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COMPUTER PROGRAM JN-METD3 TO DEAL WITH NEUTRON AND GAMMA-RAY TRANSPORT IN MULTILAYER SLABS BY THE j_N METHOD

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Takumi ASAOKA

日本原子力研究所
Japan Atomic Energy Research Institute

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Takumi ASAOKA

Division of Reactor Engineering, Tokai Research Establishment, JAERI

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Mathematical formulae of the j_N method are summarized for description of the neutron and gamma-ray transport in a multilayer slab system with anisotropic scattering in a multigroup model. A FORTRAN-IV computer program JN-METD3 is described in detail for solving the stationary transport problem within the context of j_7 approximation for space and P_3 approximation for scattering angle.

The computer code calculates the radiation flux from an extraneous source at the slab boundary as a function of space, angle and energy variables. In addition, it evaluates the eigenvalue of the integral transport equation, the effective multiplication factor or the asymptotic decay constant of neutrons for a slab reactor, as well as the eigenfunction, the space, angle and energy dependent flux distribution.

Keywords: Computer Program, Neutron Transport, Gamma-ray Transport, Multigroup Model, Multilayer Slab, Anisotropic Scattering, Stationary Problem, Radiation Flux, Effective Multiplication Factor, Asymptotic Decay Constant

j_N法による多重層平板内の中性子・ガンマ線輸送計算コード JN－METD 3

日本原子力研究所東海研究所原子炉工学部
朝岡　卓見

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多重層平板体系中の中性子とガンマ線の多群エネルギー・モデルでの輸送を，非等方散乱も考慮して記述するj_N法の基礎方程式が示された後，定常問題を，空間についてはj_7近似，散乱角に対してはP_3近似を用いて解くFORTRAN－IV計算プログラムJN－METD 3が詳細に説明されている。この計算コードでは，境界に平面線源のある平板体系中の放射線線束が，空間，角度，エネルギーの関数として計算される。また，積分型輸送方程式の固有値，すなわち，平板状原子炉の有効増倍係数あるいは中性子の漸近減衰定数，及び固有関数，すなわち，空間，角度，エネルギー依存の中性子束分布も求められるようになっている。

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1. Introduction

The j_N method has been developed from the multiple collision method[1] as a simple but accurate analytical approach to neutron transport in a finite system. The starting point to neutron transport in a finite system. The starting point of the method is an integral form of the time-dependent transport equation into which a delta function is introduced to fix the point of measurement. The rewriting of the delta function in the form of its Fourier representation is equivalent to taking a finite Fourier transform with respect to the space variable. Since the equation contains a convolution integral in time, Laplace transformation with respect to time leads to an integral equation for the Fourier-Laplace transformed emission density of neutrons (or the distribution of secondary neutrons). The transformed emission density and the kernel of the integral equation are then very conveniently expanded in spherical Bessel functions of the Fourier transformed variable. For spherical and plane geometries, the expansion of the transformed flux rather than the transformed emission density is equivalent to an expansion of the original flux in Legendre polynomials with respect to space[2][3]. A very convenient property of the expansion of the emission density is that the final expression of the flux obtained through the Fourier and Laplace inversion satisfies exactly the boundary conditions independently of the order of j_N approximation (truncation order of the expansion). The neutron flux for the stationary state can easily be written as a limiting case of time-dependent problems.

The approach outlined above has already been applied successfully to space-energy time-dependent problems in bare spheres[4] as well as space-angle energy-time dependent problems in homogeneous slabs[5], by assuming that the scattering of neutrons is spherically symmetric in the laboratory system. A computer program JN-METD1 has been developed for accurately solving both the stationary and time-dependent problems with a high computational efficiency[6]. The eigenvalues (e.g. the effective multiplication factor) and the eigenfunctions (the neutron flux distribution) of the integral equation converge to the exact values very rapidly as the order of j_N approximation increases, except for the flux at times close to the moment when the wave front of the direct neutron beam arrives. Moreover, the eigenvalues can be computed without any knowledge of the eigenfunctions which are therefore evaluated by the computer code only if required (e.g.

the flux in selected energy groups at selected space points). The calculation of the time moments of the time-dependent flux is also simple because it has been reduced to a stationary problem.

The j_N method can easily be extended to take into account anisotropic scattering of neutrons[7)] or to deal with multilayer slab systems[8)]. A complete formulation for treating neutron transport in multilayer slabs with anisotropic scattering was already given for both the stationary and time-dependent problems[9)]. A computer program JN-METD2 has then been developed for solving these problems under the assumption that the scattering of neutrons is spherically symmetric in the laboratory system[10)]. It evaluates the first three time moments of the time-dependent flux resulting from a delta function boundary source with space, angle and energy variables. In addition, it calculates the effective multiplication factor or the asymptotic decay constant of neutrons, and the space, angle and energy dependent flux distribution in multilayer slab systems.

Furthermore, the applicability of the approach to convex geometries has been demonstrated for a homogeneous medium in which the neutron scattering is isotropic[11)]. In the work, an expansion into ordinary Bessel functions of odd order was adopted for an infinite cylinder instead of the spherical Bessel functions used for the slab and spherical geometries. The method has recently been generalized to deal with an infinite cylinder with linearly anisotropic scattering of neutrons[12)].

The present report is concerned with the computer code JN-METD3 designed to solve neutron and gamma-ray transport in multilayer slab systems with anisotropic scattering in the (up to) P_3 approximation. By the use of a multigroup model and the j_N ($N \le 7$) approximation, the computer code calculates the following quantities:

(a) The space, angle and energy dependent radiation flux due to a stationary point-isotropic or monodirectional boundary source.

(b) The value of the effective multiplication factor k_{eff} of a multilayer slab reactor and the stationary neutron flux distribution as a function of space, angle and energy.

(c) The asymptotic decay constant of the fundamental neutron distribution in a multilayer slab system.

2. Mathematical Formulae

A general formulation for dealing with time-dependent neutron and gamma-ray transport in multilayer slabs with anisotropic scattering has already been shown in a previous paper[9]. We therefore only summarize here the mathematical formulae for a M-region slab within the content of a G-energy-group model and the j_N and P_L-scattering approximation.

Let x be the space coordinate, μ the direction cosine of the photon or neutron velocity, Σ_g^i the g-th group absorption coefficient or macroscopic total cross section for the i-th region, extending from $x = a_{i-1}$ to a_i (a_o=0), v_g the speed of neutron (or photon) in the g-th group and $c_i(g'\to g,\ \mu'\to\mu) = 2\pi c_i(g'\to g,\ \Omega'\to\Omega)$ the mean number of secondary photons or neutrons produced with the direction μ in the g-th group as a result of collisions in the g'-th group and i-th region with the direction μ'. The $c_i(g'\to g,\ \mu'\to\mu)$ is expanded into Legendre polynomials of the cosine of the scattering angle:

$$c_i(g'\to g,\ \mu'\to\mu) = \sum_{\ell=0}^{L} \frac{2\ell+1}{2}\, c_\ell^{\,i}(g'\to g)P_\ell(\mu_o) \ . \tag{1}$$

The g-th group gamma-ray or neutron flux in the j-th region resulting from a delta function boundary source $S_g(x,\mu,t) = S_g(\mu)\delta(x)\delta(t)$ can be written as

$$\phi_g^{\,j}(x,\mu,t) = \frac{1}{\mu}\, S_g(\mu)\,\delta(t - \frac{x}{v_g\mu})\exp[-(\sum_{k=1}^{j-1} \Sigma_g^{\,k}(a_k-a_{k-1})+\Sigma_g^{\,j}(x-a_{j-1}))/\mu]$$

$$+ \sum_{h} \exp[\Sigma_1^{\,1}v_1(s_h-1)t] \sum_{\ell=0}^{L} (2\ell+1) \sum_{p=0}^{N} \{ \sum_{i=1}^{j} F_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;\ -\sum_{k=i}^{j}\alpha_g^{\,k}+\frac{\alpha_g^{\,j}+\alpha_g^{\,i}}{2})$$

$$\times B_{p\ell}^{\,i}(g,s_h)+\sum_{i=j+1}^{M} F_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;\ \sum_{k=j}^{i}\alpha_g^{\,k} = \frac{\alpha_g^{\,j}+\alpha_g^{\,i}}{2})B_{p\ell}^{\,i}(g,s_h)\}_{s=\Sigma_1^{\,1}v_1 s_h}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ \exp[-(\Sigma v-iy)t] \sum_{\ell=0}^{L}(2\ell+1)\sum_{p=0}^{N}\{\sum_{i=1}^{j} F_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;\ -\sum_{k=i}^{j}\alpha_g^{\,k}$$

$$+\frac{\alpha_j^{\,j}+\alpha_j^{\,i}}{2})b_{p\ell}^{\,i}(g,s)$$

$$+\sum_{i=j+1}^{M} F_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;\ \sum_{k=j}^{i}\alpha_g^{\,k} - \frac{\alpha_g^{\,i}+\alpha_g^{\,j}}{2})b_{p\ell}^{\,i}(g,s)\}_{s=\Sigma_1^{\,1}v_1-\Sigma v+iy}$$

(2)

where $\alpha_g^k = P_g^k \Sigma_g^k (a_k - a_{k-1})[P_g^k = 1 - (\Sigma_1^1 v_1 - s)/(\Sigma_g^k v_g)]$, Σv stands for the minimum value of $\Sigma_g^k v_g$ for all g and k, and

$$F_{p\ell}(\alpha_g^i, \alpha_g^j, \xi, \mu, s; d) = \frac{1}{4\pi}|P_g^j| \int_{-\infty}^{\infty} dz\ j_p(\alpha_g^i z) \exp[i\alpha_g^j z(1-2\xi)] \ell^{idz}$$

$$\times \int_0^\infty dt' \exp[-P_g^j (i - iz\mu) t'] P_\ell(\mu). \quad (3)$$

The explicit expression for $F_{p\ell}$ in the j_7 and P_3 approximation ($0 \leq p \leq 7$ and $0 \leq \ell \leq 3$) is given in Appendix 1, Section 2.

In addition, $s = \Sigma_1^1 v_1 s_h$ and $B_{p\ell}^i(g, s_h)$ in Eq. (2) are respectively a pole and the residue of $b_{p\ell}^i(g,s)$ which satisfies the following linear equations:

$$\frac{1}{2q+1} b_{q\ell}^j(g',s) = \sum_{g=1}^{G} c_\ell^j(g \to g')|\alpha_g^j| S_g C_g^\ell\left(\frac{\alpha_g^j}{2}\ s;\ -\sum_{k=1}^{j} \alpha_g^k + \frac{\alpha_g^j + \alpha_g^i}{2}\right)$$
$$+ \sum_{g=1}^{G} c_\ell^j(g \to g') \sum_{m=0}^{L} (2m+1) \sum_{r=0}^{N} \left[\sum_{i=1}^{j} J_{qr}^{\ell m}\left(\frac{\alpha_g^j}{2}, \frac{\alpha_g^i}{2}, s; -\sum_{k=i}^{j} \alpha_g^k + \frac{\alpha_g^j + \alpha_g^i}{2}\right) b_{rm}^i(g,s)\right.$$
$$\left. + \sum_{i=j+1}^{M} J_{qr}^{\ell m}\left(\frac{\alpha_g^j}{2}, \frac{\alpha_g^i}{2}, s;\ \sum_{k=j}^{i} \alpha_g^k - \frac{\alpha_g^i + \alpha_g^j}{2}\right) b_{rm}^i(g,s)\right], \quad (4)$$

where

$$S_q C_q^\ell(\alpha_g^j, s; d) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz\ j_q(\alpha_g^j z) \ell^{idz} \exp(-i\alpha_g^1 z/2)$$

$$\times \int_0^\infty dt' \exp(-P_g^j t') \int_0^1 d\mu\ S_g(\mu) P_\ell(\mu) \exp(iP_g^j z t' \mu), \quad (5)$$

$$J_{qr}^{\ell m}(\alpha_g^j, \alpha_g^i, s; d) = \frac{1}{2\pi}|\alpha_g^j| \int_{-\infty}^{\infty} dz\ j_q(\alpha_g^j z) j_r(\alpha_g^i z) \ell^{idz}$$

$$\times \int_0^\infty dt' \exp(-P_g^j t') \int_{-1}^{1} d\mu P_\ell(\mu) P_m(\mu) \exp(iP_g^j z t' \mu). \quad (6)$$

The explicit expression for $J_{qr}^{\ell m}$ is shown in Appendix 1, Section 1 for $0 \leq q$, $r \leq 7$ and $0 \leq \ell, m \leq 3$.

Upon integrating Eq. (2) over μ from -1 to 1, the scalar flux is obtained in the form:

$$\phi_g^{\,j}(x,t) = \int_0^1 \frac{d\mu}{\mu} S_g(\mu)\delta(t-\frac{x}{v_g\mu})\exp[-(\sum_{k=1}^{j-1} \Sigma_g^{\,k}(a_k-a_{k-1})+\Sigma_g^{\,j}(x-a_{j-1})/\mu]$$

$$+ \sum_h \exp[\Sigma_1^{\,1}v_1(s_h-1)t] \sum_{\ell=0}^{L} (2\ell+1) \sum_{p=0}^{N} \{ \sum_{i=1}^{j} G_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},2\frac{x-a_{j-1}}{a_j-a_{j-1}} -1,s;-\sum_{k=1}^{j} \alpha_g^{\,k} + \frac{\alpha_g^{\,j}+\alpha_g^{\,i}}{2})$$

$$\times B_{p\ell}^{\,i}(g,s_h)+\sum_{i=j+1}^{M} G_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},2\frac{x-a_{j-1}}{a_j-a_{j-1}} -1,s;\sum_{k=j}^{i} \alpha_g^{\,k} - \frac{\alpha_g^{\,j}+\alpha_g^{\,i}}{2})B_{p\ell}^{\,i}(g,s_h)\}_{s=\Sigma_1^{\,1}v_1 s_h}$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\ \exp[-(\Sigma v-iy)t] \sum_{\ell=0}^{L} (2\ell+1) \sum_{p=0}^{N} \{ \sum_{i=1}^{j} G_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},2\frac{x-a_{j-1}}{a_j-a_{j-1}} -1,s;-\sum_{k=i}^{j} \alpha_g^{\,k} + \frac{\alpha_g^{\,j}+\alpha_g^{\,i}}{2})$$

$$\times b_{p\ell}^{\,i}(g,s)+\sum_{i=j+1}^{M} G_{p\ell}(\frac{\alpha_g^{\,i}}{2},\frac{\alpha_g^{\,j}}{2},2\frac{x-a_{j-1}}{a_j-a_{j-1}} -1,s;\sum_{k=j}^{i} \alpha_g^{\,k} - \frac{\alpha_g^{\,i}+\alpha_g^{\,j}}{2})b_{p\ell}^{\,i}(g,s)\}_{s=\Sigma_1^{\,1}v_1-\Sigma v+iy}, \tag{7}$$

where

$$G_{p\ell}(\alpha_g^{\,i},\alpha_g^{\,j},\xi,s;d) = \int_{-1}^{1} d\mu\ F_{p\ell}(\alpha_g^{\,i},\alpha_g^{\,j},(1+\xi)/2,\mu,s;d), \tag{8}$$

the expression for $G_{p\ell}$ in the j_7 and P_3 approximation being given in Appendix 1, Section 3.

For two cases where the angular distribution of the boundary source $S_g(\mu)$ is written as

a) $S_g(\mu) = 2S_g\mu$ (point isotropic or cosine distribution source), (9)

b) $S_g(\mu) = S_g\delta(\mu-\mu_1)$ with $\mu_1 > 0$ (monodirectional source), (10)

the integral $C_q^{\,\ell}$ defined by Eq. (5) takes respectively the following forms:

$$C_{qa}^{\,\ell}(\alpha_g^{\,j},s;d) = \frac{1}{\pi} \int_{-\infty}^{\infty} dz\ j_q(\alpha_g^{\,j}z)\ell^{\,idz}\exp(-i\alpha_g^{\,1}z/2)$$

$$\times \int_0^{\infty} dt'\exp(-P_g^{\,j}t')\int_0^1 d\mu\mu P_\ell(\mu)\exp(iP_g^{\,j}zt'\mu), \tag{11}$$

$$C_{qb}^{\,\ell}(\alpha_g^{\,j},s;d) = 2F_{q\ell}(\alpha_g^{\,j},\alpha_g^{\,1}/2,1,\mu_1,s;d)/P_g^{\,j}\ . \tag{12}$$

The explicit expression for $C_{qa}^{\,\ell}$ is shown in Appendix 1, Section 4. For the point-isotropic source (9), the first term on the right hand side of Eq. (7) is reduced to

$$\left.\begin{array}{l} 2S_g \dfrac{x}{v_g t^2} \exp[-(\sum_{k=1}^{j-1} \Sigma_g{}^k (a_k - a_{k-1}) + \Sigma_g{}^j (x - a_{j-1}))v_g t/x] \text{ for } 0<x<v_g t, \\ 2S_g \delta(t) \text{ for } x=0, \\ 0 \text{ otherwise.} \end{array}\right\} \quad (13)$$

For the monodirectional source (10), it reduces to

$$S_g \delta(t - \frac{x}{v_g \mu_1}) \exp[-(\sum_{k=1}^{j-1} \Sigma_g{}^k (a_k - a_{k-1}) + \Sigma_g{}^j (x - a_{j-1})/\mu_1]/\mu_1 . \quad (14)$$

For a stationary state, only one largest pole $s = \Sigma_1{}^1 v_1$ of $b_q{}^j\ (g,s)$ is of importance. Hence, by multiplying $s-\Sigma_1{}^1 v_1$ on both sides of Eq. (4) and taking the limit $s \to \Sigma_1{}^1 v_1$, we get the equation satisfied by $B_{q\ell}{}^j(g)$ $= \lim_{s \to \Sigma_1{}^1 v_1} (s-\Sigma_1{}^1 v_1) b_{q\ell}{}^j(g,s)$ as follows [for the case with the stationary boundary source $S_g(x,\mu) = S_g(\mu)\delta(x)$]:

$$\frac{1}{2q+1} B_{q\ell}{}^j(g') = \sum_{g=1}^{G} c_\ell{}^j(g \to g') \alpha_g{}^j S_g C_g{}^\ell (\frac{\alpha_g{}^j}{2}, \Sigma_1{}^1 v_1; -\sum_{k=1}^{j} \alpha_g{}^k + \frac{\alpha_g{}^j + \alpha_g{}^i}{2}$$
$$+ \sum_{g=1}^{G} c_\ell{}^j(g \to g') \sum_{m=0}^{L} (2m+1) \sum_{r=0}^{N} [\sum_{i=1}^{j} J_{qr}^{\ell m} (\frac{\alpha_g{}^j}{2}, \frac{\alpha_g{}^i}{2}, \Sigma_1{}^1 v_1; \sum_{k=i}^{j} \alpha_g{}^k + \frac{\alpha_g{}^j + \alpha_g{}^i}{2}) B_{rm}{}^i(g)$$
$$+ \sum_{i=j+1}^{M} J_{qr}^{\ell m} (\frac{\alpha_g{}^j}{2}, \frac{\alpha_g{}^i}{2}, \Sigma_1{}^1 v_1; \sum_{k=j}^{i} \alpha_g{}^k - \frac{\alpha_g{}^i + \alpha_g{}^j}{2}) B_{rm}{}^i(g)]. \quad (15)$$

