)

w_2(R*8) = Rotation angle about the y-axis (deg)

w_3(R*8) = Rotation angle about the x-axis (deg)

The direction cosines are calculated as follows.

DATA(1,1,1) = $\cos w_1 \cos w_2$

DATA(2,1,1) = $\sin w_1 \cos w_2$

DATA(3,1,1) = $-\sin w_2$

DATA(1,2,1) = $\cos w_1 \sin w_2 \sin w_3 - \sin w_1 \cos w_3$

DATA(2,2,1) = $\sin w_1 \sin w_2 \sin w_3 + \cos w_1 \cos w_3$

DATA(3,2,1) = $\cos w_2 \sin w_3$

(iii) If GET = Euler option, then

w_1(R*8) = Rotation angle about the z-axis (deg)

w_2(R*8) = Rotation angle about the y-axis (deg)

w_3(R*8) = Rotation angle about the z-axis (deg)

The direction cosines are calculated as follows.

DATA(1,1,1) = $\cos w_1 \cos w_2 \cos w_3 - \sin w_1 \sin w_3$

DATA(2,1,1) = $\sin w_1 \cos w_2 \cos w_3 + \cos w_1 \sin w_3$

DATA(3,2,1) = $-\sin w_2 \cos w_3$

DATA(1,2,1) = $-\cos w_1 \cos w_2 \sin w_3 - \sin w_1 \cos w_3$

DATA(2,2,1) = $-\sin w_1 \cos w_2 \sin w_3 + \cos w_1 \cos w_3$

DATA(3,2,1) = $\sin w_2 \sin w_3$

Record 7 ∿ Record 9 ---- Terminal point data or radius ----

Record 7 DATA(1,4,2) = P_x(R*8) = Terminal position of x-direction of the manipulator hand (m) or radius of circle (m)

Successive data is regarded as dummy in case of circle.

DATA(2,4,2) = P_y(R*8) = Terminal position of y-direction of the manipulator hand (m)

DATA(3,4,2) = P_z(R*8) = Terminal position of z-direction of the manipulator hand (m)

Record 8 GET('A1') = Option of orientation calculation

= '0' : user specified

= '1' : Roll-Pitch-Yaw transformation

= 'z' : Return to the initial stage Record 7 (re-trial of input data)

= excepting the above letter (default) : Euler transformation

Record 9

(i) If GET = '0', then

DATA(1,1,2) = NX(R*8) = x-component of approach vector n

DATA(2,1,2) = NY(R*8) = y-component of approach vector n

DATA(3,1,2) = NZ(R*8) = z-component of approach vector n

DATA(1,2,2) = OX(R*8) = x-component of sliding vector o

DATA(2,2,2) = OY(R*8) = y-component of sliding vector o

DATA(3,2,2) = OZ(R*8) = z-component of sliding vector o

(ii) If GET = '1', then

w_1(R*8) = Rotation angle about the z-axis (deg)

w_2(R*8) = Rotation angle about the y-axis (deg)

w_3(R*8) = Rotation angle about the x-axis (deg)

The direction cosines are calculated as follows.

DATA(1,1,2) = $\cos w_1 \cos w_2$

DATA(2,1,2) = $\sin w_1 \cos w_2$

DATA(3,1,2) = $-\sin w_2$

DATA(1,2,2) = $\cos w_1 \sin w_2 \sin w_3 - \sin w_2 \cos w_3$

DATA(2,2,2) = $\sin w_1 \sin w_2 \sin w_3 + \cos w_1 \cos w_3$

DATA(3,2,2) = $\cos w_2 \sin w_3$

(iii) If GET = Euler option, then

w_1(R*8) = Rotation angle about the z-axis (deg)

w_2(R*8) = Rotation angle about the y-axis (deg)

w_3(R*8) = Rotation angle about the z-axis (deg)

The direction cosines are calculated as follows.

DATA(1,1,2) = $\cos w_1 \cos w_2 \cos w_3 - \sin w_1 \sin w_3$

DATA(2,1,2) = $\sin w_1 \cos w_2 \cos w_3 + \cos w_1 \sin w_3$

DATA(3,1,2) = $-\sin w_2 \cos w_3$

DATA(1,2,2) = $-\cos w_1 \cos w_2 \sin w_3 - \sin w_1 \cos w_3$

DATA(2,2,2) = $-\sin w_1 \cos w_2 \sin w_3 + \cos w_1 \cos w_3$

DATA(3,2,2) = $\sin w_2 \sin w_3$

Appendix 3. List of Input Data for Sample Problems

(1) Sample problem 1

```
 INPUT --- TITLE
+++++++++++++++++ T   I   T   L   E ++++++++++++++++++
+                                                     +
+ +++  BENCH  MARK  20  +++                           +
+                                                     +
+++++++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>   41

 INPUT --- EPS , EPS1

 EPS   ===>  1.00000D-04     EPS1   ===>  1.00000D-03

  MC        0 : STRAIGHT LINE
            1 : CIRCLE ( X-Y PLANE )
            2 : CIRCLE ( Y-Z PLANE )
            3 : CIRCLE ( Z-X PLANE )

     MC  ===>  0

---------- INITIAL   POINT ----------

 INPUT --- PX : PY : PZ

 PX ==> -1.00000D-01   PY ==>  3.50000D-01   PZ ==>  1.63100D+00 ( M )

 KEYIN      0 : USER SPECIFIED
            1 : ROLL-YAW-PITCH ANGLE
            2 : RE-TRIAL
     DEFAULTS : EULER ANGLE

 DIRECTION CALCULATION OPTION = 0
 INPUT --- NX , NY , NZ

 NX ==>  0.0             NY ==>  0.0            NZ ==>  1.00000D+00

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0            OZ ==>  0.0

---------- TERMINAL  POINT ----------

 INPUT --- PX : PY : PZ

 PX ==>  1.00000D-01   PY ==>  3.50000D-01   PZ ==>  1.63100D+00 ( M )

 KEYIN      0 : USER SPECIFIED
            1 : ROLL-YAW-PITCH ANGLE
            2 : RE-TRIAL
     DEFAULTS : EULER ANGLE

 DIRECTION CALCULATION OPTION = 0
 INPUT --- NX , NY , NZ

 NX ==>  0.0             NY ==>  0.0            NZ ==>  1.00000D+00

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0            OZ ==>  0.0
```

(2) Sample problem 2

```
 INPUT --- TITLE
++++++++++++++++++ T  I  T  L  E +++++++++++++++++++
+                                                  +
+ +++  BENCH  MARK  3D +++                         +
+                                                  +
++++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>   40

 INPUT --- EPS , EPS1

 EPS    ===>  1.000000D-04    EPS1    ===>  1.000000D-03


  MC        0 : STRAIGHT LINE
            1 : CIRCLE ( X-Y PLANE )
            2 : CIRCLE ( Y-Z PLANE )
            3 : CIRCLE ( Z-X PLANE )

     MC  ===>  0

---------- INITIAL  POINT ----------


 INPUT --- PX : PY : PZ

 PX ==>  0.0            PY ==>  4.30000D-01   PZ ==>  1.35100D+00 ( M )


 KEYIN       0 : USER SPECIFIED
             1 : ROLL-YAW-PITCH ANGLE
             2 : RE-TRIAL
     DEFAULTS : EULER ANGLE


 DIRECTION CALCULATION OPTION = 0
 INPUT --- NX , NY , NZ

 NX ==>  0.0            NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00    OY ==>  0.0           OZ ==>  0.0

---------- TERMINAL  POINT ----------


 INPUT --- PX : PY : PZ

 PX ==>  0.0            PY ==>  4.30000D-01   PZ ==>  1.55100D+00 ( M )


 KEYIN       0 : USER SPECIFIED
             1 : ROLL-YAW-PITCH ANGLE
             2 : RE-TRIAL
     DEFAULTS : EULER ANGLE


 DIRECTION CALCULATION OPTION = 0
 INPUT --- NX , NY , NZ

 NX ==>  0.0            NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  1.000000D+00   OY ==>  0.0           OZ ==>  0.0
```