The stationary vector and scalar fluxes can thus be written respectively as

$$\phi_g{}^j(x,\mu) = \frac{1}{\mu} S_g(\mu) \exp[-(\sum_{k=1}^{j-1} \Sigma_g{}^k (a_k - a_{k-1}) + \Sigma_g{}^j (x - a_{j-1}))/\mu]$$
$$+ \sum_{\ell=0}^{L} (2\ell+1) \sum_{p=0}^{N} \{ \sum_{i=1}^{j} F_{p\ell} (\frac{\alpha_g{}^i}{2}, \frac{\alpha_g{}^j}{2}, \frac{x-a_{j-1}}{a_j - a_{j-1}}, \mu, \Sigma_1{}^1 v_1; -\sum_{k=i}^{j} \alpha_g{}^k + \frac{\alpha_g{}^j + \alpha_g{}^i}{2}) B_{p\ell}{}^i(g)$$
$$+ F_{p\ell} (\frac{\alpha_g{}^i}{2}, \frac{\alpha_g{}^j}{2}, \frac{x-a_{j-1}}{a_j - a_{j-1}}, \mu, \Sigma_1{}^1 v_1; \sum_{k=j}^{i} \alpha_g{}^k - \frac{\alpha_g{}^j + \alpha_g{}^i}{2}) B_{p\ell}{}^i(g) \}, \quad (16)$$

$$\phi_g{}^j(x) = \int_o^1 \frac{d\mu}{\mu} S_g(\mu) \exp[-(\sum_{k=1}^{j-1} \Sigma_g{}^k (a_k - a_{k-1}) + \Sigma_g{}^j (x - a_{j-1}))/\mu]$$
$$+ \sum_{\ell=0}^{L} (2\ell+1) \sum_{p=0}^{N} \{ \sum_{i=1}^{j} G_{p\ell} (\frac{\alpha_g{}^i}{2}, \frac{\alpha_g{}^j}{2}, 2\frac{x-a_{j-1}}{a_j - a_{j-1}} - 1, \Sigma_1{}^1 v_1; -\sum_{k=i}^{j} \alpha_g{}^k + \frac{\alpha_g{}^j + \alpha_g{}^i}{2}) B_{p\ell}{}^i(g)$$
$$+ \sum_{i=j+1}^{M} G_{p\ell} (\frac{\alpha_g{}^i}{2}, \frac{\alpha_g{}^j}{2}, 2\frac{x-a_{j-1}}{a_j - a_{j-1}} - 1, \Sigma_1{}^1 v_1; \sum_{k=j}^{i} \alpha_g{}^k - \frac{\alpha_g{}^i + \alpha_g{}^j}{2}) B_{p\ell}{}^i(g) . \quad (17)$$

It is noted that the expression (16) or (17) can be obtained also from the integration of Eq. (2) or (7) over t from 0 to infinity. For the two cases with the point-isotropic source (9) and monodirectional source (10), the first term on the right hand side of Eq. (17) reduces respectively to

$$2S_g E_2[\sum_{k=1}^{j-1} \Sigma_g{}^k(a_k-a_{k-1})+\Sigma_g{}^j(x-a_{j-1})], \tag{18}$$

$$S_g \exp[-(\sum_{k=1}^{j-1} \Sigma_g{}^k(a_k-a_{k-1})+\Sigma_g{}^j(x-a_{j-1}))/\mu_1]/\mu_1. \tag{19}$$

It is seen from Eq. (15) that also the critical condition for a system without extraneous source $S_g = 0$ can easily be written as follows:

$$\left| \frac{\delta_{gg'}\delta_{qr}\delta_{\ell m}\delta_{ji}}{2q+1} - (2m+1)c_\ell{}^j(g\to g')J_{qr}{}^{\ell m}(\frac{\alpha_g{}^j}{2},\frac{\alpha_g{}^i}{2},\Sigma_1{}^1v_1;\mp(\sum_k \alpha_g{}^k - \frac{\alpha_g{}^j+\alpha_g{}^i}{2}))\right| = 0, \tag{20}$$

$$g,g' = 1,2,\ldots,G;\quad q,r = 0,1,2,\ldots,N;$$
$$\ell,m = 0,1,2,\ldots,L;\quad j,i = 1,2,\ldots,M.$$

In order to get the value of the effective multiplication factor k_{eff} for a given reactor, $c_o{}^j(g\to g')$ is divided into two parts. These are the scattering part $c_{os}{}^j(g\to g') = \Sigma_{so}{}^j(g\to g')/\Sigma_g{}^j$ and the fission part $c_f{}^j(g\to g') = \chi_{g'}(\nu\Sigma_f)_g{}^j/\Sigma_g{}^j$ where χ_g stands for the proportion of fission neutrons born in the g-th group. By the use of this separation, the value of k_{eff} is obtained by solving Eq. (20) with

$$c_o{}^j(g\to g') = c_{os}{}^j(g\to g') + c_f{}^j(g\to g')/k_{eff}. \tag{21}$$

The ratio between the residues $B_{r\ell}{}^i(g)$'s can now be obtained, under the condition (20), from Eq. (15) with $S_g = 0$ and $c_o{}^j(g\to g')$ given by Eq. (21) for obtaining the flux distribution in a multilayer slab reactor according to Eq. (16) or (17) with $S_g = 0$. In addition, Eq. (20) with $s = \Sigma_1{}^1v_1s_1$ instead of $\Sigma_1{}^1v_1$ gives the decay constant of the fundamental mode $\lambda = \Sigma_1{}^1v_1(1-s_1)$ which governs the asymptotic behavior of neutrons as $t\to\infty$ [see Eqs. (1) and (7)].

For a non-multiplying system in which there is no up-scattering of neutrons, Eq. (4) [or (15)] can be simplified to the form which is solved in the same way as for a one-group model:

$$(2q+1)c_\ell^j(g'\to g')\sum_{m=0}^{L}(2m+1)\sum_{r=0}^{N}\Big[\sum_{i=1}^{j}J_{qr}^{\ell m}\Big(\frac{\alpha_{g'}^j}{2},\frac{\alpha_{g'}^i}{2},s;\mp\big(\sum_k\alpha_{g'}^k-\frac{\alpha_{g'}^j+\alpha_{g'}^i}{2}\big)b_{rm}^i(g',s)$$
$$-b_{q\ell}^i(g',s) = -(2q+1)\Big[\sum_{g=1}^{g'}c_\ell^j(g\to g')|\alpha_g^j|S_gC_q^\ell\Big(\frac{\alpha_g^j}{2},s;-\sum_{k=1}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\Big)$$
$$+\sum_{g=1}^{g'-1}c_\ell^j(g\to g')\sum_{m=0}^{L}(2m+1)\sum_{r=0}^{N}\Big[\sum_{i=1}^{M}J_{qr}^{\ell m}\Big(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;\mp\big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\big)\Big)b_{rm}^i(g,s)\Big]. \quad (22)$$

3. JN-METD3 Computer Code

3.1 Input data

After a title card with a 20A4 format, 19 integers are read with a 25I3 format. These input integers are defined as follows:

IIO	3, 5 or 7 for the j_3, j_5 or j_7 approximation (0 to stop the execution; see Appendix 2, Section 1)
IIII	0 or 1 for solving the problem with a point-isotropic or mono-directional external source at x = 0 (for NSOUCE=1)
NSOUCE	-1, 0 or 1 for the problem to obtain the asymptotic decay constant (LLL=0; if NSLOWD=1 the decay constant of neutrons belonging to the lowest energy-group being calculated), to compute the value of the effective multiplication factor (NSLOWD=0) or to deal with a subcritical system with an external source for obtaining the flux distribution
NSLOWD	1 for non-multiplying system without up-scattering of neutrons (0 otherwise)
IA(11)	Length of the cross section (XSEC) mixing tables of the data 11$ and 12*
IA(13)	Number of the cross section sets to be read from cards (the data 14*)
IGRP	Total number of energy groups

in the same way as for a one-group model:

$$(2q+1)c_{\ell}^{j}(g'\to g')\sum_{m=0}^{L}(2m+1)\sum_{r=0}^{N}\Big[\sum_{i=1}^{j}J_{qr}^{\ell m}\Big(\frac{\alpha_g^{\,j}}{2},\frac{\alpha_g^{\,i}}{2},s;\mp\Big(\sum_k\alpha_{g'}^{\,k}-\frac{\alpha_{g'}^{\,j}+\alpha_{g'}^{\,i}}{2}\Big)b_{rm}^{\,i}(g',s)$$
$$-b_{q\ell}^{\,i}(g',s) = -(2q+1)\Big[\sum_{g=1}^{g'}c_{\ell}^{j}(g\to g')|\alpha_g^{\,j}|S_gC_q^{\,\ell}\Big(\frac{\alpha_g^{\,j}}{2},s;-\sum_{k=1}^{j}\alpha_g^{\,k}+\frac{\alpha_g^{\,j}+\alpha_g^{\,i}}{2}\Big)$$
$$+\sum_{g=1}^{g'-1}c_{\ell}^{j}(g\to g')\sum_{m=0}^{L}(2m+1)\sum_{r=0}^{N}\Big[\sum_{i=1}^{M}J_{qr}^{\ell m}\Big(\frac{\alpha_g^{\,j}}{2},\frac{\alpha_g^{\,i}}{2},s;\mp\Big(\sum_k\alpha_g^{\,k}-\frac{\alpha_g^{\,j}+\alpha_g^{\,i}}{2}\Big)\Big)b_{rm}^{\,i}(g,s)\Big]. \qquad (22)$$

3. JN-METD3 Computer Code

3.1 Input data

After a title card with a 20A4 format, 19 integers are read with a 25I3 format. These input integers are defined as follows:

IIO	3, 5 or 7 for the j_3, j_5 or j_7 approximation (0 to stop the execution; see Appendix 2, Section 1)
IIII	0 or 1 for solving the problem with a point-isotropic or mono-directional external source at x = 0 (for NSOUCE=1)
NSOUCE	-1, 0 or 1 for the problem to obtain the asymptotic decay constant (LLL=0; if NSLOWD=1 the decay constant of neutrons belonging to the lowest energy-group being calculated), to compute the value of the effective multiplication factor (NSLOWD=0) or to deal with a subcritical system with an external source for obtaining the flux distribution
NSLOWD	1 for non-multiplying system without up-scattering of neutrons (0 otherwise)
IA(11)	Length of the cross section (XSEC) mixing tables of the data 11$ and 12*
IA(13)	Number of the cross section sets to be read from cards (the data 14*)
IGRP	Total number of energy groups

IHT IHS JHL	Arrangement of reaction type of the P_ℓ component of the cross section for the g-th group and i-th material; XSEC(.....,IHT-2, IHT-1, IHT,....., IHS-1, IHS, IHS+1,....., JHL; g,ℓ,i) = ..., $\Sigma_{ag}{}^{i}$, $\nu\Sigma_{fg}{}^{i}$, $\Sigma_{tg}{}^{i}$,.....,$\Sigma^{i}(g+1\to g)$, $\Sigma^{i}(g\to g)$, $\Sigma^{i}(g-1\to g)$,...... $\Sigma^{i}(g-JHL+IHS\to g)$[the P_ℓ component should have been multiplied by $(2\ell+1)$]
LP	Order of Legendre expansion of the scattering cross section plus 1 (≤ 4)
NNNN	Total number of homogeneous regions in the multilayer slab system
LLL	1 for computing the flux distribution (0 otherwise)
IAA	Total number of input cards for the present problem
NENRGY	Number of energy groups for which the flux distributions are to be calculated (see the array NGRUP mentioned below)
NTFLUX	1 for computing the total flux (0 otherwise)
NTSPAC	Number of space points at which the angular and/or total flux are to be obtained (see the array NSPACE mentioned below)
NANGL	Number of angular points at which the angular flux is to be calculated (total range of μ from -1 to 1 is divided into NANGL-1 to have an equal spacing, and only μ=1 if NANGL=1)
IA(14)	Number of activities to be calculated (only when NTFLUX=1)

The maximum numbers allowed for IGRP, NNNN and so on depend only on the size of a blank COMMON (see §3.2).

Next, in the subroutine JNMETD, the following data depending on the input integer NSOUCE are read with 7F10.6 (energy-dependent quantities are ordered respectively by energy-group beginning with the first or highest group):

NSOUCE=1	SOCE(I),I=1,IGRP; boundary source intensity S_g
	DANGL; direction cosine μ_1 of monodirectional source at x=0 (only if IIII>0)
NSOUCE=0	CK1,CK2,EPSK; the first and second guess for the value of k_{eff} and the required relative accuracy
NSOUCE=-1	VG(I),I=1,IGRP; speed of neutrons v_g

	CK1,CK2,EPSK; the first and second guess for the asymptotic time constant $1-s_1$ and the required relative accuracy

If NSLOWD=0, these data are followed by

XG(I) I=1,IGRP	Fission spectrum χ_g

which are read with 7F10.6 in the same manner as VG. The following data are then read in the FIDO input format[13]:

XSEC	14* followed by the cross sections for all types of reactions arranged as mentioned above in the first group, then for those in the second group and so on, for the zeroth, first, ... Legendre components. These data are followed by T
IMIX	11$ followed by material numbers to mix the cross sections for obtaining those for mixtures, with T at the end (only when IA(11)>0)
DEN	12* followed by number densities of materials to be mixed as specified by IMIX, with T at the end (only when IA(11)>0)

Then, the following data are read with 7F10.6 in the order of space region (K):

A(K)	Thickness of the region
BUCLG(I,K), I=1,IGRP	Buckling for taking into account the finite extention of the system in y and z directions, $(B_y{}^2+B_z{}^2)_g$, ordered by group g

Finally, if LLL=1, the following data are read with a 25I3 format in the subroutine FLUXCA:

NGRUP(I) I=1,NENRGY	Energy-group indices of NENRGY groups for which the flux distributions are to be calculated (in increasing order)
NSPACE(I), I=1,NNNN	NNNN numbers of space points at which the flux is to be calculated. (The first integer is the number of space points for the first region, the second integer is for the second region and so on.) If NSPACE(I)>1 the I-th region is divided

	into NSPACE(I)-1 to have an equal spacing and if NSPACE(I)=1 one space point is selected at the boundary between the I-th and (I+1)-th region.

The input data for three sample problems are given in Appendix 2. The first sample problem calculates the value of k_{eff} of 2-region slab reactor with the P_1-scattering in the 1-energy-group j_5 approximation. In addition, it obtains the angular and total flux at 3 angle and/or 6 space points in the second slab. The second problem, on the other hand, computes only the decay constant of neutrons belonging to the lowest energy-group in a 1-region slab system, with the P_1-scattering and a point-isotropic boundary source, in the 7-group j_7 approximation. The third problem calculates, in the context of a 7-group j_5 approximation, the angular and total flux of the highest and lowest group neutrons, respectively, and one activity at 3 angle and/or 6 space points in a 2-region slab system with the P_1-scattering and a point-isotropic boundary source.

3.2 Computer programs

The JN-METD3 package consists of 14 programs: MAIN, JNMETD, FLUXCA, FIDO, FCAL, FSCAL, SGMOD, CCALC, DET, ITRTON, SOLEQ, GCAL, FMCAL and EP. In addition, the code makes use of the library subprograms, DEXP, DLOG, DATAN, DSIN, DCOS and LEGDD[14].

Almost all subscript variables and their dimension informations are stored in a blank COMMON for the use of the adjustable dimensioning. The present size of the COMMON for subscript variables is 84 K words so that the program requires the core storage less than 128 K words in FORTRAN-D on the FACOM-230/75. For altering the dimension of the COMMON to fit the core storage, 12 statements should be adjusted. (All 14 program decks are respectively numbered.) These are 5 cards in the MAIN program: the 26th (dimension of ACOM), 27th (dimension of ICOM), 28th (dimension of BCOM), 32nd (clear COMMON) and 122nd card (available $\gtrless$ required storage?), and 8 COMMON statements (dimension of ACOM) which are the 27th card of JNMETD, 13th of FLUXCA, 5th of FIDO, 12th of FCAL, 10th of CCALC, 5th of ITRTON, 10th of GCAL and 5th of EP.

In the MAIN program, as can be seen from the flow chart shown in Appendix 3, Section 1, sizes of the required arrays are computed based on the input parameters and then first-word addresses are calculated for these arrays. The locations of these pointers and the associated arrays with their dummy dimensions are given in Table 1 which shows also the fact that the storage locations bigger than IA(38) are used in two different ways, once in JNMETD and then again in FLUXCA. The actual values of the integer variables specifying the sizes of arrays are summarized in Table 2. The first-word addresses and the dimension informations are transferred through a call statement and a part of vector in the blank COMMON is treated as a multi-dimensional array in subprograms.

Table 1 Location of the first elements of Real*8 (·Real*4 or * Integer) arrays stored in the blank COMMON and their dimensions† (see Table 2)

Location	Array name (dimension)		
IA(31)	ALPHA(IGRP,NNNN)		
IA(51)	XV(IGRP,NNNN)		
IA(32)	RES(IIO,IGRP,NNNN,LP)		
IA(33)	· A(NNNN)		
IA(34)	· SOCE(IA(16))		
IA(35)	· XSEC(JHL,IGRP,LP,IA(10))††		
IA(36)	· VG(IGRP)		
IA(52)	· XG(IA(3))		
IA(54)	· DEN(IA(11))		
IA(55)	* IMIX(IA(11))		
IA(38)	ED(IGO,IA(1))	IA(38)	X(NTSPAC)
IA(39)	E(IGO,IA(2))	IA(40)	· ANGL(NANGL)
IA(40)	C1(IGRP,IGRP,IA(6),LP)	IA(47)	· TFLUX(IA(9),NENRGY)
IA(53)	ALS(IGRP,IA(12))	IA(48)	· VFLUX(NANGL,NTSPAC,NENRGY)
IA(41)	C2(IA(5),IA(5),IA(6))	IA(30)	· DOSE(IA(9),IA(14))
IA(42)	SC(IA(7),IA(8))	IA(49)	* NGRUP(NENRGY)
IA(43)	· BUCLG(IGRP,NNNN)	IA(50)	* NSPACE(NNNN)
IA(44)	· CS(IGRP,IGRP,NNNN,LP)		
IA(45)	· CF(IGRP,IGRP,IA(4))		
IA(37)	· XBSEC(IGRP)		

† IGO=(IIO+1)*IGRP*NNNN or (IIO+1)*NNNN for NSLOWD=0 or 1

†† IA(10)=IA(13) or IA(13)+NNNN for IA(11)=0 or positive

Table 2 Computed integers for specifying the array dimensions

<table>
<tr><td>NSLOWD</td><td colspan="3">0</td><td colspan="2">1</td></tr>
<tr><td>NSOUCE</td><td>-1</td><td>0</td><td>1</td><td>-1</td><td>1</td></tr>
<tr><td>NSPH</td><td colspan="3">(IIO+1)*NNNN*IGRP</td><td colspan="2">(IIO+1)*NNNN</td></tr>
<tr><td>IA(1)</td><td>IGO†</td><td>IGO+1</td><td>0</td><td colspan="2">IGO</td></tr>
<tr><td>IA(2)</td><td colspan="2">IGO</td><td>IGO+2</td><td>IGO</td><td>IGO+2</td></tr>
<tr><td>IA(3)</td><td colspan="3">IGRP</td><td colspan="2">0</td></tr>
<tr><td>IA(4)</td><td colspan="3">NNNN</td><td colspan="2">0</td></tr>
<tr><td>IA(5)</td><td>0</td><td>IGRP</td><td>0</td><td colspan="2">0</td></tr>
<tr><td>IA(6)</td><td colspan="2">NNNN</td><td>0</td><td>NNNN</td><td>0</td></tr>
<tr><td>IA(7)</td><td colspan="3">0</td><td>0</td><td>IGO</td></tr>
<tr><td>IA(8)</td><td colspan="3">0</td><td>0</td><td>IGRP</td></tr>
<tr><td>IA(9)</td><td colspan="5">NTSPAC (only if NTFLUX>0)</td></tr>
<tr><td>IA(12)</td><td>NNNN</td><td colspan="2">0</td><td>NNNN</td><td>0</td></tr>
<tr><td>IA(16)</td><td colspan="2">0</td><td>IGRP</td><td>0</td><td>IGRP</td></tr>
</table>

† IGO=NSPH*LP

In the subroutine JNMETD, firstly the input data are read by calling, for reading the FIDO format data, XSEC, IMIX and DEN, the subroutine FIDO[13] as seen from the flow chart of Appendix 3, Section 2, and then the following quantities are computed:

(a) The residue $B_{q\ell}{}^{j}(g)$ according to Eq. (15) [or Eq. (22) for a non-multiplying system without up-scattering of neutrons] for a multilayer slab with a stationary boundary source (NSOUCE=1).

(b) The value of k_{eff} for a multilayer slab reactor (NSOUCE=0 and NSLOWD =0) and if LLL>0 the ratios between $B_{r\ell}{}^{1}(g)$'s from Eqs. (15) with S_g=0 and (20) and $c_o{}^{j}(g\to g')$ given by Eq. (21).

(c) The asymptotic time constant $1-s_1$ for obtaining the asymptotic decay constant $\Sigma_1{}^{1}v_1(1-s_1)$ for a multilayer slab (NSOUCE=-1 and LLL=0) or if NSLOWD=1 the asymptotic decay constant of neutrons belonging to the lowest energy group.

For the cases (b) and (c), the values of $c_o{}^{k}(i\to j)$ or $c_\ell{}^{k}(i\to j)$ for all ℓ are first modified according to the guess of k_{eff} or $1-s_1$, and for the case (c) the values of $\alpha_i{}^{k} = P_i{}^{k}\Sigma_i{}^{k}(a_k-a_{k-1})$ are calculated. With these values of $c_\ell^k(i\to j)$ and α_i^k, the matrix elements for Eq. (15) or (22) are then computed by calling the subroutine FCAL (α_g^j,α_g^i,d,LP,IIO) which obtains the value $(\pm i)P_g^j J_{qr}^{\ell m}(\alpha_g^j,\alpha_g^i,s;d)$ for ℓ,m=0∿LP-1 by the use of their explicit expressions shown in Appendix 1, Section 1 [see formulae (A.11)∿(A.25)]. In the case where $\alpha_g^j+\alpha_g^i+|d|\leq 5$, FCAL calls the subroutine FSCAL (α_g^j,α_g^i,d,LP,n-1,IIO,JII) in which the series expansions shown by the formulae (A.28)∿(A.41) are used for the calculation depending on the values of parameters α_g^j,α_g^i and d (JII stands for the parameter range). The FCAL and FSCAL use the subroutine SGMOD (SSI,I,.....) in which when I>0 $X_{n,s+n,L}(\alpha_j,\alpha_i,d)$ is modified to $X_{n+1,s+n+1,L}$ [see Eq. (A.10) in Appendix 1] and when I=0 the summation of (A.20) or (A.21) is performed. The exponential integral $E_n(x)$ appeared on the right hand side of Eq. (A.11) is evaluated by the function subprogram EP(n,x)[10]. At the end of FCAL, the recurrence relation (A.6) given in Appendix 1 is adopted for computing the functions J_{qr} with q=2∿7 and r=1∿6 from the values of J_{0r} and J_{1r} with r=0∿7 and J_{q0} and J_{q7} with q=2∿7.

After having been obtained the matrix elements, JNMETD calls, for the cases (b) and (c), the subroutine DET to evaluate the determinant (20) [with $s=\Sigma_l{}^l v_l s_l$ for the case (c) instead of $s=\Sigma_l{}^l v_l$ for the case (b)] or the corresponding equation for the case (c) with NSLOWD=1. The subroutine ITRTON is then used for iterating the process to make the value of the determinant zero until the relative difference between two successive values of k_{eff} or $1-s_l$ becomes smaller than EPSK. For the case (b) with LLL>0, after obtained the converged value of k_{eff}, the ratios between the residues are calculated by evaluating the cofactors of the determinant by the use of the subroutine SOLEQ which solves a system of simultaneous linear equations.

For the case (a), in addition to the matrix elements, the first term on the right hand side of Eq. (15) or if NSLOWD=1 the right hand side of Eq. (22) is evaluated with the help of the subroutine CCALC ($\alpha_g^j,\alpha_g^1/2$,d, LP,IIO) or FMCAL ($\alpha_g^i,\alpha_g^j(2\xi-1)-d,\mu$,IIO). The CCALC computes $(-i)2\alpha_g^j P_g^j \times C_{qa}^\ell(\alpha_g^j,s;d)$ by using their explicit expressions if $\alpha_g^j+\alpha_g^1/2+|d|>5$ or the series expansions otherwise. As is seen from the expression (A.69), it uses EP for evaluating E_1. On the other hand, FMCAL computes $(i)F_{p\ell}(\alpha_g^i,\alpha_g^j,\xi,\mu,s;d)/P_\ell(\mu)$ for obtaining $C_{pb}^\ell(\alpha_g^j,s;d)$ of Eq. (12) with the help of the subroutine in the FACOM SSL (Scientific Subroutine Library), LEGDD ($\mu,\ell,P_\ell(\mu)$,ILL) for evaluating $P_\ell(\mu)$. The FMCAL uses the series expansions given in Appendix 1, Section 2, if $(\alpha_g^{\,i}+|\alpha_g^{\,j}(2\xi-1)-d|)/|\mu|\leq 5$. The residues are then obtained in JNMETD by calling SOLEQ to solve Eq. (15) if NSLOWD=0 or Eq. (22) if NSLOWD=1.

The subroutine FLUXCA computes for NTFLUX>0 the total flux and/or for NANGL>0 the angular flux by using the values of the residues (or the ratios between them) obtained as mentioned above in JNMETD. As is seen from the flow diagram of FLUXCA shown in Appendix 3, Section 3, after having calculated the angle points (values of μ) at which the angular fluxes are to be computed if NANGL>0, the space points ($0\leq\xi\leq 1$) are determined in each region and the total fluxes are calculated at these points with the help of the subroutine GCAL($\alpha_g^{\,i}$,$\alpha_g^{\,j}(2\xi-1)-d$,ℓ,IIO) if NTFLUX>0. The GCAL computes (i)$G_{p\ell}(\alpha_g^{\,i},\alpha_g^{\,j},2\xi-1,s;d)$ on the right hand side of Eq. (17) by adopting the explicit expressions with the help of EP when $|\alpha_g^{\,i}|+|\alpha_g^{\,j}(2\xi-1)-d|>5$ or the series expansions otherwise (see Appendix 1, Section 3). For NANGL>0, FLUXCA calls FMCAL and LEGDD for calculating the second term on the right hand side of Eq. (16). In the case where NSOUCE=1, FLUXCA evaluates also the contribution of uncollided source radiations to the total or/and angular flux according to the first term on the right hand side of Eq. (17) with the help of EP [see Eq. (18)] or/and Eq. (16).

4. Concluding Remarks

Since we have already developed a general formulation of the j_N method for dealing with time-dependent transport in a multilayer slab system with anisotropic scattering of neutrons, it is hoped that the present computer program can easily be extended to obtain a detailed time evolution of radiations. However, as having been seen in Appendix 1, the analytical expressions for the functions appeared in the formulation are rather complicated and hence the programming of the computer code needs care upon keeping always the rounding error reasonably small. In the present code JN-METD3, the functions are evaluated on the basis of either their explicit expressions or series expansions obtained under the assumption that the values of all arguments of the function are small. Therefore, in the case where the ratio between the arguments is very large, it is possible that the function is evaluated with a large rounding error. In such a case, it will be a crucial point for obtaining an accurate result which order of the j_N approximation should be applied to the calculation, because given in Appendix 1, Section 2, if $(\alpha_g{}^i+|\alpha_g{}^j(2\xi-1)-d|)/|\mu|\leq 5$. The residues are then obtained in JNMETD by calling SOLEQ to solve Eq. (15) if NSLOWD=0 or Eq. (22) if NSLOWD=1.

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The CPU time on the FACOM-230/75 is about 18 sec for solving all three sample problems given in Appendix 2.

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The author wishes to express his gratitude to Mrs. E. Caglioti Bonanni for her assistance to derive the analytical expressions for the functions appeared in the present method during his stay in EURATOM-CCR, Ispra, Italy. He is also indebted to Mr. T. Tsutsui of System Analysis Laboratory, JAERI, for his help about the programming work of JN-METD3.

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Appendix 1. Analytical Expressions of Functions in the j_7-P_3 Approximation

1. $J_{qr}^{\ell m}(\alpha_j,\alpha_i,s;d)$

As shown already in a previous paper[9], for $\alpha_j \geq \alpha_i > 0$ and $P_j > 0$, Eq. (6) is rewritten as

$$J_{qr}^{\ell m}(\alpha_j,\alpha_i,s;d) = \frac{1}{2\pi}\frac{\alpha_i}{P_j}\int_{-1}^{1} d\mu P_\ell(\mu)P_m(\mu)\int_0^\infty dt'\exp(-t')$$
$$\times \int_{-\infty}^{\infty} dz\ j_q(\alpha_j z)j_r(\alpha_i z)\exp[i(d+\mu t')z]. \tag{A.1}$$

On dividing the parameter range into five:

(a) $-(\alpha_j+\alpha_i+d) > 0$,

(b) $-(\alpha_j+\alpha_i+d) < 0$ and $-(\alpha_j-\alpha_i+d) > 0$,

(c) $-(\alpha_j-\alpha_i+d) < 0$ and $-(-\alpha_j+\alpha_i+d) > 0$,

(d) $-(-\alpha_j+\alpha_i+d) < 0$ and $-(-\alpha_j-\alpha_i+d) > 0$,

(e) $-(-\alpha_j-\alpha_i+d) < 0$,

$J_{qr}^{\ell m}(\alpha_j,\alpha_i,s;d)$ for the range (d) or (e) is equal to $(-1)^{r+q+\ell+m}J_{qr}^{\ell m}(\alpha_j,\alpha_i,s;-d)$ for the range (b) or (a). For the parameter ranges (a), (b) and (c), we have

$$8(i)^{q+r}\frac{P_j}{\alpha_j}J_{qr}^{\ell m}(\alpha_j,\alpha_i,s;d) = \sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s \sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!}$$
$$\times \sum_{p=0}^{[\ell/2]}\sum_{v=0}^{[m/2]}\left(-\frac{1}{2}\right)^{[\ell/2]-p+[m/2]-v}\frac{(2\ell+2p)-1-2[\ell/2])!!(2m+2v-1-2[m/2])!!}{([\ell/2]-p)!(\ell+2p-2[\ell/2])!([m/2]-v)!(m+2v-2[m/2])!}$$
$$\times\begin{cases}[X(\alpha_j,\alpha_i,d)+(-1)^r X(\alpha_j,-\alpha_i,d)+(-1)^q X(-\alpha_j,\alpha_i,d)+(-1)^{r+q}X(-\alpha_j,-\alpha_i,d) \\ -Y_1(\alpha_j,\alpha_i,d)-(-1)^r Y_1(\alpha_j,-\alpha_i,d)-(-1)^q Y_1(-\alpha_j,\alpha_i,d) \\ -(-1)^{r+q}Y_1(-\alpha_j,-\alpha_i,d)], & \text{for (a)}, \\ [(-1)^{\ell+m}X(-\alpha_j,-\alpha_i,-d)+(-1)^r X(\alpha_j,-\alpha_i,d)+(-1)^q X(-\alpha_j,\alpha_i,d) \\ +(-1)^{r+q}X(-\alpha_j,-\alpha_i,d)+Y_2(\alpha_j,\alpha_i,d)-(-1)^r Y_2(\alpha_j,-\alpha_i,d) \\ -(-1)^q Y_2(-\alpha_j,\alpha_i,d)-(-1)^{r+q}Y_2(-\alpha_j,-\alpha_i,d)], & \text{for (b)}, \\ [(-1)^{\ell+m}X(-\alpha_j,-\alpha_i,-d)+(-1)^{\ell+m+r}X(-\alpha_j,\alpha_i,-d)+(-1)^q X(-\alpha_j,\alpha_i,d) \\ +(-1)^{r+q}X(-\alpha_j,-\alpha_i,d)+Y_2(\alpha_j,\alpha_i,d)+(-1)^r Y_2(\alpha_j,-\alpha_i,d) \\ -(-1)^q Y_2(-\alpha_j,\alpha_i,d)-(-1)^{r+q}Y_2(-\alpha_j,-\alpha_i,d)], & \text{for (c)},\end{cases} \tag{A.2}$$

$$X(\alpha_j,\alpha_i,d) = X_{nsL}(\alpha_j,\alpha_i,d) = \frac{1}{(\alpha_j)^{n+1}(\alpha_i)^{s-n+1}} \frac{2}{(L+s+2)!}\{ \exp(\alpha_j+\alpha_i+d)$$

$$\times \sum_{u=1}^{L+s+2} (L+s+2-u)!(\alpha_j+\alpha_i+d)^{u-1}+(\alpha_j+\alpha_i+d)^{L+s+2}E_1[-(\alpha_j+\alpha_i+d)]\}, \qquad (A.3)$$

$$Y_2(\alpha_j,\alpha_i,d) = Y_2^{nsL}(\alpha_j,\alpha_i,d) = Y_1(\alpha_j,\alpha_i,d)-(-1)^{\ell+m}Y_1(-\alpha_j,-\alpha_i,-d), \qquad (A.4)$$

$$Y_1(\alpha_j,\alpha_i,d) = Y_1^{nsL}(\alpha_j,\alpha_i,d) = \frac{1}{(\alpha_j)^{n+1}(\alpha_i)^{s-n+1}} \sum_{u=0}^{s+1} \frac{(\alpha_j+\alpha_i+d)^{s+1-u}}{(L+u+1)(s+1-u)!} \qquad (A.5)$$

$$L = \ell-2[\ell/2]+2p+m-2[m/2]+2v.$$

Since

$$J_{q+1,r}(\alpha_j,\alpha_i,s;d) = \frac{\alpha_i}{\alpha_j}\frac{2q+1}{2r+1}[J_{q,r+1}+J_{q,r-1}]-J_{q-1,r}, \qquad (A.6)$$

the functions to be evaluated in the j_7 approximation (requiring J_{qr}'s with $q=0\sim7$ and $r=0\sim7$) are J_{0r} and J_{1r} with $r=0\sim7$, and J_{q0} and J_{q7} with $q=2\sim7$. For these functions,

$$\sum_{s=0}^{q+r}(-\frac{1}{2})^s \sum_{n=s-r\geq0}^{s\leq q} \frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!} X_{nsL}(\alpha_j,\alpha_i,d)$$

$$= \sum_{m=0}^{q} \frac{(q+n)!}{n!(q-n)!} \sum_{s=n}^{n+r}(-\frac{1}{2})^s \frac{(r+s-n)!}{(s-n)!(r-s+n)!} X_{nsL}(\alpha_j,\alpha_i,d)$$

$$= \begin{cases} \sum_{s=0}^{r}(-\frac{1}{2})^s \frac{(r+s)!}{s!(r-s)!} X_{osL}(\alpha_j,\alpha_i,d), \quad \text{for } q=0, & (A.7) \\ \sum_{s=0}^{q}(-\frac{1}{2})^s \frac{(q+s)!}{s!(q-s)!} X_{osL}(\alpha_j,\alpha_i,d), \quad \text{for } r=0, & (A.8) \\ \sum_{s=0}^{r}(-\frac{1}{2})^s \frac{(r+s)!}{s!(r-s)!}[X_{osL}(\alpha_j,\alpha_i,d)-X_{1,s+1,L}(\alpha_j,\alpha_i,d)], \quad \text{for } q=1, & (A.9) \\ \sum_{s=0}^{7}(-\frac{1}{2})^s \frac{(7+s)!}{s!(7-s)!} \sum_{n=0}^{q}(-1)^n \frac{(q+n)!}{(2n)!!(q-n)!} X_{n,s+n,L}(\alpha_j,\alpha_i,d), \quad \text{for } r=7. & (A.10) \end{cases}$$

Explicit expressions on the right hand side of Eq. (A.7) or (A.8) are written as follows:

$$\alpha_j\alpha_i \sum_{s=0}^{r} (-\frac{1}{2})^s \frac{(r+s)!}{s!(r-s)!} X_{osL}(\alpha_j,\alpha_i,d) =$$

$$W_{L+1}(x)\ell^x + c_{L+1}x^{L+2}E_1(-x), \qquad \text{for } r=0,$$

$$[W_{L+1}(x) - \frac{1}{3\alpha_i} W_{L+2}(x)]\ell^x + (c_{L+1} - \frac{c_{L+2}}{3\alpha_i} x)x^{L+2}E_1(-x), \qquad \text{for } r=1,$$

$$[W_{L+1}(x) - \frac{1}{\alpha_i} W_{L+2}(x) + \frac{1}{4\alpha_i^2} W_{L+3}(x)]\ell^x + (c_{L+1} - \frac{c_{L+2}}{\alpha_i} x + \frac{c_{L+3}}{4\alpha_i^2} x^2)x^{L+2}E_1(-x),$$
$$\text{for } r=2,$$

$$[W_{L+1}(x) - \frac{2}{\alpha_i} W_{L+2}(x) + \frac{5}{4\alpha_i^2} W_{L+3}(x) - \frac{1}{4\alpha_i^3} W_{L+4}(x)]\ell^x$$
$$+(c_{L+1} - \frac{2}{\alpha_i} c_{L+2}x + \frac{5}{4\alpha_i^2} c_{L+3}x^2 - \frac{c_{L+4}}{4\alpha_i^3} x^3)x^{L+2}E_1(-x), \qquad \text{for } r=3,$$

$$[W_{L+1}(x) - \frac{10}{3\alpha_i} W_{L+2}(x) + \frac{15}{4\alpha_i^2} W_{L+3}(x) - \frac{7}{4\alpha_i^3} W_{L+4}(x) + \frac{7}{24\alpha_i^3} W_{L+5}(x)]\ell^x$$
$$+(c_{L+1} - \frac{10}{3\alpha_i} c_{L+2}x + \frac{15}{4\alpha_i^2} c_{L+3}x^2 - \frac{7}{4\alpha_i^3} c_{L+4}x^3 + \frac{7}{24\alpha_i^4} c_{L+5}x^4)x^{L+2}E_1(-x),$$
$$\text{for } r=4,$$

$$[W_{L+1}(x) - \frac{5}{\alpha_i} W_{L+2}(x) + \frac{35}{4\alpha_i^2} W_{L+3}(x) - \frac{7}{\alpha_i^3} W_{L+4}(x) + \frac{21}{8\alpha_i^4} W_{L+5}(x) - \frac{3}{8\alpha_i^5} W_{L+6}(x)]\ell^x$$
$$+(c_{L+1} - \frac{5}{\alpha_i} c_{L+2}x + \frac{35}{4\alpha_i^2} c_{L+3}x^2 - \frac{7}{\alpha_i^3} c_{L+4}x^3 + \frac{21}{8\alpha_i^4} c_{L+5}x^4 - \frac{3}{8\alpha_i^5} c_{L+6}x^5)$$
$$\times x^{L+2}E_1(-x), \qquad \text{for } r=5,$$

$$[W_{L+1}(x) - \frac{7}{\alpha_i} W_{L+2}(x) + \frac{35}{2\alpha_i^2} W_{L+3}(x) - \frac{21}{\alpha_i^3} W_{L+4}(x) + \frac{105}{8\alpha_i^4} W_{L+5}(x) - \frac{33}{8\alpha_i^5} W_{L+6}(x)$$
$$+ \frac{33}{64\alpha_i^6} W_{L+7}(x)]\ell^x + (c_{L+1} - \frac{7}{\alpha_i} c_{L+2}x + \frac{35}{2\alpha_i^2} c_{L+3}x^2 - \frac{21}{\alpha_i^3} c_{L+4}x^3 + \frac{105}{8\alpha_i^4} c_{L+5}x^4$$
$$- \frac{33}{8\alpha_i^5} c_{L+6}x^5 + \frac{33}{64\alpha_i^6} c_{L+7}x^6)x^{L+2}E_1(-x), \qquad \text{for } r=6,$$

$$[W_{L+1}(x) - \frac{28}{3\alpha_i} W_{L+2}(x) + \frac{63}{2\alpha_i^2} W_{L+3}(x) - \frac{105}{2\alpha_i^3} W_{L+4}(x) + \frac{385}{8\alpha_i^4} W_{L+5}(x)$$
$$- \frac{99}{4\alpha_i^5} W_{L+6}(x) + \frac{429}{64\alpha_i^6} W_{L+7}(x) - \frac{143}{192\alpha_i^7} W_{L+8}(x)]\ell^x + (c_{L+1} - \frac{28}{3\alpha_i} c_{L+2}x$$
$$+ \frac{63}{2\alpha_i^2} c_{L+3}x^2 - \frac{105}{2\alpha_i^3} c_{L+4}x^3 + \frac{385}{8\alpha_i^4} c_{L+5}x^4 - \frac{99}{4\alpha_i^5} c_{L+6}x^5 + \frac{429}{64\alpha_i^6} c_{L+7}x^6$$
$$- \frac{143}{192\alpha_i^7} c_{L+8}x^7)x^{L+2}E_1(-x), \qquad \text{for } r=7, \qquad (A.11)$$

where

$$W_{L+m}(x) = c_{L+m}Z_{L+m}(x), \tag{A.12}$$

$$c_{L+m} = (m+1)!/(L+m+1)! \ ,$$

$$Z_{L+m}(x) = \sum_{u=1}^{L+m+1} (L+m+1-u)!x^{u-1},$$

$$x = \alpha_j+\alpha_i+d.$$

The $Z_m(x)$ satisfies the following recurrence relation:

$$Z_m(x) = xZ_{m-1}(x)+m!, \qquad Z_1(x) = x+1. \tag{A.13}$$

The $X_{1,s+1,L}$ term on the right hand side of Eq. (A.9):

$$\alpha_j\alpha_i \sum_{s=0}^{r} (-\frac{1}{2})^s \frac{(r+s)!}{s!(r-s)!} X_{1,s+1,L}(\alpha_j,\alpha_i,d),$$

for r=0∿7, can easily be written down by replacing c_{L+m} and Z_{L+m} in Eq. (A.11) by $c_{L+m+1}x/[(m+2)\alpha_j]$ and Z_{L+m+1}/x, respectively. In the same manner, the explicit expression for

$$\alpha_j\alpha_i \sum_{s=0}^{7} (-\frac{1}{2})^s \frac{(7+s)!}{s!(7-s)!} X_{n,s+n,L}(\alpha_j,\alpha_i,d)$$

of Eq. (A.10) can be obtained from the last equation of (A.11) by replacing repeatedly c_{L+m} and Z_{L+m} in the expression for $X_{n,s+n,L}$ by $c_{L+m+1}x/[(m+2)\alpha_j]$ and Z_{L+m+1}/x, respectively, to get that for $X_{n+1,s+n+1,L}$.