(3) Sample problem 3

```
 INPUT --- TITLE
++++++++++++++++++ T  I  T  L  E +++++++++++++++++++
+                                                   +
+ +++  BENCH  MARK  4  +++                          +
+                                                   +
+++++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>   50

 INPUT --- EPS , EPS1

 EPS    ===>  1.00000D-04    EPS1    ===>  1.00000D-03


  MC        0 : STRAIGHT LINE
            1 : CIRCLE ( X-Y PLANE )
            2 : CIRCLE ( Y-Z PLANE )
            3 : CIRCLE ( Z-X PLANE )

     MC  ===>  0

---------- INITIAL   POINT ----------


 INPUT --- PX : PY : PZ

 PX ==> -2.50000D-01   PY ==>  6.50000D-01   PZ ==>  1.30000D+00 ( M )


 KEYIN       0 : USER SPECIFIED
             1 : ROLL-YAW-PITCH ANGLE
             2 : RE-TRIAL
     DEFAULTS : EULER ANGLE


 DIRECTION CALCULATION OPTION = 0
 INPUT --- NX , NY , NZ

 NX ==>  0.0             NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  0.0             OY ==>  0.0           OZ ==>  1.00000D+00

---------- TERMINAL  POINT ----------


 INPUT --- PX : PY : PZ

 PX ==>  1.00000D-01   PY ==>  7.00000D-01   PZ ==>  1.55000D+00 ( M )


 KEYIN       0 : USER SPECIFIED
             1 : ROLL-YAW-PITCH ANGLE
             2 : RE-TRIAL
     DEFAULTS : EULER ANGLE


 DIRECTION CALCULATION OPTION = 0
 INPUT --- NX , NY , NZ

 NX ==>  0.0             NY ==>  1.00000D+00   NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  0.0             OY ==>  0.0           OZ ==>  1.00000D+00
```

(4) Sample problem 4

```
 INPUT --- TITLE
+++++++++++++++++++ T   I   T   L   E +++++++++++++++++++
+                                                        +
+ +++  CIRCLE CASE  1  +++                               +
+                                                        +
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>   60

 INPUT --- EPS , EPS1

 EPS    ===>  1.00000D-04    EPS1    ===>  1.000000D-03

  MC        0 : STRAIGHT LINE
            1 : CIRCLE ( X-Y PLANE )
            2 : CIRCLE ( Y-Z PLANE )
            3 : CIRCLE ( Z-X PLANE )

     MC  ===>  1

  -------  CENTER OF CIRCLE   -------

 INPUT --- PX : PY : PZ

 PX ==>  1.500000-01   PY ==>  2.000000-01   PZ ==>  1.680000+00 ( M )

 INPUT --- NX , NY , NZ

 NX ==>  0.0           NY ==>  0.0           NZ ==>  1.000000D+00

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0           OZ ==>  0.0

  -------       RADIUS       -------

 RADIUS ===>  1.000000D-01 ( M )

 INPUT --- NX , NY , NZ

 NX ==>  0.0           NY ==>  0.0           NZ ==>  1.000000D+00

 INPUT --- OX , OY , OZ

 OX ==>  1.00000D+00   OY ==>  0.0           OZ ==>  0.0
```

(5) Sample problem 5

```
 INPUT --- TITLE
+++++++++++++++++++ T  I  T  L  E +++++++++++++++++++
+                                                   +
+ +++  CIRCLE CASE  2 _ +++                          +
+                                                   +
+++++++++++++++++++++++++++++++++++++++++++++++++++++

 INPUT --- N

 N      ===>   60

 INPUT --- EPS , EPS1

 EPS    ===>  1.000000-04    EPS1    ===>  1.00000D-03


  MC        0 : STRAIGHT LINE
            1 : CIRCLE ( X-Y PLANE )
            2 : CIRCLE ( Y-Z PLANE )
            3 : CIRCLE ( Z-X PLANE )

     MC  ===>  3

  -------  CENTER OF CIRCLE   -------

 INPUT --- PX : PY : PZ

 PX ==> -1.00000D-01    PY ==>  6.00000D-01    PZ ==>   1.40000D+00 ( M )

 INPUT --- NX , NY , NZ

 NX ==>  0.0            NY ==>  1.00000D+00    NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  0.0            OY ==>  0.0            OZ ==>  1.00000D+00

  -------        RADIUS        -------

 RADIUS ===>  1.20000D-01 ( M )

 INPUT --- NX , NY , NZ

 NX ==>  0.0            NY ==>  1.00000D+00    NZ ==>  0.0

 INPUT --- OX , OY , OZ

 OX ==>  0.0            OY ==>  0.0            OZ ==>  1.00000D+00
```