The extra terms consisting of Y_1 or Y_2 on the right hand side of Eq. (A.2) give the following expressions; On changing the order of summation as

$$\sum_{s=0}^{q+r} \sum_{n=s-r\geq 0}^{s\leq q} = \sum_{n=0}^{q} \sum_{s=n}^{n+r},$$

for the parameter range (a), we get

$$\sum_{s=n}^{n+r} (-\frac{1}{2})^s \frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!} [Y_1(\alpha_j,\alpha_i,d)+(-1)^r Y_1(\alpha_j,-\alpha_i,d)$$

$$+(-1)^q Y_1(-\alpha_j,\alpha_i,d)+(-1)^{r+q} Y_1(-\alpha_j,-\alpha_i,d)]=0. \tag{A.14}$$

For the parameter range (b),

$$\alpha_j\alpha_i \sum_{s=0}^{q+r} (-\frac{1}{2})^s \sum_{n=s-r\geq 0}^{s\leq q} \frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!}[Y_2(\alpha_j,\alpha_i,d)-(-1)^r Y_2(\alpha_j,-\alpha_i,d)$$

$$-(-1)^q Y_2(-\alpha_j,\alpha_i,d)-(-1)^{r+q} Y_2(-\alpha_j,-\alpha_i,d)]$$

$$= 2\alpha_j\alpha_i \sum_{n=0}^{q} \sum_{s=n}^{n+r} (-\frac{1}{2})^s \frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!}$$

$$\times[Y_1^{nsL}(\alpha_j,\alpha_i,d)-(-1)^{\ell+m} Y_1^{nsL}(-\alpha_j,-\alpha_i,-d)], \qquad (A.15)$$

where the relation (A.14) has been used. The explicit expressions for Eq. (A.15) with q=0 and L=even are written as follows:

$$4(\alpha_j+\alpha_i+d)/(L+1), \qquad \text{for } r=0,$$

$$-\frac{1}{\alpha_i}\{\frac{4}{L+3}+\frac{2}{L+1}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)\}, \qquad \text{for } r=1,$$

$$-\frac{1}{\alpha_i}\{\frac{12}{L+3}-\frac{2}{\alpha_i}(\alpha_j+\alpha_i+d)[(\alpha_j-\alpha_i+d)\frac{\alpha_j+d}{L+1}+\frac{6}{L+3}]\}, \qquad \text{for } r=2,$$

$$-\frac{1}{\alpha_i}\{\frac{24}{L+3}+\frac{60}{(L+5)\alpha_i^2}+\frac{1}{2\alpha_i^2}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)[\frac{1}{L+1}(5(\alpha_j+d)^2-\alpha_i^2)+\frac{60}{L+3}]\}$$

$$\text{for } r=3,$$

$$-\frac{1}{\alpha_i}\{\frac{40}{L+3}+\frac{420}{(L+5)\alpha_i^2}-\frac{1}{\alpha_i^4}(\alpha_j+\alpha_i+d)[(\alpha_j-\alpha_i+d)(\alpha_j+d)(\frac{7(\alpha_j+d)^2-3\alpha_i^2}{2(L+1)}+\frac{70}{L+3})]$$

$$+40\alpha_i^2/(L+3)+420/(L+5)]\}, \qquad \text{for } r=4,$$

$$-\frac{1}{\alpha_i}\{\frac{60}{L+3}+\frac{1680}{(L+5)\alpha_i^2}+\frac{3780}{(L+7)\alpha_i^4}+\frac{1}{\alpha_i^4}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)$$

$$\times[\frac{21(\alpha_j+d)^4-14(\alpha_j+d)^2\alpha_i^2+\alpha_i^4}{4(L+1)}+\frac{315(\alpha_j+d)^2+105\alpha_i^2}{2(L+3)}+\frac{1890}{L+5}]\}, \text{for } r=5,$$

$$-\frac{1}{\alpha_i}\{\frac{84}{L+3}+\frac{5040}{(L+5)\alpha_i^2}+\frac{41580}{(L+7)\alpha_i^4}-\frac{1}{\alpha_i^6}(\alpha_j+\alpha_i+d)[(\alpha_j-\alpha_i+d)(\alpha_j+d)$$

$$\times[\frac{33(\alpha_j+d)^4-30(\alpha_j+d)^2\alpha_i^2+5\alpha_i^4}{4(L+1)}+\frac{693(\alpha_j+d)^2+63\alpha_i^2}{2(L+3)}+\frac{6930}{L+5}]$$

$$+84\alpha_i^4/(L+3)+5040\alpha_i^2/(L+5)+41580/(L+7)\}, \qquad \text{for } r=6,$$

$$-\frac{1}{\alpha_i}\{\frac{112}{L+3}+\frac{12600}{(L+5)\alpha_i^2}+\frac{249480}{(L+7)\alpha_i^4}+\frac{630630}{(L+9)\alpha_i^6}+\frac{1}{\alpha_i^6}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)$$

$$\times[\frac{429(\alpha_j+d)^6-495(\alpha_j+d)^4\alpha_i^2+135(\alpha_j+d)^2\alpha_i^4-5\alpha_i^6}{32(L+1)}+\frac{3003(\alpha_j+d)^4-462(\alpha_j+d)^2\alpha_i^2+483\alpha_i^4}{4(L+3)}$$

$$+\frac{45045(\alpha_j+d)^2+24255\alpha_i^2}{2(L+5)}+\frac{270270}{L+7}]\},\qquad \text{for } r=7\ .$$

(A.16)

For q=0 and L=odd, we have the following expressions:

$$4/(L+2),\qquad \text{for } r=0,$$

$$-\frac{4}{L+2}\frac{\alpha_j+d}{\alpha_i}\ ,\qquad \text{for } r=1,$$

$$-\frac{2}{L+2}\ [1-3(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{12}{(L+4)\alpha_i^2}\ ,\qquad \text{for } r=2,$$

$$\frac{\alpha_j+d}{\alpha_i}\ \{\frac{1}{L+2}[6-10(\frac{\alpha_j+d}{\alpha_i})^2]-\frac{60}{(L+4)\alpha_i^2}\}\ ,\qquad \text{for } r=3,$$

$$\frac{1}{L+2}[\frac{3}{2}-15(\frac{\alpha_j+d}{\alpha_i})^2+\frac{35}{8}(\frac{\alpha_j+d}{\alpha_i})^4]-\frac{30}{(L+4)\alpha_i^2}[1-7(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{420}{(L+6)\alpha_i^4}\ ,$$

for r=4,

$$-\frac{\alpha_j+d}{\alpha_i}\{\frac{1}{L+2}[\frac{15}{2}-35(\frac{\alpha_j+d}{\alpha_i})^2+\frac{63}{2}(\frac{\alpha_j+d}{\alpha_i})^4]-\frac{210}{(L+4)\alpha_i^2}[1-3(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{3780}{(L+6)\alpha_i^4}\},$$

for r=5,

$$-\frac{1}{4(L+2)}[5-105(\frac{\alpha_j+d}{\alpha_i})^2+315(\frac{\alpha_j+d}{\alpha_i})^4-231(\frac{\alpha_j+d}{\alpha_i})^6]+\frac{1}{(L+4)\alpha_i^2}[\frac{105}{2}-945(\frac{\alpha_j+d}{\alpha_i})^2$$

$$+\frac{3465}{2}(\frac{\alpha_j+d}{\alpha_i})^4]-\frac{1890}{(L+6)\alpha_i^4}[1-11(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{41580}{(L+8)\alpha_i^6}\ ,\qquad \text{for } r=6,$$

$$\frac{\alpha_j+d}{\alpha_i}\{\frac{1}{4(L+2)}[35-315(\frac{\alpha_j+d}{\alpha_i})^2+693(\frac{\alpha_j+d}{\alpha_i})^4-429(\frac{\alpha_j+d}{\alpha_i})^6]-\frac{1}{(L+4)\alpha_i^2}[\frac{945}{2}-3465(\frac{\alpha_j+d}{\alpha_i})^2$$

$$+\frac{9009}{2}(\frac{\alpha_j+d}{\alpha_i})^4]+\frac{1}{(L+6)\alpha_i^4}[20790-90090(\frac{\alpha_j+d}{\alpha_i})^2]-\frac{540540}{(L+8)\alpha_i^6}\},\qquad \text{for } r=7.$$

(A.17)

The explicit expressions for (A.15) with r=0 and q=1∿7 are the same as those for q=0 and r=1∿7 shown in Eqs. (A.16) and (A.17), except for interchanging α_j with α_i (and vice versa).

The expressions for q=1 and r=1∿7 are obtained by taking respectively the sum of those for q=0 and r=1∿7 and the following formulae; For L=even,

$$-\frac{1}{\alpha_j}\{\frac{4}{L+3}+\frac{1}{3\alpha_i}\ (\alpha_j+\alpha_i+d)[\frac{2}{L+1}(\alpha_j-\alpha_i+d)(\alpha_j+d)+\frac{12}{L+3}-\frac{4\alpha_i^2}{L+1}]\},\qquad \text{for } r=1,$$

$$-\frac{1}{\alpha_j}\{\frac{4}{L+3}+\frac{12}{(L+5)\alpha_i^2}+\frac{1}{2\alpha_i^2}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)[\frac{(\alpha_j+d)^2-\alpha_i^2}{L+1}+\frac{12}{L+3}]\},$$
for r=2,

$$-\frac{1}{\alpha_j}\{\frac{4}{L+3}+\frac{60}{(L+5)\alpha_i^2}-\frac{1}{\alpha_i^3}(\alpha_j+\alpha_i+d)[\frac{1}{L+1}(\alpha_j-\alpha_i+d)(\alpha_j+d)(\frac{1}{2}(\alpha_j+d)^2-\frac{1}{2}\alpha_i^2+\frac{10}{3})$$
$$+4\alpha_i^2/(L+3)+60/(L+5)\},$$
for r=3,

$$-\frac{1}{\alpha_j}\{\frac{4}{L+3}+\frac{180}{(L+5)\alpha_i^2}+\frac{420}{(L+7)\alpha_i^4}+\frac{1}{\alpha_i^4}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)[\frac{1}{12(L+1)}(7(\alpha_j+d)^4-8(\alpha_j+d)^2\alpha_i^2$$
$$+\alpha_i^4)+\frac{5}{2(L+3)}(7(\alpha_j+d)^2+\alpha_i^2)+\frac{210}{L+5}]\},$$
for r=4,

$$-\frac{1}{\alpha_j}\{\frac{4}{L+3}+\frac{420}{(L+5)\alpha_i^2}+\frac{3780}{(L+7)\alpha_i^4}-\frac{1}{\alpha_i^5}(\alpha_j+\alpha_i+d)[(\alpha_j-\alpha_i+d)(\alpha_j+d)$$
$$\times(\frac{3(\alpha_j+d)^4-4(\alpha_j+d)^2\alpha_i^2+\alpha_i^4}{4(L+1)}+\frac{63(\alpha_j+d)^2-7\alpha_i^2}{2(L+3)}+\frac{630}{L+5})$$
$$+4\alpha_i^4/(L+3)+420\alpha_i^2/(L+5)+3780/(L+7)]\},$$
for r=5,

$$-\frac{1}{\alpha_j}\{\frac{4}{L+3}+\frac{840}{(L+5)\alpha_i^2}+\frac{18900}{(L+7)\alpha_i^4}+\frac{41580}{(L+9)\alpha_i^6}+\frac{1}{\alpha_i^6}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)$$
$$\times[\frac{33(\alpha_j+d)^6-51(\alpha_j+d)^4\alpha_i^2+19(\alpha_j+d)^2\alpha_i^4-\alpha_i^6}{32(L+1)}+\frac{231(\alpha_j+d)^4-84(\alpha_j+d)^2\alpha_i^2+21\alpha_i^4}{4(L+3)}$$
$$+\frac{3465(\alpha_j+d)^2+1575\alpha_i^2}{2(L+5)}+\frac{20790}{L+7}]\},$$
for r=6,

$$-\frac{1}{\alpha_j}\{\frac{4}{L+3}+\frac{1512}{(L+5)\alpha_i^2}+\frac{69300}{(L+7)\alpha_i^4}+\frac{540540}{(L+9)\alpha_i^6}-\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)[(\alpha_j-\alpha_i+d)(\alpha_j+d)$$
$$\times(\frac{143(\alpha_j+d)^6-253(\alpha_j+d)^4\alpha_i^2+125(\alpha_j+d)^2\alpha_i^4-15\alpha_i^6}{96(L+1)}+\frac{429(\alpha_j+d)^4-264(\alpha_j+d)^2\alpha_i^2+51\alpha_i^4}{4(L+3)}$$
$$+\frac{9009(\alpha_j+d)^2+2079\alpha_i^2}{2(L+5)}+\frac{90090}{L+7})+\frac{4\alpha_i^6}{L+3}+\frac{1512\alpha_i^4}{L+5}+\frac{69300\alpha_i^2}{L+7}+\frac{540540}{L+9}]\},$$
for r=7.

(A.18)

For L=odd, the expressions are written as follows:

$$-\frac{1}{\alpha_j}\{\frac{2\alpha_i}{L+2}[1-(\frac{\alpha_j+d}{\alpha_i})^2]-\frac{4}{(L+4)\alpha_i}\},$$
for r=1,

$$\frac{\alpha_j+d}{\alpha_i\alpha_j}\{\frac{2\alpha_i}{L+2}[1-(\frac{\alpha_j+d}{\alpha_i})^2]-\frac{12}{(L+4)\alpha_i}\},$$
for r=2,

$$\frac{1}{\alpha_j}\{[\frac{\alpha_i}{2(L+2)}(1-(\frac{\alpha_j+d}{\alpha_i})^2)-\frac{6}{(L+4)\alpha_i}][1-5(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{60}{(L+6)\alpha_i^3}\}, \qquad \text{for } r=3,$$

$$-\frac{\alpha_j+d}{\alpha_i\alpha_j}\{[\frac{\alpha_i}{2(L+2)}(1-(\frac{\alpha_j+d}{\alpha_i})^2)-\frac{10}{(L+4)\alpha_i}][3-7(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{420}{(L+6)\alpha_i^3}\}, \qquad \text{for } r=4,$$

$$-\frac{1}{\alpha_j}\{[\frac{\alpha_i}{4(L+2)}(1-(\frac{\alpha_j+d}{\alpha_i})^2)-\frac{15}{2(L+4)\alpha_i}][1-14(\frac{\alpha_j+d}{\alpha_i})^2+21(\frac{\alpha_j+d}{\alpha_i})^4]$$

$$+\frac{210}{(L+6)\alpha_i^3}[1-9(\frac{\alpha_j+d}{\alpha_i})^2]-\frac{3780}{(L+8)\alpha_i^5}\}, \qquad \text{for } r=5.$$

$$\frac{\alpha_j+d}{\alpha_i\alpha_j}\{[\frac{\alpha_i}{4(L+2)}(1-(\frac{\alpha_j+d}{\alpha_i})^2)-\frac{21}{2(L+4)\alpha_i}][5-30(\frac{\alpha_j+d}{\alpha_i})^2+33(\frac{\alpha_j+d}{\alpha_i})^4]$$

$$+\frac{630}{(L+6)\alpha_i^3}[3-11(\frac{\alpha_j+d}{\alpha_i})^2]-\frac{41580}{(L+8)\alpha_i^5}\} \qquad \text{for } r=6,$$

$$\frac{1}{\alpha_j}\{[\frac{\alpha_i}{32(L+2)}(1-(\frac{\alpha_j+d}{\alpha_i})^2)-\frac{7}{4(L+4)\alpha_i}][5-135(\frac{\alpha_j+d}{\alpha_i})^2+495(\frac{\alpha_j+d}{\alpha_i})^4-429(\frac{\alpha_j+d}{\alpha_i})^6]$$

$$+\frac{315}{(L+6)\alpha_i^3}[\frac{3}{2}-33(\frac{\alpha_j+d}{\alpha_i})^2+\frac{143}{2}(\frac{\alpha_j+d}{\alpha_i})^4]-\frac{20790}{(L+8)\alpha_i^5}[1-13(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{540540}{(L+10)\alpha_i^7}\},$$

$$\text{for } r=7. \qquad \text{(A.19)}$$

The expressions for (A.15) with r=7 and q=2∿7 are written, similarly to Eq. (A.10), as follows; For L=even,

$$-\frac{1}{\alpha_j^2\alpha_i^7}\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)}{10(L+1)}[\frac{143}{32}(\alpha_j+d)^8-11(\alpha_j+d)^6\alpha_i^2+\frac{139}{16}(\alpha_j+d)^4\alpha_i^4-\frac{9}{4}(\alpha_j+d)^2\alpha_i^6$$

$$+\frac{3}{32}\alpha_i^8]+\frac{3}{\alpha_j^2}\sum_{k=0}^{4}\frac{L+2k+1}{L+2k+3}[\frac{1}{L+2k+1}\text{ terms in the expression for r=7 of Eq.(A.16)}],$$

$$\text{for } q=2$$

$$\frac{1}{\alpha_j^3\alpha_i^7}\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)(\alpha_j+d)}{4(L+1)}[\frac{13}{16}(\alpha_j+d)^8-\frac{21}{8}(\alpha_j+d)^6\alpha_i^2+3(\alpha_j+d)^4\alpha_i^4-\frac{11}{8}(\alpha_j+d)^2\alpha_i^6$$

$$+\frac{3}{16}\alpha_i^8]+\frac{15}{\alpha_j^2}\sum_{k=0}^{4}\frac{L+2k+1}{L+2k+3}[\frac{1}{L+2k+1}\text{ terms in the expression for r=7 of Eq.(A.18)}],$$

$$\text{for } q=3,$$

$$-\frac{1}{\alpha_j^4\alpha_i^7}\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)}{384(L+1)}[\frac{91}{2}(\alpha_j+d)^{10}-\frac{371}{2}(\alpha_j+d)^8\alpha_i^2+287(\alpha_j+d)^6\alpha_i^4-203(\alpha_j+d)^4\alpha_i^6$$

$$+\frac{119}{2}(\alpha_j+d)^2\alpha_i^8-\frac{7}{2}\alpha_i^{10}]+\frac{35}{\alpha_j^2}\sum_{k=0}^{5}\frac{L+2k+1}{L+2k+3}[\frac{1}{L+2k+1}\text{ terms of the above expression}$$

for q=2], for q=4,

$$\frac{21}{256}\frac{1}{\alpha_j^5\alpha_i^7}\frac{\alpha_j+d}{L+1}[(\alpha_j+d)^2-\alpha_i^2]^6+\frac{63}{\alpha_j^2}\sum_{k=0}^{5}\frac{L+2k+1}{L+2k+3}[\frac{1}{L+2k+1}\text{ terms of the above}$$

expression for q=3], for q=5,

$$-\frac{33}{512}\frac{1}{\alpha_j^6\alpha_i^7}\frac{1}{L+1}[(\alpha_j+d)^2-\alpha_i^2]^7+\frac{99}{\alpha_j^2}\sum_{k=0}^{6}\frac{L+2k+1}{L+2k+3}[\frac{1}{L+2k+1}\text{ terms of the above}$$

expression for q=4], for q=6,

$$\frac{1}{\alpha_j^7\alpha_i^7}\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)(\alpha_j+d)}{80(L+1)}[\frac{143}{32}(\alpha_j+d)^{12}-\frac{253}{80}(\alpha_j+d)^{10}\alpha_i^2+\frac{3083}{32}(\alpha_j+d)^8\alpha_i^4$$

$$-\frac{493}{3}(\alpha_j+d)^6\alpha_i^6+\frac{16399}{96}(\alpha_j+d)^4\alpha_i^8-\frac{2657}{24}(\alpha_j+d)^2\alpha_i^{10}+\frac{4387}{96}\alpha_i^2]$$

$$+\frac{143}{\alpha_j^2}\sum_{k=0}^{6}\frac{L+2k+1}{L+2k+3}[\frac{1}{L+2k+1}\text{ terms of the above expression for q=5}],$$

for q=7.

(A.20)

For L=odd,

$$-\frac{1}{32(L+2)}\frac{\alpha_i(\alpha_j+d)}{\alpha_j^2}[1-(\frac{\alpha_j+d}{\alpha_i})^2]^2[15-110(\frac{\alpha_j+d}{\alpha_i})^2+143(\frac{\alpha_j+d}{\alpha_i})^4]$$

$$+\frac{3}{\alpha_j^2}\sum_{k=1}^{4}\frac{L+2k}{L+2k+2}[\frac{1}{L+2k}\text{ terms in the expression for r=7 of Eq. (A.17)}],$$

for q=2,

$$-\frac{1}{64(L+2)}\frac{\alpha_i^3}{\alpha_j^3}[1-(\frac{\alpha_j+d}{\alpha_i})^2]^3[3-66(\frac{\alpha_j+d}{\alpha_i})^2+143(\frac{\alpha_j+d}{\alpha_i})^4]+\frac{15}{\alpha_j^2}\sum_{k=1}^{5}\frac{L+2k}{L+2k+2}$$

$$\times[\frac{1}{L+2k}\text{ terms of the above expression for r=7 of Eq.(A.19)}],$$ for q=3

$$-\frac{1}{64(L+2)}\frac{\alpha_i^3(\alpha_j+d)}{\alpha_j^4}[1-(\frac{\alpha_j+d}{\alpha_i})^2]^4[21-91(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{35}{\alpha_j^2}\sum_{k=1}^{5}\frac{L+2k}{L+2k+2}$$

$$\times[\frac{1}{L+2k}\text{ terms of the above expression for q=2}],$$ for q=4,

$$\frac{1}{256(L+2)}\frac{\alpha_i^5}{\alpha_j^5}[1-(\frac{\alpha_j+d}{\alpha_i})^2]^5[21-273(\frac{\alpha_j+d}{\alpha_i})^2]+\frac{63}{\alpha_j^2}\sum_{k=1}^{6}\frac{L+2k}{L+2k+2}$$

$\times[\frac{1}{L+2k}$ terms of the above expression for q=3], for q=5,

$$-\frac{231}{256(L+2)}\frac{\alpha_i^5(\alpha_j+d)}{\alpha_j^6}[1-(\frac{\alpha_j+d}{\alpha_i})^2]^6+\frac{99}{\alpha_j^2}\sum_{k=1}^{6}\frac{L+2k}{L+2k+2}[\frac{1}{L+2k}$$ terms of the above

expression for q=4], for q=6,

$$-\frac{429}{512(L+2)}\frac{\alpha_i^7}{\alpha_j^7}[1-(\frac{\alpha_j+d}{\alpha_i})^2]^7+\frac{143}{\alpha_j^2}\sum_{k=1}^{7}\frac{L+2k}{L+2k+2}[\frac{1}{L+2k}$$ terms of the above

expression for q=5], for q=7.

(A.21)

The extra terms of Eq. (A.2) for the parameter range (c):

$$\alpha_j\alpha_i\sum_{s=0}^{q+r}(-\frac{1}{2})^s\sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!(r+s-n)!}{n(q-n)!(s-n)!(r-s+n)!}[Y_2(\alpha_j,\alpha_i,d)+(-1)^rY_2(\alpha_j,-\alpha_i,d)$$

$$-(-1)^qY_2(-\alpha_j,\alpha_i,d)-(-1)^{r+q}Y_2(-\alpha_j,-\alpha_i,d)] \qquad (A.22)$$

are written as follows for r=0 and L=even:

$8\alpha_i/(L+1)$, for q=0,

$-8\alpha_i d/[(L+1)\alpha_j]$, for q=1,

$$-\frac{4\alpha_i}{L+1}[1-(\frac{\alpha_i}{\alpha_j})^2-3(\frac{d}{\alpha_j})^2]+\frac{24}{L+3}\frac{\alpha_i}{\alpha_j^2},$$ for q=2,

$$\frac{\alpha_i d}{\alpha_j}\{\frac{1}{L+1}[12-20(\frac{\alpha_i}{\alpha_j})^2-20(\frac{d}{\alpha_j})^2]-\frac{120}{L+3}\frac{1}{\alpha_j^2}\},$$ for q=3,

$$\frac{\alpha_i}{L+1}\{3-10(\frac{\alpha_i}{\alpha_j})^2+7(\frac{\alpha_i}{\alpha_j})^4-[30-70(\frac{\alpha_i}{\alpha_j})^2](\frac{d}{\alpha_j})^2+35(\frac{d}{\alpha_j})^4\}$$

$$-\frac{\alpha_i}{(L+3)\alpha_j}[60-140(\frac{\alpha_i}{\alpha_j})^2-420(\frac{d}{\alpha_j})^2]+\frac{840}{L+5}\frac{\alpha_i}{\alpha_j^4},$$ for q=4,

$$-\frac{\alpha_i d}{\alpha_j}\{\frac{1}{L+1}[15-70(\frac{\alpha_i}{\alpha_j})^2+63(\frac{\alpha_i}{\alpha_j})^4-70(1-3(\frac{\alpha_i}{\alpha_j})^2)(\frac{d}{\alpha_j})^2+63(\frac{d}{\alpha_j})^4]$$

$$-\frac{420}{(L+3)\alpha_j^2}[1-3(\frac{\alpha_i}{\alpha_j})^2-3(\frac{d}{\alpha_j})^2]+\frac{7560}{(L+5)\alpha_j^4}\},$$ for q=5,

$$-\frac{\alpha_i}{2(L+1)}\{5-35(\frac{\alpha_i}{\alpha_j})^2+63(\frac{\alpha_i}{\alpha_j})^4-33(\frac{\alpha_i}{\alpha_j})^6-[105-630(\frac{\alpha_i}{\alpha_j})^2+693(\frac{\alpha_i}{\alpha_j})^4](\frac{d}{\alpha_j})^2$$

$$+[315-1155(\frac{\alpha_i}{\alpha_j})^2](\frac{d}{\alpha_j})^4-231(\frac{d}{\alpha_j})^6\}+\frac{\alpha_i}{(L+3)\alpha_j^2}\{105-630(\frac{\alpha_i}{\alpha_j})^2+693(\frac{\alpha_i}{\alpha_j})^4$$

$$-[1890-6930(\frac{\alpha_i}{\alpha_j})^2](\frac{d}{\alpha_j})^2+3465(\frac{d}{\alpha_j})^4\}-\frac{1260\alpha_i}{(L+5)\alpha_j^4}[3-11(\frac{\alpha_i}{\alpha_j})^2-33(\frac{d}{\alpha_j})^2]$$

$$+\frac{83160}{L+7}\frac{\alpha_i}{\alpha_j^6}\ , \qquad \text{for } q=6,$$

$$\frac{\alpha_i d}{\alpha_j}\{\frac{1}{2(L+1)}[35-315(\frac{\alpha_i}{\alpha_j})^2+693(\frac{\alpha_i}{\alpha_j})^4-429(\frac{\alpha_i}{\alpha_j})^6-(315-2310(\frac{\alpha_i}{\alpha_j})^2+3003(\frac{\alpha_i}{\alpha_j})^4)(\frac{d}{\alpha_j})^2$$

$$+(693-3003(\frac{\alpha_i}{\alpha_j})^2)(\frac{d}{\alpha_j})^4-429(\frac{d}{\alpha_j})^6]-\frac{1}{(L+3)\alpha_j^2}[945-6930(\frac{\alpha_i}{\alpha_j})^2+9009(\frac{\alpha_i}{\alpha_j})^4$$

$$-2310(3-13(\frac{\alpha_i}{\alpha_j})^2)(\frac{d}{\alpha_j})^2+9009(\frac{d}{\alpha_j})^4]+\frac{13860}{(L+5)\alpha_j^4}[3-13(\frac{\alpha_i}{\alpha_j})^2-13(\frac{d}{\alpha_j})^2]$$

$$-\frac{1081080}{(L+7)\alpha_j^6}\}\ , \qquad \text{for } q=7. \qquad \text{(A.23)}$$

For r=0 and L=odd, we have the following expressions:

$$0, \qquad \text{for } q=0,$$

$$-8\alpha_i/[(L+2)\alpha_j], \qquad \text{for } q=1,$$

$$24\alpha_i d/[(L+2)\alpha_j^2], \qquad \text{for } q=2,$$

$$\frac{\alpha_i}{\alpha_j}\{\frac{1}{L+2}[12-20(\frac{\alpha_i}{\alpha_j})^2-60(\frac{d}{\alpha_j})^2]-\frac{120}{(L+4)\alpha_j^2}\}, \qquad \text{for } q=3,$$

$$-\frac{\alpha_i d}{\alpha_j^2}\{\frac{1}{L+2}[60-140(\frac{\alpha_i}{\alpha_j})^2-140(\frac{d}{\alpha_j})^2]-\frac{840}{(L+4)\alpha_j^2}\}\ , \qquad \text{for } q=4,$$

$$-\frac{\alpha_i}{\alpha_j}\{\frac{1}{L+2}[15-70(\frac{\alpha_i}{\alpha_j})^2+63(\frac{\alpha_i}{\alpha_j})^4-210(1-3(\frac{\alpha_i}{\alpha_j})^2)(\frac{d}{\alpha_j})^2+315(\frac{d}{\alpha_j})^4]$$

$$-\frac{420}{(L+4)\alpha_j^2}[1-3(\frac{\alpha_i}{\alpha_j})^2-9(\frac{d}{\alpha_j})^2]+\frac{7560}{(L+6)\alpha_j^4}\}\ , \qquad \text{for } q=5,$$

$$\frac{\alpha_i d}{\alpha_j^2}\{\frac{1}{L+2}[105-630(\frac{\alpha_i}{\alpha_j})^2+693(\frac{\alpha_i}{\alpha_j})^4-210(3-11(\frac{\alpha_i}{\alpha_j})^2)(\frac{d}{\alpha_j})^2+693(\frac{d}{\alpha_j})^4]$$

$$-\frac{1260}{(L+4)\alpha_j^2}[3-11(\frac{\alpha_i}{\alpha_j})^2-11(\frac{d}{\alpha_j})^2]+\frac{83160}{(L+6)\alpha_j^4}\}, \qquad \text{for } q=6,$$

$$\frac{\alpha_i}{\alpha_j}\{\frac{1}{2(L+2)}[35-315(\frac{\alpha_i}{\alpha_j})^2+693(\frac{\alpha_i}{\alpha_j})^4-429(\frac{\alpha_i}{\alpha_j})^6-(945-6930(\frac{\alpha_i}{\alpha_j})^2+9009(\frac{\alpha_i}{\alpha_j})^4)(\frac{d}{\alpha_j})^2$$

$$+1155(3-13(\frac{\alpha_i}{\alpha_j})^2)(\frac{d}{\alpha_j})^4+3003(\frac{d}{\alpha_j})^6]-\frac{63}{(L+4)\alpha_j^2}[15-110(\frac{\alpha_i}{\alpha_j})^2+143(\frac{\alpha_i}{\alpha_j})^4$$

$$-110(3-13(\frac{\alpha_i}{\alpha_j})^2)(\frac{d}{\alpha_j})^2+715(\frac{d}{\alpha_j})^4]+\frac{13860}{(L+6)\alpha_j^4}[3-13(\frac{\alpha_i}{\alpha_j})^2-39(\frac{d}{\alpha_j})^2]$$

$$-\frac{1081080}{(L+8)\alpha_j^6}\}, \qquad \text{for } q=7. \tag{A.24}$$

The expression (A.22) with r=1 or 7 gives the value zero except for the cases where it gives the following forms:

$$-8\alpha_i^2/[3(L+1)\alpha_j], \qquad \text{for } r=q=1 \text{ and } L=\text{even},$$

$$-8\alpha_i^8/[15(L+1)\alpha_j^7], \qquad \text{for } r=q=7 \text{ and } L=\text{even}. \tag{A.25}$$

In the case where $\alpha_j+\alpha_i+|d|$ is small, we can use the following series expansion:

$$\sum_{s=0}^{q+r}(-\frac{1}{2})^s\sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!}X_{nsL}(\alpha_j,\alpha_i,d)$$

$$=\sum_{m=0}^{L+1+q+r}\frac{2}{m+1}\alpha_i^{L-m-1}\sum_{p=0}^{q+r+L+1-m}\frac{1}{p!}(\frac{d}{\alpha_i})^p\sum_{t=-q-1}^{L+r-m-p}(\frac{\alpha_j}{\alpha_i})^t\sum_{n=-t-1\geq 0}^{q}\frac{(q+n)!}{n!(q-n)!}\frac{1}{(t+n+1)!}$$

$$\times\sum_{s=t+m+p+n-L\geq n}^{n+r}(-\frac{1}{2})^s\frac{(r+s-n)!}{(s-n)!(r-s+n)!}\frac{1}{(L+s-m-t-p-n)!}$$

$$+\sum_{s=0}^{q+r}(-\frac{1}{2})^s\frac{2}{(L+s+2)!}[\sum_{u=1}^{L+s+2}\frac{1}{u}-\gamma-\ell n(-(\alpha_j+\alpha_i+d))]\sum_{n=s-r}^{s\leq q}\frac{(q+n(!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!}$$

$$\times\frac{(\alpha_j+\alpha_i+d)^{L+s+2}}{\alpha_j^{n+1}\alpha_i^{s-n+1}}-\sum_{m=2}^{\infty}\frac{1}{m-1}(\alpha_j+\alpha_i+d)^{m-1}\sum_{s=0}^{q+r}(-\frac{1}{2})^s\frac{2}{(L+s+m+1)!}$$

$$\times\sum_{n=s-r}^{s\leq q}\frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!}\frac{(\alpha_j+\alpha_i+d)^{L+s+2}}{\alpha_j^{n+1}\alpha_i^{s-n+1}}, \tag{A.26}$$

where γ is the Euler-Mascheroni constant. On taking into account 4 X_{nsL} terms on the right hand side of Eq. (A.2), the first term on the right hand side of Eq. (A.26) is to be multiplied by the following factors for the parameter ranges (a), (b) and (c), respectively:

$$[1+(-1)^{q+t}][1+(-1)^{L+r-m-1-p-t}], \qquad \text{for (a)},$$

$$\{(-1)^{m+1}+(-1)^{q+t}+(-1)^{L+r-m-1-p-t}[1+(-1)^{q+t}]\} \qquad \text{for (b)},$$

$$[(-1)^{m+1}+(-1)^{q+t}][1+(-1)^{L+r-m-1-p-t}], \qquad \text{for (c)}. \qquad (A.27)$$

Thus, for the range (a), the first term on the right hand side of (A.26) gives, by using the expression (A.27), the following forms: For q=r=0, 0 for L=0, and

$$8, \qquad \text{for } L=1,$$

$$8d+4, \qquad \text{for } L=2,$$

$$\frac{4}{3}\alpha_i^2[1+(\frac{\alpha_j}{\alpha_i})^2+3(\frac{d}{\alpha_i})^2]+4d+\frac{8}{3}, \qquad \text{for } L=3,$$

$$\frac{4}{3}\alpha_i^2 d[1+(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{2}{3}\alpha_i^2[1+(\frac{\alpha_j}{\alpha_i})^2+3(\frac{d}{\alpha_i})^2]+\frac{8}{3}d+2, \qquad \text{for } L=4,$$

$$\frac{1}{3}\alpha_i^4[\frac{1}{5}+\frac{2}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{1}{5}(\frac{\alpha_j}{\alpha_i})^4+2(\frac{d}{\alpha_i})^2(1+(\frac{\alpha_j}{\alpha_i})^2)+(\frac{d}{\alpha_i})^4]+\frac{2}{3}\alpha_i^2 d[1+(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]$$
$$+\frac{4}{9}\alpha_i^2[1+(\frac{\alpha_j}{\alpha_i})^2+3(\frac{d}{\alpha_i})^2]+2d+\frac{8}{5}, \qquad \text{for } L=5,$$

$$\frac{1}{15}\alpha_i^4 d[1+\frac{10}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{\alpha_j}{\alpha_i})^4+\frac{10}{3}(\frac{d}{\alpha_i})^2(1+(\frac{\alpha_j}{\alpha_i})^2)+(\frac{d}{\alpha_i})^4]+\frac{1}{30}\alpha_i^4[1+\frac{10}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{\alpha_j}{\alpha_i})^4$$
$$+10(\frac{d}{\alpha_i})^2(1+(\frac{\alpha_j}{\alpha_i})^2)+5(\frac{\alpha_j}{\alpha_i})^4]+\frac{4}{9}\alpha_i^2 d[1+(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{1}{3}\alpha_i^2[1+(\frac{\alpha_j}{\alpha_i})^2+3(\frac{d}{\alpha_i})^2]$$
$$+\frac{8}{5}d+\frac{4}{3}, \qquad \text{for } L=6. \qquad (A.28)$$

For q=0 and r=1, 0 for L=0 and 1, and

$$\frac{8}{3}\alpha_i, \qquad \text{for } L=2,$$

$$\frac{8}{3}\alpha_i(d+\frac{1}{2}), \qquad \text{for } L=3,$$

$$\frac{4}{3}\alpha_i^3[\frac{1}{5}+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{4}{3}\alpha_i(d+\frac{2}{3}), \qquad \text{for } L=4,$$

$\frac{4}{9}\ \alpha_i{}^3 d[\frac{3}{5}+(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{2}{3}\ \alpha_i{}^3[\frac{1}{5}+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{8}{9}\ \alpha_i(d+\frac{3}{4}),$ for L=5,

$\frac{1}{9}\ \alpha_i{}^5[\frac{3}{35}+\frac{2}{5}(\frac{\alpha_j}{\alpha_i})^2+\frac{1}{5}(\frac{\alpha_j}{\alpha_i})^4+2(\frac{d}{\alpha_i})^2(\frac{3}{5}+(\frac{\alpha_j}{\alpha_i})^2)+(\frac{d}{\alpha_i})^4]+\frac{2}{9}\ \alpha_i{}^3 d[\frac{3}{5}+(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]$

$+\frac{4}{9}\ \alpha_i{}^3[\frac{1}{5}+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{2}{3}\ \alpha_i(d+\frac{4}{5}),$ for L=6.

(A.29)

For q=0 and r=2, 0 for L=0, 1 and 2, and

$\frac{8}{15}\ \alpha_i{}^2,$ for L=3,

$\frac{8}{15}\ \alpha_i{}^2(d+\frac{1}{2})$, for L=4,

$\frac{4}{15}\ \alpha_i{}^4[\frac{1}{7}+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{4}{15}\ \alpha_i{}^2(d+\frac{2}{3}),$ for L=5,

$\frac{4}{45}\ \alpha_i{}^4 d[\frac{3}{7}+(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{2}{15}\ \alpha_i{}^4[\frac{1}{7}+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{8}{45}\ \alpha_i{}^2(d+\frac{3}{4}),$

for L=6.

(A.30)

For q=0 and r=3, 0 for L=0∿3, and

$\frac{8}{105}\ \alpha_i{}^3,$ for L=4,

$\frac{8}{105}\ \alpha_i{}^3(d+\frac{1}{2}),$ for L=5,

$\frac{4}{105}\ \alpha_i{}^5[\frac{1}{9}+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{4}{105}\ \alpha_i{}^3(d+\frac{2}{3}),$ for L=6.

(A.31)

For q=0 and r=4, 0 for L=0∿4, and

$\frac{8}{945}\ \alpha_i{}^4,$ for L=5,

$\frac{8}{945}\ \alpha_i{}^4(d+\frac{1}{2}),$ for L=6. (A.32)

For q=0 and r=5, 0 for L=0∿5, and

$$\frac{8}{10395}\alpha_i^5 , \qquad \text{for } L=6. \qquad (A.33)$$

For q=0 and r≥6, 0 for L=0∿6. The expressions for q=q and r=0 are obtained by interchanging α_i with α_j in the expressions (A.28) ∿ (A.33) for r=r and q=0, respectively. For q=1 and r=1, we have 0 for L=0∿2, and

$$\frac{8}{9}\alpha_i\alpha_j , \qquad \text{for } L=3,$$

$$\frac{8}{9}\alpha_i\alpha_j(d+\frac{1}{2}), \qquad \text{for } L=4,$$

$$\frac{4}{45}\alpha_i^3\alpha_j[1+(\frac{\alpha_j}{\alpha_i})^2+5(\frac{d}{\alpha_i})^2]+\frac{4}{9}\alpha_i\alpha_j(d+\frac{2}{3}), \qquad \text{for } L=5,$$

$$\frac{4}{45}\alpha_i^3\alpha_j d[1+(\frac{\alpha_j}{\alpha_i})^2+\frac{5}{3}(\frac{d}{\alpha_i})^2]+\frac{2}{45}\alpha_i^3\alpha_j[1+(\frac{\alpha_j}{\alpha_i})^2+5(\frac{d}{\alpha_i})^2]+\frac{8}{27}\alpha_i\alpha_j(d+\frac{3}{4}),$$

$$\text{for } L=6. \qquad (A.34)$$

For q=1 and r=2, 0 for L=0∿3, and

$$\frac{8}{45}\alpha_i^2\alpha_j, \qquad \text{for } L=4,$$

$$\frac{8}{45}\alpha_i^2\alpha_j(d+\frac{1}{2}), \qquad \text{for } L=5,$$

$$\frac{4}{45}\alpha_i^4\alpha_j[\frac{1}{7}+\frac{1}{5}(\frac{\alpha_j}{\alpha_i})^2+(\frac{d}{\alpha_i})^2]+\frac{4}{45}\alpha_i^2\alpha_j(d+\frac{2}{3}), \qquad \text{for } L=6. \qquad (A.35)$$

For q=1 and r=3, 0 for L=0∿4, and

$$\frac{8}{315}\alpha_i^3\alpha_j, \qquad \text{for } L=5,$$

$$\frac{8}{315}\alpha_i^3\alpha_j(d+\frac{1}{2}), \qquad \text{for } L=6. \qquad (A.36)$$

For q=1 and r=4, 0 for L=0∿5, and

$$\frac{8}{2835}\alpha_i^4\alpha_j, \qquad \text{for } L=6. \qquad (A.37)$$

For q=1 and r≥5, 0 for L=0∿6. In addition, as seen from the above-mentioned expressions, we get 0 for all cases with r=7 and L≤6.

For the parameter range (b), the first term on the right hand side of Eq. (A.26) gives the following expressions by taking into account the factor (A.27) and the contribution of Y_2 terms written by Eq. (A.15) divided by $\alpha_j\alpha_i$ [the expressions (A.16) ∿ (A.21) divided by $\alpha_j\alpha_i$]:

$$\sum_{m=0}^{[(L-1)/2]} \left[\frac{1}{2m+1} U_{qr}^{L-2m} + \frac{1}{m+1} V_{qr}^{L-2m}\right], \qquad (A.38)$$

$$U_{qr}^{L} = \sum_{n=0}^{L+r+q+1} \frac{1}{n!} d^n W_{qr}^{L-n}, \quad V_{qr}^{L} = \sum_{n=0}^{L-r-q-2\geq 0} \frac{1}{n!} d^n Z_{qr}^{L-n},$$

where

$$W_{00}^{n} = -\frac{4}{(n+1)!}\left[1+(-1)^n \frac{\alpha_j}{\alpha_i}\right]^{n+1} \frac{\alpha_i^n}{\alpha_j},$$

$$Z_{00}^{2} = 4, \quad Z_{00}^{3}=0, \quad Z_{00}^{4}= \frac{2}{3}\left[1+\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\alpha_i^2,$$

$$Z_{00}^{5} = 0, \quad Z_{00}^{6}= \frac{1}{30}\left[1+ \frac{10}{3}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\alpha_i^4,$$

$$W_{01}^{n} = -\frac{4}{(n+2)n!}\left[1-(-1)^n \frac{\alpha_j}{\alpha_i}\right]^{n+1}\left[1+(-1)^n \frac{1}{n+1} \frac{\alpha_j}{\alpha_i}\right]\frac{\alpha_i^n}{\alpha_j}, \quad n\geq 0,$$

$$W_{01}^{-1} = 4/\alpha_i^2, \quad W_{01}^{-2} = 4/(\alpha_i^2\alpha_j),$$

$$Z_{01}^{3} = \frac{4}{3}\alpha_i, \quad Z_{01}^{4} = 0, \quad Z_{01}^{5} = \frac{2}{15}\left[1+ \frac{5}{3}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\alpha_i^3, \quad Z_{01}^{6} = 0,$$

$$W_{02}^{n} = -\frac{4}{(n+3)(n+1)(n-1)!}\left[1+(-1)^n \frac{\alpha_j}{\alpha_i}\right]^{n+1}\left[1- \frac{3}{n(n+2)} \frac{\alpha_j}{\alpha_i}\left((-1)^n(n+1)- \frac{\alpha_j}{\alpha_i}\right)\right]\frac{\alpha_i^n}{\alpha_j},$$

$$n>0,$$

$$W_{02}^{0} = 2\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{1}{\alpha_i}, \quad W_{02}^{-1} = 2\left[1-3\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{1}{\alpha_i\alpha_j},$$

$$W_{02}^{-2} = -12/\alpha_i^3, \quad W_{02}^{-3} = -12/(\alpha_i^3\alpha_j),$$

$$Z_{02}^{4} = \frac{4}{15}\alpha_i^2, \quad Z_{02}^{5} = 0, \quad Z_{02}^{6} = \frac{2}{105}\alpha_i^4\left[1+ \frac{7}{3}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right],$$

$$W_{0\bar{3}}^{4} = 60/(\alpha_i^{4}\alpha_j), \quad W_{0\bar{3}}^{3} = 60/\alpha_i^{4},$$

$$W_{0\bar{3}}^{2} = -6[1-5(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^2\alpha_j}, \quad W_{0\bar{3}}^{1} = -6[1-\frac{5}{3}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^2},$$

$$W_{03}^{0} = \frac{1}{2}[1-(\frac{\alpha_j}{\alpha_i})^2][1-5(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_j}, \quad W_{03}^{1} = \frac{1}{2}[1-(\frac{\alpha_j}{\alpha_i})^2]^2,$$

$$W_{03}^{2} = -\frac{1}{12}[1-(\frac{\alpha_j}{\alpha_i})^2]^3\frac{\alpha_i^2}{\alpha_j},$$

$$W_{03}^{3} = -\frac{4}{105}(1+\frac{\alpha_j}{\alpha_i})^4[1-\frac{29}{16}\frac{\alpha_j}{\alpha_i}+\frac{5}{4}(\frac{\alpha_j}{\alpha_i})^2-\frac{5}{16}(\frac{\alpha_j}{\alpha_i})^3]\frac{\alpha_i^3}{\alpha_j},$$

$$W_{03}^{4} = -\frac{1}{96}(1-\frac{\alpha_j}{\alpha_i})^5[1+\frac{47}{35}\frac{\alpha_j}{\alpha}+\frac{5}{7}(\frac{\alpha_j}{\alpha_i})^2+\frac{1}{7}(\frac{\alpha_j}{\alpha_i})^3]\frac{\alpha_i^4}{\alpha_j},$$

$$W_{03}^{5} = -\frac{2}{945}(1+\frac{\alpha_j}{\alpha_i})^6[1-\frac{69}{64}\frac{\alpha_j}{\alpha_i}+\frac{15}{32}(\frac{\alpha_j}{\alpha_i})^2-\frac{5}{64}(\frac{\alpha_j}{\alpha_i})^3]\frac{\alpha_i^5}{\alpha_j},$$

$$W_{03}^{6} = -\frac{1}{2880}(1-\frac{\alpha_j}{\alpha_i})^7[1+\frac{19}{21}\frac{\alpha_j}{\alpha_i}+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{1}{21}(\frac{\alpha_j}{\alpha_i})^3]\frac{\alpha_i^6}{\alpha_j},$$

$$Z_{03}^{5} = \frac{4}{105}\alpha_i^3, \quad Z_{03}^{6} = 0,$$

$$W_{0\bar{4}}^{5} = -420/(\alpha_i^5\alpha_j), \quad W_{0\bar{4}}^{4} = -420/\alpha_i^5,$$

$$W_{0\bar{4}}^{3} = 30[1-7(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^3\alpha_j}, \quad W_{0\bar{4}}^{2} = 30[1-\frac{7}{3}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^3},$$

$$W_{0\bar{4}}^{1} = -\frac{3}{2}[1-10(\frac{\alpha_j}{\alpha_i})^2+\frac{35}{3}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i\alpha_j}, \quad W_{04}^{0} = -\frac{3}{2}[1-(\frac{\alpha_j}{\alpha_i})^2][1-\frac{7}{3}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i},$$

$$W_{04}^{1} = \frac{1}{12}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1-7(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i}{\alpha_j}, \quad W_{04}^{2} = \frac{1}{12}[1-(\frac{\alpha_j}{\alpha_i})^2]^3\alpha_i,$$

$$W_{04}^{3} = -\frac{1}{96}[1-(\frac{\alpha_j}{\alpha_i})^2]^4\frac{\alpha_i^3}{\alpha_j},$$

$$W_{04}^{4} = -\frac{4}{945}(1+\frac{\alpha_j}{\alpha_i})^5[1-\frac{325}{128}\frac{\alpha_j}{\alpha_i}+\frac{345}{128}(\frac{\alpha_j}{\alpha_i})^2-\frac{175}{128}(\frac{\alpha_j}{\alpha_i})^3+\frac{35}{128}(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_i^4}{\alpha_j},$$

$$W_{04}^{5} = -\frac{1}{960}(1-\frac{\alpha_j}{\alpha_i})^6[1+\frac{122}{63}\frac{\alpha_j}{\alpha_i}+\frac{34}{21}(\frac{\alpha_j}{\alpha_i})^2+\frac{2}{3}(\frac{\alpha_j}{\alpha_i})^3+\frac{1}{9}(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_i^5}{\alpha_j},$$

$$W_{04}^{6} = -\frac{2}{10395}(1+\frac{\alpha_j}{\alpha_i})^7[1-\frac{203}{128}\frac{\alpha_j}{\alpha_i}+\frac{141}{128}(\frac{\alpha_j}{\alpha_i})^2-\frac{49}{128}(\frac{\alpha_j}{\alpha_i})^3+\frac{7}{128}(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_i^6}{\alpha_j}$$

$$Z_{04}^{6} = \frac{4}{945}\alpha_i^4,$$

$$W_{0\bar{5}}^{6} = 3780/(\alpha_i^6\alpha_j), \quad W_{0\bar{5}}^{5} = 3780/\alpha_i^6,$$

$$W_{05}^{-4} = -210[1-9(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^4\alpha_j} , \quad W_{05}^{-3} = -210[1-3(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^4} ,$$

$$W_{05}^{-2} = \frac{15}{2}[1-14(\frac{\alpha_j}{\alpha_i})^2+21(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^2\alpha_j} , \quad W_{05}^{-1} = \frac{15}{2}[1-\frac{14}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{21}{5}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^2} ,$$

$$W_{05}^{0} = -\frac{1}{4}[1-(\frac{\alpha_j}{\alpha_i})^2][1-14(\frac{\alpha_j}{\alpha_i})^2+21(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_j} ,$$

$$W_{05}^{1} = -\frac{1}{4}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1-3(\frac{\alpha_j}{\alpha_i})^2],$$

$$W_{05}^{2} = \frac{1}{96}[1-(\frac{\alpha_j}{\alpha_i})^2]^3[1-9(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i^2}{\alpha_j} , \quad W_{05}^{3} = \frac{1}{96}[1-(\frac{\alpha_j}{\alpha_i})^2]^4\alpha_i^2 ,$$

$$W_{05}^{4} = -\frac{1}{960}[1-(\frac{\alpha_j}{\alpha_i})^2]^5\frac{\alpha_i^4}{\alpha_j} ,$$

$$W_{05}^{5} = -\frac{4}{10395}(1+\frac{\alpha_j}{\alpha_i})^6[1-\frac{843}{256}\frac{\alpha_j}{\alpha_i}+\frac{609}{128}(\frac{\alpha_j}{\alpha_i})^2-\frac{469}{128}(\frac{\alpha_j}{\alpha_i})^3+\frac{189}{128}(\frac{\alpha_j}{\alpha_i})^4-\frac{63}{256}(\frac{\alpha_j}{\alpha_i})^5]\frac{\alpha_i^5}{\alpha_j} ,$$

$$W_{05}^{6} = -\frac{1}{11520}(1-\frac{\alpha_j}{\alpha_i})^7[1+\frac{593}{231}\frac{\alpha_j}{\alpha_i}+\frac{98}{33}(\frac{\alpha_j}{\alpha_i})^2+\frac{62}{33}(\frac{\alpha_j}{\alpha_i})^3+\frac{7}{11}(\frac{\alpha_j}{\alpha_i})^4+\frac{1}{11}(\frac{\alpha_j}{\alpha_i})^5]\frac{\alpha_i^6}{\alpha_j} ,$$

$$W_{06}^{-7} = -41580/(\alpha_i^7\alpha_j) , \quad W_{06}^{-6} = -41580/\alpha_i^7,$$

$$W_{06}^{-5} = 1890[1-11(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^5\alpha_j} , \quad W_{06}^{-4} = 1890[1-\frac{11}{3}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^5} ,$$

$$W_{06}^{-3} = -\frac{105}{2}[1-18(\frac{\alpha_j}{\alpha_i})^2+33(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^3\alpha_j} , \quad W_{06}^{-2} = -\frac{105}{2}[1-6(\frac{\alpha_j}{\alpha_i})^2+\frac{33}{5}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^3} ,$$

$$W_{06}^{-1} = \frac{5}{4}[1-21(\frac{\alpha_j}{\alpha_i})^2+63(\frac{\alpha_j}{\alpha_i})^4-\frac{231}{5}(\frac{\alpha_j}{\alpha_i})^6]\frac{1}{\alpha_i\alpha_j} ,$$

$$W_{06}^{0} = \frac{5}{4}[1-(\frac{\alpha_j}{\alpha_i})^2][1-6(\frac{\alpha_j}{\alpha_i})^2+\frac{33}{5}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i} ,$$

$$W_{06}^{1} = -\frac{1}{32}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1-18(\frac{\alpha_j}{\alpha_i})^2+33(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_i}{\alpha_j} ,$$

$$W_{06}^{2} = -\frac{1}{32}[1-(\frac{\alpha_j}{\alpha_i})^2]^3[1-\frac{11}{3}(\frac{\alpha_j}{\alpha_i})^2]\alpha_i ,$$

$$W_{06}^{3} = \frac{1}{960}[1-(\frac{\alpha_j}{\alpha_i})^2]^4[1-11(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i^3}{\alpha_j} , \quad W_{06}^{4} = \frac{1}{960}[1-(\frac{\alpha_j}{\alpha_i})^2]^5\alpha_i^3,$$

$$W_{06}^{5} = -\frac{1}{11520}[1-(\frac{\alpha_j}{\alpha_i})^2]^6\frac{\alpha_i^5}{\alpha_j} ,$$

$$W_{06}^{6} = -\frac{4}{135135}(1+\frac{\alpha_j}{\alpha_i})^7[1-\frac{4165}{1024}\frac{\alpha_j}{\alpha_i}+\frac{7651}{1024}(\frac{\alpha_j}{\alpha_i})^2-\frac{3969}{512}(\frac{\alpha_j}{\alpha_i})^3+\frac{2415}{512}(\frac{\alpha_j}{\alpha_i})^4$$
$$-\frac{1617}{1024}(\frac{\alpha_j}{\alpha_i})^5+\frac{231}{1024}(\frac{\alpha_j}{\alpha_i})^6]\frac{\alpha_i^6}{\alpha_j} ,$$

$W_{0\bar{7}}^{8} = 540540/(\alpha_i^{8}\alpha_j)$, $W_{0\bar{7}}^{7} = 540540/\alpha_i^{8}$,

$W_{0\bar{7}}^{6} = -20790[1-13(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^6\alpha_j}$, $W_{0\bar{7}}^{5} = -20790[1-\frac{13}{3}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^6}$

$W_{0\bar{7}}^{4} = \frac{945}{2}[1-22(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{3}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^4\alpha_j}$,

$W_{0\bar{7}}^{3} = \frac{945}{2}[1-\frac{22}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{15}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^4}$,

$W_{0\bar{7}}^{2} = -\frac{35}{4}[1-27(\frac{\alpha_j}{\alpha_i})^2+99(\frac{\alpha_j}{\alpha_i})^4-\frac{429}{5}(\frac{\alpha_j}{\alpha_i})^6]\frac{1}{\alpha_i^2\alpha_j}$,

$W_{0\bar{7}}^{1} = -\frac{35}{4}[1-9(\frac{\alpha_j}{\alpha_i})^2+\frac{99}{5}(\frac{\alpha_j}{\alpha_i})^4-\frac{429}{35}(\frac{\alpha_j}{\alpha_i})^6]\frac{1}{\alpha_i^2}$,

$W_{07}^{0} = \frac{5}{32}[1-(\frac{\alpha_j}{\alpha_i})^2][1-27(\frac{\alpha_j}{\alpha_i})^2+99(\frac{\alpha_j}{\alpha_i})^4-\frac{429}{5}(\frac{\alpha_j}{\alpha_i})^6]\frac{1}{\alpha_j}$,

$W_{07}^{1} = \frac{5}{32}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1-\frac{22}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{15}(\frac{\alpha_j}{\alpha_i})^4]$,

$W_{07}^{2} = -\frac{1}{320}[1-(\frac{\alpha_j}{\alpha_i})^2]^3[1-22(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{3}(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_i^2}{\alpha_j}$,

$W_{07}^{3} = -\frac{1}{320}[1-(\frac{\alpha_j}{\alpha_i})^2]^4[1-\frac{13}{3}(\frac{\alpha_j}{\alpha_i})^2]\alpha_i^2$,

$W_{07}^{4} = \frac{1}{11520}[1-(\frac{\alpha_j}{\alpha_i})^2]^5[1-13(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i^4}{\alpha_j}$

$W_{07}^{5} = \frac{1}{11520}[1-(\frac{\alpha_j}{\alpha_i})^2]^6\alpha_i^4$, $W_{07}^{6} = -\frac{1}{161280}[1-(\frac{\alpha_j}{\alpha_i})^2]^7\frac{\alpha_i^6}{\alpha_j}$,

$W_{10}^{n} = \frac{4}{(n+2)!}[1-(-1)^n\frac{\alpha_j}{\alpha_i}]^{n+1}[1+(-1)^n(n+1)\frac{\alpha_j}{\alpha_i}]\frac{\alpha_i^{n+1}}{\alpha_j^2}$, $n\geq 0$,

which can be obtained from W_{01}^{n} by interchanging α_i with α_j,

$W_{1\bar{0}}^{1} = 4/\alpha_j^2$, $W_{1\bar{0}}^{2} = 4/(\alpha_j^2\alpha_i)$,

$Z_{10}^{3} = \frac{4}{3}\alpha_j$, $Z_{10}^{4} = 0$, $Z_{10}^{5} = \frac{2}{9}[1+\frac{3}{5}(\frac{\alpha_j}{\alpha_i})^2]\alpha_i^2\alpha_j$, $Z_{10}^{6} = 0$,

$W_{11}^{n} = \frac{4}{(n+3)(n+1)!}[1+(-1)^n\frac{\alpha_j}{\alpha_i}]^{n+1}[1-(-1)^n(n+1)\frac{\alpha_j}{\alpha_i}+(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i^{n+1}}{\alpha_j^2}$, $n\geq 0$,

$W_{1\bar{1}}^{1} = 2[1+(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_j^2}$, $W_{1\bar{1}}^{2} = 0$, $W_{1\bar{1}}^{3} = 4/(\alpha_i^2\alpha_j^2)$,

$Z_{11}^{4} = \frac{4}{9}\alpha_i\alpha_j$, $Z_{11}^{5} = 0$, $Z_{11}^{6} = \frac{2}{45}[1+(\frac{\alpha_j}{\alpha_i})^2]\alpha_i^3\alpha_j$,

$$W_{12}^{-4} = 12/(\alpha_i^3\alpha_j^2)\ ,\quad W_{12}^{-3} = 0\ ,\quad W_{12}^{-2} = -2[1+3(\frac{\alpha_j}{\alpha_i})^2]\ \frac{1}{\alpha_i\alpha_j^2}\ ,$$

$$W_{12}^{-1} = -4\alpha_j/\alpha_i^3,\quad W_{12}^{0} = \frac{1}{2}[1-(\frac{\alpha_j}{\alpha_i})^2][1+3(\frac{\alpha_j}{\alpha_i})^2]\ \frac{\alpha_i}{\alpha_j^2}\ ,$$

$$W_{12}^{1} = \frac{4}{15}(1+\frac{\alpha_j}{\alpha_i})^2[1-2\frac{\alpha_j}{\alpha_i}+3(\frac{\alpha_j}{\alpha_i})^2-\frac{3}{2}(\frac{\alpha_j}{\alpha_i})^3](\frac{\alpha_i}{\alpha_j})^2,$$

$$W_{12}^{2} = \frac{1}{12}[1-(\frac{\alpha_j}{\alpha_i})^2]^3\ \frac{\alpha_i^3}{\alpha_j^2}\ ,$$

$$W_{12}^{3} = \frac{2}{105}(1+\frac{\alpha_j}{\alpha_i})^4[1-4\frac{\alpha_j}{\alpha_i}+3(\frac{\alpha_j}{\alpha_i})^2-\frac{3}{4}(\frac{\alpha_j}{\alpha_i})^3]\ \frac{\alpha_i^4}{\alpha_j^2}\ ,$$

$$W_{12}^{4} = \frac{1}{288}(1-\frac{\alpha_j}{\alpha_i})^5[1+5\frac{\alpha_j}{\alpha_i}+3(\frac{\alpha_j}{\alpha_i})^2+\frac{3}{5}(\frac{\alpha_j}{\alpha_i})^3]\ \frac{\alpha_i^5}{\alpha_j^2}\ ,$$

$$W_{12}^{5} = \frac{1}{1890}(1+\frac{\alpha_j}{\alpha_i})^6[1-6\frac{\alpha_j}{\alpha_i}+3(\frac{\alpha_j}{\alpha_i})^2-\frac{1}{2}(\frac{\alpha_j}{\alpha_i})^3]\ \frac{\alpha_i^6}{\alpha_j^2}\ ,$$

$$W_{12}^{6} = \frac{1}{14400}(1-\frac{\alpha_j}{\alpha_i})^7[1+7\frac{\alpha_j}{\alpha_i}+3(\frac{\alpha_j}{\alpha_i})^2+\frac{3}{7}(\frac{\alpha_j}{\alpha_i})^3]\ \frac{\alpha_i^7}{\alpha_j^2}\ ,$$

$$Z_{12}^{5} = \frac{4}{45}\ \alpha_i^2\alpha_j\ ,\quad Z_{12}^{6} = 0,$$

$$W_{13}^{-5} = -60/(\alpha_i^4\alpha_j^2),\quad W_{13}^{-4} = 0\ ,\quad W_{13}^{-3} = 6[1+5(\frac{\alpha_j}{\alpha_i})^2]\ \frac{1}{\alpha_i^2\alpha_j^2}\ ,$$

$$W_{13}^{-2} = 20\alpha_j/\alpha_i^4,\qquad W_{13}^{-1} = -\frac{1}{2}[1+6(\frac{\alpha_j}{\alpha_i})^2-15(\frac{\alpha_j}{\alpha_i})^4]\ \frac{1}{\alpha_j^2}\ ,$$

$$W_{13}^{0} = -2[1-(\frac{\alpha_j}{\alpha_i})^2]\ \frac{\alpha_j}{\alpha_i^2}\ ,\quad W_{13}^{1} = \frac{1}{12}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1+5(\frac{\alpha_j}{\alpha_i})^2](\frac{\alpha_i}{\alpha_j})^2\ ,$$

$$W_{13}^{2} = \frac{4}{105}(1+\frac{\alpha_j}{\alpha_i})^3[1-3\frac{\alpha_j}{\alpha_i}+6(\frac{\alpha_j}{\alpha_i})^2-\frac{45}{8}(\frac{\alpha_j}{\alpha_i})^3+\frac{15}{8}(\frac{\alpha_j}{\alpha_i})^4]\ \frac{\alpha_i^3}{\alpha_j^2}\ ,$$

$$W_{13}^{3} = \frac{1}{96}[1-(\frac{\alpha_j}{\alpha_i})^2]^4\ \frac{\alpha_i^4}{\alpha_j^2}\ ,$$

$$W_{13}^{4} = \frac{2}{945}(1+\frac{\alpha_j}{\alpha_i})^5[1-5\frac{\alpha_j}{\alpha_i}+6(\frac{\alpha_j}{\alpha_i})^2-\frac{25}{8}(\frac{\alpha_j}{\alpha_i})^3+\frac{5}{8}(\frac{\alpha_j}{\alpha_i})^4]\ \frac{\alpha_i^5}{\alpha_j^2}\ ,$$

$$W_{13}^{5} = \frac{1}{2880}(1-\frac{\alpha_j}{\alpha_i})^6[1+6\frac{\alpha_j}{\alpha_i}+6(\frac{\alpha_j}{\alpha_i})^2+\frac{18}{7}(\frac{\alpha_j}{\alpha_i})^3+\frac{3}{7}(\frac{\alpha_j}{\alpha_i})^4]\ \frac{\alpha_i^6}{\alpha_j^2}\ ,$$

$$W_{13}^{6} = \frac{1}{20790}(1+\frac{\alpha_j}{\alpha_i})^7[1-7\frac{\alpha_j}{\alpha_i}+6(\frac{\alpha_j}{\alpha_i})^2-\frac{35}{16}(\frac{\alpha_j}{\alpha_i})^3+\frac{5}{16}(\frac{\alpha_j}{\alpha_i})^4]\ \frac{\alpha_i^7}{\alpha_j^2}\ ,$$

$$Z_{13}^{6} = \frac{4}{315}\ \alpha_i^3\alpha_j\ ,$$

$$W_{14}^{-6} = 420/(\alpha_i^5\alpha_j^2),\quad W_{14}^{-5} = 0,\quad W_{14}^{-4} = -30[1+7(\frac{\alpha_j}{\alpha_i})^2]\ \frac{1}{\alpha_i^3\alpha_j^2}\ ,$$

$$W_{14}^{-3} = -140\alpha_j/\alpha_i^5 \quad , \quad W_{14}^{-2} = \frac{3}{2}[1+10(\frac{\alpha_j}{\alpha_i})^2-35(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i\alpha_j^2} \; ,$$

$$W_{14}^{-1} = 10[1-\frac{7}{5}(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_j}{\alpha_i^3} \; , \quad W_{14}^{0} = -\frac{1}{12}[1-(\frac{\alpha_j}{\alpha_i})^2][1+10(\frac{\alpha_j}{\alpha_i})^2-35(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_i}{\alpha_j^2}$$

$$W_{14}^{1} = -\frac{1}{2}[1-(\frac{\alpha_j}{\alpha_i})^2]^2\frac{\alpha_j}{\alpha_i} \; ,$$

$$W_{14}^{2} = \frac{1}{96}[1-(\frac{\alpha_j}{\alpha_i})^2]^3[1+7(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i^3}{\alpha_j^2},$$

$$W_{14}^{3} = \frac{4}{945}(1+\frac{\alpha_j}{\alpha_i})^4[1-4\frac{\alpha_j}{\alpha_i}+10(\frac{\alpha_j}{\alpha_i})^2-\frac{215}{16}(\frac{\alpha_j}{\alpha_i})^3+\frac{35}{4}(\frac{\alpha_j}{\alpha_i})^4-\frac{35}{16}(\frac{\alpha_j}{\alpha_i})^5]\frac{\alpha_i^4}{\alpha_j^2} \; ,$$

$$W_{14}^{4} = \frac{1}{960}[1-(\frac{\alpha_j}{\alpha_i})^2]^5\frac{\alpha_i^5}{\alpha_j^2} \; ,$$

$$W_{14}^{5} = \frac{2}{10395}(1+\frac{\alpha_j}{\alpha_i})^6[1-6\frac{\alpha_j}{\alpha_i}+10(\frac{\alpha_j}{\alpha_i})^2-\frac{515}{64}(\frac{\alpha_j}{\alpha_i})^3+\frac{105}{32}(\frac{\alpha_j}{\alpha_i})^4-\frac{35}{64}(\frac{\alpha_j}{\alpha_i})^5]\frac{\alpha_i^6}{\alpha_j^2} \; ,$$

$$W_{14}^{6} = \frac{1}{34560}(1-\frac{\alpha_j}{\alpha_i})^7[1+7\frac{\alpha_j}{\alpha_i}+10(\frac{\alpha_j}{\alpha_i})^2+\frac{142}{21}(\frac{\alpha_j}{\alpha_i})^3+\frac{7}{3}(\frac{\alpha_j}{\alpha_i})^4+\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^5]\frac{\alpha_i^7}{\alpha_j^2} \; ,$$

$$W_{15}^{-7} = 3780/(\alpha_i^6\alpha_j^2), \quad W_{15}^{-6} = 0, \quad W_{15}^{-5} = 210[1+9(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^4\alpha_j^2} \; ,$$

$$W_{15}^{-4} = 1260\alpha_j/\alpha_i^6, \quad W_{15}^{-3} = -\frac{15}{2}[1+14(\frac{\alpha_j}{\alpha_i})^2-63(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^2\alpha_j^2} \; ,$$

$$W_{15}^{-2} = -70[1-\frac{9}{5}(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_j}{\alpha_i^4} \; , \quad W_{15}^{-1} = \frac{1}{4}[1+15(\frac{\alpha_j}{\alpha_i})^2-105(\frac{\alpha_j}{\alpha_i})^4+105(\frac{\alpha_j}{\alpha_i})^6]\frac{1}{\alpha_j^2} \; ,$$

$$W_{15}^{0} = \frac{5}{2}[1-(\frac{\alpha_j}{\alpha_i})^2][1-\frac{9}{5}(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_j}{\alpha_i^2} \; ,$$

$$W_{15}^{1} = -\frac{1}{96}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1+14(\frac{\alpha_j}{\alpha_i})^2-63(\frac{\alpha_j}{\alpha_i})^4](\frac{\alpha_i}{\alpha_j})^2,$$

$$W_{15}^{2} = -\frac{1}{12}[1-(\frac{\alpha_j}{\alpha_i})^2]^3\alpha_j, \quad W_{15}^{3} = \frac{1}{960}[1-(\frac{\alpha_j}{\alpha_i})^2]^4[1+9(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i^4}{\alpha_j^2} \; ,$$

$$W_{15}^{4} = \frac{4}{10395}(1+\frac{\alpha_j}{\alpha_i})^5[1-5\frac{\alpha_j}{\alpha_i}+15(\frac{\alpha_j}{\alpha_i})^2-\frac{3325}{128}(\frac{\alpha_j}{\alpha_i})^3+\frac{3185}{128}(\frac{\alpha_j}{\alpha_i})^4-\frac{1575}{128}(\frac{\alpha_j}{\alpha_i})^5+\frac{315}{128}(\frac{\alpha_j}{\alpha_i})^6]\frac{\alpha_i^5}{\alpha_j^2},$$

$$W_{15}^{5} = \frac{1}{11520}[1-(\frac{\alpha_j}{\alpha_i})^2]^6\frac{\alpha_i^6}{\alpha_j^2}$$

$$W_{15}^{6} = \frac{2}{135135}(1+\frac{\alpha_j}{\alpha_i})^7[1-7\frac{\alpha_j}{\alpha_i}+15(\frac{\alpha_j}{\alpha_i})^2-\frac{2107}{128}(\frac{\alpha_j}{\alpha_i})^3+\frac{1309}{128}(\frac{\alpha_j}{\alpha_i})^4-\frac{441}{128}(\frac{\alpha_j}{\alpha_i})^5+\frac{63}{128}(\frac{\alpha_j}{\alpha_i})^6]\frac{\alpha_i^7}{\alpha_j^2},$$

$$W_{15}^{-8} = 41580/(\alpha_i^7\alpha_j^2), \quad W_{16}^{-7} = 0, \quad W_{16}^{-6} = -1890[1+11(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^5\alpha_j^2} \; ,$$

$$W_{16}^{-5} = -13860\alpha_j/\alpha_i^7 \; , \quad W_{16}^{-4} = \frac{105}{2}[1+18(\frac{\alpha_j}{\alpha_i})^2-99(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^3\alpha_j^2} \; ,$$

$$W_{16}^{\bar{3}} = 630\left[1-\frac{11}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_j}{\alpha_i^5}, \quad W_{16}^{\bar{2}} = -\frac{5}{4}\left[1+21\left(\frac{\alpha_j}{\alpha_i}\right)^2-189\left(\frac{\alpha_j}{\alpha_i}\right)^4+231\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\frac{1}{\alpha_i\alpha_j^2},$$

$$W_{16}^{\bar{1}} = -\frac{35}{2}\left[1-\frac{18}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{99}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{\alpha_j}{\alpha_i^3}$$

$$W_{16}^{0} = \frac{1}{32}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\left[1+21\left(\frac{\alpha_j}{\alpha_i}\right)^2-189\left(\frac{\alpha_j}{\alpha_i}\right)^4+231\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\frac{\alpha_i}{\alpha_j^2},$$

$$W_{16}^{1} = \frac{5}{12}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^2\left[1-\frac{11}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_j}{\alpha_i},$$

$$W_{16}^{2} = -\frac{1}{960}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^3\left[1+18\left(\frac{\alpha_j}{\alpha_i}\right)^2-99\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{\alpha_i^3}{\alpha_j^2},$$

$$W_{16}^{3} = -\frac{1}{96}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^4\alpha_i\alpha_j, \quad W_{16}^{4} = \frac{1}{11520}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^5\left[1+11\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_i^5}{\alpha_j^2},$$

$$W_{16}^{5} = \frac{4}{135135}\left(1+\frac{\alpha_j}{\alpha_i}\right)^6\left[1-6\frac{\alpha_j}{\alpha_i}+21\left(\frac{\alpha_j}{\alpha_i}\right)^2-\frac{11333}{256}\left(\frac{\alpha_j}{\alpha_i}\right)^3+\frac{7119}{128}\left(\frac{\alpha_j}{\alpha_i}\right)^4-\frac{5229}{128}\left(\frac{\alpha_j}{\alpha_i}\right)^5+\frac{2079}{128}\left(\frac{\alpha_j}{\alpha_i}\right)^6\right.$$
$$\left.-\frac{693}{256}\left(\frac{\alpha_j}{\alpha_i}\right)^7\right],$$

$$W_{16}^{6} = \frac{1}{161280}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^7\frac{\alpha_i^7}{\alpha_j^2},$$

$$W_{17}^{\bar{9}} = -540540/(\alpha_i^8\alpha_j^2), \quad W_{17}^{\bar{8}} = 0, \quad W_{17}^{\bar{7}} = 20790\left[1+13\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{1}{\alpha_i^6\alpha_j^2},$$

$$W_{17}^{\bar{6}} = 180180\alpha_j/\alpha_i^8, \quad W_{17}^{\bar{5}} = -\frac{945}{2}\left[1+22\left(\frac{\alpha_j}{\alpha_i}\right)^2-143\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{1}{\alpha_i^4\alpha_j^2},$$

$$W_{17}^{\bar{4}} = -6930\left[1-\frac{13}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_j}{\alpha_i^6}, \quad W_{17}^{\bar{3}} = \frac{35}{4}\left[1+27\left(\frac{\alpha_j}{\alpha_i}\right)^2-297\left(\frac{\alpha_j}{\alpha_i}\right)^4+429\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\frac{1}{\alpha_i^2\alpha_j^2},$$

$$W_{17}^{\bar{2}} = \frac{315}{2}\left[1-\frac{22}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{143}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{\alpha_j}{\alpha_i^3},$$

$$W_{17}^{\bar{1}} = -\frac{5}{32}\left[1+28\left(\frac{\alpha_j}{\alpha_i}\right)^2-378\left(\frac{\alpha_j}{\alpha_i}\right)^4+924\left(\frac{\alpha_j}{\alpha_i}\right)^6-\frac{3003}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^8\right]\frac{1}{\alpha_j^2},$$

$$W_{17}^{0} = -\frac{35}{12}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\left[1-\frac{22}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{143}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{\alpha_j}{\alpha_i^2},$$

$$W_{17}^{1} = \frac{1}{320}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^2\left[1+27\left(\frac{\alpha_j}{\alpha_i}\right)^2-297\left(\frac{\alpha_j}{\alpha_i}\right)^4+429\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\left(\frac{\alpha_i}{\alpha_j}\right)^2,$$

$$W_{17}^{2} = \frac{5}{96}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^3\left[1-\frac{13}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\alpha_j,$$

$$W_{17}^{3} = -\frac{1}{11520}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^4\left[1+22\left(\frac{\alpha_j}{\alpha_i}\right)^2-143\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{\alpha_i^4}{\alpha_j^2},$$

$$W_{17}^{4} = -\frac{1}{960}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^5\alpha_j\alpha_i^2, \quad W_{17}^{5} = \frac{1}{161280}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^6\left[1+13\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_i^6}{\alpha_j^2},$$

$$W_{17}^{6} = \frac{4}{2027025}(1+\frac{\alpha_j}{\alpha_i})^7[1-7\frac{\alpha_j}{\alpha_i}+28(\frac{\alpha_j}{\alpha_i})^2-\frac{71001}{1024}(\frac{\alpha_j}{\alpha_i})^3+\frac{109935}{1024}(\frac{\alpha_j}{\alpha_i})^4-\frac{53361}{512}(\frac{\alpha_j}{\alpha_i})^5$$

$$+\frac{31647}{512}(\frac{\alpha_j}{\alpha_i})^6-\frac{21021}{1024}(\frac{\alpha_j}{\alpha_i})^7+\frac{3003}{1024}(\frac{\alpha_j}{\alpha_i})^8] ,$$

$$W_{27}^{-10} = 1621620/(\alpha_i^{8}\alpha_j^{3}), \quad W_{27}^{-9} = 0, \quad W_{27}^{-8} = -62370[1+\frac{13}{3}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^6\alpha_j^3} ,$$

$$W_{27}^{-7} = 0, \quad W_{27}^{-6} = \frac{2835}{2}[1+\frac{22}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{3}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^4\alpha_j^3} ,$$

$$W_{27}^{-5} = 36036\alpha_j^2/\alpha_i^8, \quad W_{27}^{-4} = -\frac{105}{4}[1+9(\frac{\alpha_j}{\alpha_i})^2+99(\frac{\alpha_j}{\alpha_i})^4-429(\frac{\alpha_j}{\alpha_i})^6]\frac{1}{\alpha_i^2\alpha_j^3} ,$$

$$W_{27}^{-3} = -1386[1-\frac{13}{7}(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_j^2}{\alpha_i^6}$$

$$W_{27}^{-2} = \frac{15}{32}[1+\frac{28}{3}(\frac{\alpha_j}{\alpha_i})^2+126(\frac{\alpha_j}{\alpha_i})^4-924(\frac{\alpha_j}{\alpha_i})^6+1001(\frac{\alpha_j}{\alpha_i})^8]\frac{1}{\alpha_j^3},$$

$$W_{27}^{-1} = \frac{63}{2}[1-\frac{22}{7}(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{63}(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_j^2}{\alpha_i^4} ,$$

$$W_{27}^{0} = -\frac{3}{320}[1-(\frac{\alpha_j}{\alpha_i})^2][1+\frac{28}{3}(\frac{\alpha_j}{\alpha_i})^2+126(\frac{\alpha_j}{\alpha_i})^4-924(\frac{\alpha_j}{\alpha_i})^6+1001(\frac{\alpha_j}{\alpha_i})^8]\frac{\alpha_i^2}{\alpha_j^3} ,$$

$$W_{27}^{1} = -\frac{7}{12}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1-\frac{13}{7}(\frac{\alpha_j}{\alpha_i})^2](\frac{\alpha_j}{\alpha_i})^2,$$

$$W_{27}^{2} = \frac{1}{3840}[1-(\frac{\alpha_j}{\alpha_i})^2]^3[1+9(\frac{\alpha_j}{\alpha_i})^2+99(\frac{\alpha_j}{\alpha_i})^4-429(\frac{\alpha_j}{\alpha_i})^6]\frac{\alpha_i^4}{\alpha_j^3} ,$$

$$W_{27}^{3} = \frac{1}{96}[1-(\frac{\alpha_j}{\alpha_i})^2]^4\ \alpha_j^2,$$

$$W_{27}^{4} = -\frac{1}{53760}[1-(\frac{\alpha_j}{\alpha_i})^2]^5[1+\frac{22}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{3}(\frac{\alpha_j}{\alpha_i})^4]\frac{\alpha_i^6}{\alpha_j^3} ,$$

$$W_{27}^{5} = -\frac{4}{675675}(1+\frac{\alpha_j}{\alpha_i})^6[1-6\frac{\alpha_j}{\alpha_i}+21(\frac{\alpha_j}{\alpha_i})^2-56(\frac{\alpha_j}{\alpha_i})^3+126(\frac{\alpha_j}{\alpha_i})^4-\frac{55503}{256}(\frac{\alpha_j}{\alpha_i})^5$$

$$+\frac{32109}{128}(\frac{\alpha_j}{\alpha_i})^6-\frac{22869}{128}(\frac{\alpha_j}{\alpha_i})^7+\frac{9009}{128}(\frac{\alpha_j}{\alpha_i})^8-\frac{3003}{256}(\frac{\alpha_j}{\alpha_i})^9]\frac{\alpha_i^7}{\alpha_j^3} ,$$

$$W_{27}^{6} = -\frac{1}{2580480}[1-(\frac{\alpha_j}{\alpha_i})^2]^7[3+13(\frac{\alpha_j}{\alpha_i})^2]\frac{\alpha_i^8}{\alpha_j^3} ,$$

$$W_{37}^{-11} = -8108100/(\alpha_i^8\alpha_j^4), \quad W_{37}^{-10} = 0,$$

$$W_{37}^{-9} = 142560[1+\frac{13}{5}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^6\alpha_j^4} , \quad W_{37}^{-8} = 0,$$

$$W_{37}^{-7} = -\frac{14175}{2}[1+\frac{22}{5}(\frac{\alpha_j}{\alpha_i})^2+\frac{143}{15}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^4\alpha_j^4} , \quad W_{37}^{-6} = 0,$$

$$W_{37}^{-5} = \frac{525}{4}\left[1+\frac{27}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{99}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{429}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\frac{1}{\alpha_i^2\alpha_j^4}, \quad W_{37}^{-4} = 5148\alpha_j^3/\alpha_i^8$$

$$W_{37}^{-3} = -\frac{75}{32}\left[1+\frac{28}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{126}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{924}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^6-\frac{3003}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^8\right]\frac{1}{\alpha_j^4},$$

$$W_{37}^{-2} = -198\left[1-\frac{13}{9}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_j^3}{\alpha_i^6},$$

$$W_{37}^{-1} = \frac{3}{64}\left[1+5\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{70}{3}\left(\frac{\alpha_j}{\alpha_i}\right)^4+210\left(\frac{\alpha_j}{\alpha_i}\right)^6-1155\left(\frac{\alpha_j}{\alpha_i}\right)^8+1001\left(\frac{\alpha_j}{\alpha_i}\right)^{10}\right]\frac{\alpha_i^2}{\alpha_j^4},$$

$$W_{37}^{0} = \frac{9}{2}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\left[1-\frac{13}{9}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_j^3}{\alpha_i^4},$$

$$W_{37}^{1} = -\frac{1}{768}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^2\left[1+\frac{28}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{126}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{924}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^6-\frac{3003}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^8\right]\left(\frac{\alpha_i}{\alpha_j}\right)^4,$$

$$W_{37}^{2} = -\frac{1}{12}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^3\frac{\alpha_j^3}{\alpha_i^2},$$

$$W_{37}^{3} = \frac{1}{10752}\left(1-\frac{\alpha_j}{\alpha_i}\right)^4\left[1+4\frac{\alpha_j}{\alpha_i}+\frac{57}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{128}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^3+\frac{266}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{504}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^5+210\left(\frac{\alpha_j}{\alpha_i}\right)^6\right.$$
$$\left.+\frac{2112}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^7+\frac{2673}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^8+\frac{1716}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^9+\frac{429}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^{10}\right]\frac{\alpha_i^6}{\alpha_j^4},$$

$$W_{37}^{4} = \frac{1}{135135}\left(1+\frac{\alpha_j}{\alpha_i}\right)^5\left[1-5\frac{\alpha_j}{\alpha_i}+15\left(\frac{\alpha_j}{\alpha_i}\right)^2-35\left(\frac{\alpha_j}{\alpha_i}\right)^3+70\left(\frac{\alpha_j}{\alpha_i}\right)^4-126\left(\frac{\alpha_j}{\alpha_i}\right)^5+210\left(\frac{\alpha_j}{\alpha_i}\right)^6\right.$$
$$\left.-\frac{35805}{128}\left(\frac{\alpha_j}{\alpha_i}\right)^7+\frac{31185}{128}\left(\frac{\alpha_j}{\alpha_i}\right)^8-\frac{15015}{128}\left(\frac{\alpha_j}{\alpha_i}\right)^9+\frac{3003}{128}\left(\frac{\alpha_j}{\alpha_i}\right)^{10}\right]\frac{\alpha_i^7}{\alpha_j^4},$$

$$W_{37}^{5} = \frac{1}{172032}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^6\left[1+\frac{22}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{143}{15}\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{\alpha_i^8}{\alpha_j^4},$$

$$W_{37}^{6} = \frac{2}{2297295}\left(1+\frac{\alpha_j}{\alpha_i}\right)^7\left[1-7\frac{\alpha_j}{\alpha_i}+\frac{123}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^2-\frac{301}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^3+\frac{574}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^4-\frac{882}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^5+210\left(\frac{\alpha_j}{\alpha_i}\right)^6\right.$$
$$\left.-\frac{114807}{640}\left(\frac{\alpha_j}{\alpha_i}\right)^7+\frac{64449}{640}\left(\frac{\alpha_j}{\alpha_i}\right)^8-\frac{21021}{640}\left(\frac{\alpha_j}{\alpha_i}\right)^9+\frac{3003}{640}\left(\frac{\alpha_j}{\alpha_i}\right)^{10}\right]\frac{\alpha_i^9}{\alpha_j^4},$$

$$W_{47}^{-12} = 56756700/(\alpha_i^8\alpha_j^5), \quad W_{47}^{-11} = 0,$$

$$W_{47}^{-10} = -2182950\left[1+\frac{13}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{1}{\alpha_i^6\alpha_j^5}, \quad W_{47}^{-9} = 0,$$

$$W_{47}^{-8} = \frac{99225}{2}\left[1+\frac{22}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{143}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^3\right]\frac{1}{\alpha_i^4\alpha_j^5}, \quad W_{47}^{-7} = 0,$$

$$W_{47}^{-6} = -\frac{3675}{4}\left[1+\frac{27}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{297}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{429}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\frac{1}{\alpha_i^2\alpha_j^5}, \quad W_{37}^{-5} = 0,$$

$$W_{47}^{-4} = \frac{525}{32}\left[1+4\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{54}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{132}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^6+\frac{429}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^8\right]\frac{1}{\alpha_j^5}, \quad W_{47}^{-3} = -572\alpha_j^4/\alpha_i^8,$$

$$W_{47}^{-2} = -\frac{21}{64}\left[1+\frac{25}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2+10\left(\frac{\alpha_j}{\alpha_i}\right)^4+30\left(\frac{\alpha_j}{\alpha_i}\right)^6+165\left(\frac{\alpha_j}{\alpha_i}\right)^8-429\left(\frac{\alpha_j}{\alpha_i}\right)^{10}\right]\frac{\alpha_i^2}{\alpha_j^5},$$

$$W_{47}^{-1} = -22\left[1-\frac{13}{11}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_j^4}{\alpha_i^6},$$

$$W_{47}^{0} = \frac{7}{768}\left[1-\frac{18}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{45}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^4-20\left(\frac{\alpha_j}{\alpha_i}\right)^6+135\left(\frac{\alpha_j}{\alpha_i}\right)^8-594\left(\frac{\alpha_j}{\alpha_i}\right)^{10}+429\left(\frac{\alpha_j}{\alpha_i}\right)^{12}\right]\frac{\alpha_i^4}{\alpha_j^5},$$

$$W_{47}^{1} = \frac{1}{2}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^2\frac{\alpha_j^4}{\alpha_i^4}$$

$$W_{47}^{2} = -\frac{1}{1536}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^3\left[1+4\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{54}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{132}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^6+\frac{429}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^8\right]\frac{\alpha_i^6}{\alpha_j^5},$$

$$W_{47}^{3} = -\frac{4}{19305}\left(1+\frac{\alpha_j}{\alpha_i}\right)^4\left[1-4\frac{\alpha_j}{\alpha_i}+10\left(\frac{\alpha_j}{\alpha_i}\right)^2-20\left(\frac{\alpha_j}{\alpha_i}\right)^3+35\left(\frac{\alpha_j}{\alpha_i}\right)^4-56\left(\frac{\alpha_j}{\alpha_i}\right)^5+84\left(\frac{\alpha_j}{\alpha_i}\right)^6-120\left(\frac{\alpha_j}{\alpha_i}\right)^7\right.$$
$$\left.+165\left(\frac{\alpha_j}{\alpha_i}\right)^8-\frac{2805}{16}\left(\frac{\alpha_j}{\alpha_i}\right)^9+\frac{429}{4}\left(\frac{\alpha_j}{\alpha_i}\right)^{10}-\frac{429}{16}\left(\frac{\alpha_j}{\alpha_i}\right)^{11}\right]\frac{\alpha_i^7}{\alpha_j^5},$$

$$W_{47}^{4} = -\frac{1}{24576}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^5\left[1+\frac{27}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{297}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{429}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\frac{\alpha_i^8}{\alpha_j^5},$$

$$W_{47}^{5} = -\frac{2}{328185}\left(1+\frac{\alpha_j}{\alpha_i}\right)^6\left[1-6\frac{\alpha_j}{\alpha_i}+\frac{130}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2-\frac{290}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^3+75\left(\frac{\alpha_j}{\alpha_i}\right)^4-116\left(\frac{\alpha_j}{\alpha_i}\right)^5+156\left(\frac{\alpha_j}{\alpha_i}\right)^6\right.$$
$$\left.-180\left(\frac{\alpha_j}{\alpha_i}\right)^7+165\left(\frac{\alpha_j}{\alpha_i}\right)^8-\frac{6765}{64}\left(\frac{\alpha_j}{\alpha_i}\right)^9+\frac{1287}{32}\left(\frac{\alpha_j}{\alpha_i}\right)^{10}-\frac{429}{64}\left(\frac{\alpha_j}{\alpha_i}\right)^{11}\right]\frac{\alpha_i^9}{\alpha_j^5},$$

$$W_{47}^{6} = -\frac{1}{1327104}\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^7\left[1+\frac{22}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{143}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{\alpha_i^{10}}{\alpha_j^5},$$

$$W_{57}^{-13} = -510810300/(\alpha_i^8\alpha_j^6), \quad W_{57}^{-12} = 0,$$

$$W_{57}^{-11} = 19646550\left[1+\frac{13}{9}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{1}{\alpha_i^6\alpha_j^6}, \quad W_{57}^{-10} = 0,$$

$$W_{57}^{-9} = \frac{893025}{2}\left[1+\frac{22}{9}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{143}{63}\left(\frac{\alpha_j}{\alpha_i}\right)^4\right]\frac{1}{\alpha_i^4\alpha_j^6}, \quad W_{57}^{-8} = 0,$$

$$W_{57}^{-7} = \frac{33075}{4}\left[1+3\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{33}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{143}{35}\left(\frac{\alpha_j}{\alpha_i}\right)^6\right]\frac{1}{\alpha_i^2\alpha_j^6}, \quad W_{57}^{-6} = 0,$$

$$W_{57}^{-5} = -\frac{4725}{32}\left[1+\frac{28}{9}\left(\frac{\alpha_j}{\alpha_i}\right)^2+6\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{44}{5}\left(\frac{\alpha_j}{\alpha_i}\right)^6+\frac{143}{15}\left(\frac{\alpha_j}{\alpha_i}\right)^8\right]\frac{1}{\alpha_j^6}, \quad W_{57}^{-4} = 0,$$

$$W_{57}^{-3} = \frac{189}{64}\left[1+\frac{25}{9}\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{50}{9}\left(\frac{\alpha_j}{\alpha_i}\right)^4+10\left(\frac{\alpha_j}{\alpha_i}\right)^6+\frac{55}{3}\left(\frac{\alpha_j}{\alpha_i}\right)^8+\frac{143}{3}\left(\frac{\alpha_j}{\alpha_i}\right)^{10}\right]\frac{\alpha_i^2}{\alpha_j^6}, \quad W_{57}^{-2} = 52\frac{\alpha_j^5}{\alpha_i^8},$$

$$W_{57}^{-1} = -\frac{21}{256}\left[1+2\left(\frac{\alpha_j}{\alpha_i}\right)^2+\frac{25}{7}\left(\frac{\alpha_j}{\alpha_i}\right)^4+\frac{20}{3}\left(\frac{\alpha_j}{\alpha_i}\right)^6+15\left(\frac{\alpha_j}{\alpha_i}\right)^8+66\left(\frac{\alpha_j}{\alpha_i}\right)^{10}-143\left(\frac{\alpha_j}{\alpha_i}\right)^{12}\right]\frac{\alpha_i^4}{\alpha_j^6},$$

$$W_{57}^{0} = -2\left[1-\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]\frac{\alpha_j^5}{\alpha_i^6},$$

$$W_{57}^{1} = \frac{3}{512}[1-(\frac{\alpha_j}{\alpha_i})^2]^2[1+\frac{25}{9}(\frac{\alpha_j}{\alpha_i})^2+\frac{50}{9}(\frac{\alpha_j}{\alpha_i})^4+10(\frac{\alpha_j}{\alpha_i})^6+\frac{55}{3}(\frac{\alpha_j}{\alpha_i})^8+\frac{143}{3}(\frac{\alpha_j}{\alpha_i})^{10}](\frac{\alpha_i}{\alpha_j})^6,$$

$$W_{57}^{2} = \frac{4}{2145}(1+\frac{\alpha_j}{\alpha_i})^3[1-3\frac{\alpha_j}{\alpha_i}+6(\frac{\alpha_j}{\alpha_i})^2-10(\frac{\alpha_j}{\alpha_i})^3+15(\frac{\alpha_j}{\alpha_i})^4-21(\frac{\alpha_j}{\alpha_i})^5+28(\frac{\alpha_j}{\alpha_i})^6-36(\frac{\alpha_j}{\alpha_i})^7$$
$$+45(\frac{\alpha_j}{\alpha_i})^8-55(\frac{\alpha_j}{\alpha_i})^9+66(\frac{\alpha_j}{\alpha_i})^{10}-\frac{429}{8}(\frac{\alpha_j}{\alpha_i})^{11}+\frac{143}{8}(\frac{\alpha_j}{\alpha_i})^{12}]\frac{\alpha_i^7}{\alpha_j^6},$$

$$W_{57}^{3} = \frac{3}{8192}[1-(\frac{\alpha_j}{\alpha_i})^2]^4[1+\frac{28}{9}(\frac{\alpha_j}{\alpha_i})^2+6(\frac{\alpha_j}{\alpha_i})^4+\frac{44}{5}(\frac{\alpha_j}{\alpha_i})^6+\frac{143}{15}(\frac{\alpha_j}{\alpha_i})^8]\frac{\alpha_i^8}{\alpha_j^6},$$

$$W_{57}^{4} = \frac{2}{36465}(1+\frac{\alpha_j}{\alpha_i})^5[1-5\frac{\alpha_j}{\alpha_i}+\frac{118}{9}(\frac{\alpha_j}{\alpha_i})^2-\frac{230}{9}(\frac{\alpha_j}{\alpha_i})^3+\frac{125}{3}(\frac{\alpha_j}{\alpha_i})^4-\frac{539}{9}(\frac{\alpha_j}{\alpha_i})^5+\frac{700}{9}(\frac{\alpha_j}{\alpha_i})^6$$
$$-92(\frac{\alpha_j}{\alpha_i})^7+\frac{295}{3}(\frac{\alpha_j}{\alpha_i})^8-\frac{275}{3}(\frac{\alpha_j}{\alpha_i})^9+66(\frac{\alpha_j}{\alpha_i})^{10}-\frac{715}{24}(\frac{\alpha_j}{\alpha_i})^{11}+\frac{143}{24}(\frac{\alpha_j}{\alpha_i})^{12}]\frac{\alpha_i^9}{\alpha_j^6},$$

$$W_{57}^{5} = \frac{1}{147456}[1-(\frac{\alpha_j}{\alpha_i})^2]^6[1+3(\frac{\alpha_j}{\alpha_i})^2+\frac{33}{7}(\frac{\alpha_j}{\alpha_i})^4+\frac{143}{35}(\frac{\alpha_j}{\alpha_i})^6]\frac{\alpha_i^{10}}{\alpha_j^6},$$

$$W_{57}^{6} = \frac{1}{1385670}(1+\frac{\alpha_j}{\alpha_i})^7[1-7\frac{\alpha_j}{\alpha_i}+\frac{214}{9}(\frac{\alpha_j}{\alpha_i})^2-\frac{490}{9}(\frac{\alpha_j}{\alpha_i})^3+\frac{2035}{21}(\frac{\alpha_j}{\alpha_i})^4-\frac{1289}{9}(\frac{\alpha_j}{\alpha_i})^5+\frac{1628}{9}(\frac{\alpha_j}{\alpha_i})^6$$
$$-196(\frac{\alpha_j}{\alpha_i})^7+\frac{535}{3}(\frac{\alpha_j}{\alpha_i})^8-\frac{385}{3}(\frac{\alpha_j}{\alpha_i})^9+66(\frac{\alpha_j}{\alpha_i})^{10}-\frac{1001}{48}(\frac{\alpha_j}{\alpha_i})^{11}+\frac{1001}{336}(\frac{\alpha_j}{\alpha_i})^{12}]\frac{\alpha_i^{11}}{\alpha_j^6},$$

$$W_{67}^{-14} = 5618913300/(\alpha_i^8\alpha_j^7), \quad W_{67}^{-13} = 0,$$

$$W_{67}^{-12} = -216112050[1+\frac{13}{11}(\frac{\alpha_j}{\alpha_i})^2]\frac{1}{\alpha_i^6\alpha_j^7}, \quad W_{67}^{-11} = 0,$$

$$W_{67}^{-10} = \frac{9823275}{2}[1+2(\frac{\alpha_j}{\alpha_i})^2+\frac{13}{9}(\frac{\alpha_j}{\alpha_i})^4]\frac{1}{\alpha_i^4\alpha_j^7}, \quad W_{67}^{-9} = 0,$$

$$W_{67}^{-8} = -\frac{363825}{4}[1+\frac{27}{11}(\frac{\alpha_j}{\alpha_i})^2+3(\frac{\alpha_j}{\alpha_i})^4+\frac{143}{77}(\frac{\alpha_j}{\alpha_i})^6]\frac{1}{\alpha_i^2\alpha_j^7}, \quad W_{67}^{-7} = 0,$$

$$W_{67}^{-6} = \frac{51975}{32}[1+\frac{28}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{42}{11}(\frac{\alpha_j}{\alpha_i})^4+4(\frac{\alpha_j}{\alpha_i})^6+\frac{13}{5}(\frac{\alpha_j}{\alpha_i})^8]\frac{1}{\alpha_j^7}, \quad W_{67}^{-5} = 0,$$

$$W_{67}^{-4} = -\frac{2079}{64}[1+\frac{25}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{350}{99}(\frac{\alpha_j}{\alpha_i})^4+\frac{50}{11}(\frac{\alpha_j}{\alpha_i})^6+5(\frac{\alpha_j}{\alpha_i})^8+\frac{13}{3}(\frac{\alpha_j}{\alpha_i})^{10}]\frac{\alpha_i^2}{\alpha_j^7}, \quad W_{67}^{-3} = 0,$$

$$W_{67}^{-2} = \frac{231}{256}[1+\frac{18}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{25}{11}(\frac{\alpha_j}{\alpha_i})^4+\frac{100}{33}(\frac{\alpha_j}{\alpha_i})^6+\frac{45}{11}(\frac{\alpha_j}{\alpha_i})^8+6(\frac{\alpha_j}{\alpha_i})^{10}+13(\frac{\alpha_j}{\alpha_i})^{12}]\frac{\alpha_i^4}{\alpha_j^7},$$

$$W_{67}^{-1} = 4\alpha_j^6/\alpha_i^8,$$

$$W_{67}^{0} = -\frac{33}{512}[1-(\frac{\alpha_j}{\alpha_i})^2][1+\frac{18}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{25}{11}(\frac{\alpha_j}{\alpha_i})^4+\frac{100}{33}(\frac{\alpha_j}{\alpha_i})^6+\frac{45}{11}(\frac{\alpha_j}{\alpha_i})^8+6(\frac{\alpha_j}{\alpha_i})^{10}+13(\frac{\alpha_j}{\alpha_i})^{12}]\frac{\alpha_i^6}{\alpha_j^7},$$

$$W_{67}^{1} = -\frac{4}{195}(1+\frac{\alpha_j}{\alpha_i})^2[1-2\frac{\alpha_j}{\alpha_i}+3(\frac{\alpha_j}{\alpha_i})^2-4(\frac{\alpha_j}{\alpha_i})^3+5(\frac{\alpha_j}{\alpha_i})^4-6(\frac{\alpha_j}{\alpha_i})^5+7(\frac{\alpha_j}{\alpha_i})^6-8(\frac{\alpha_j}{\alpha_i})^7+9(\frac{\alpha_j}{\alpha_i})^8$$

$$-10(\frac{\alpha_j}{\alpha_i})^9+11(\frac{\alpha_j}{\alpha_i})^{10}-12(\frac{\alpha_j}{\alpha_i})^{11}+13(\frac{\alpha_j}{\alpha_i})^{12}-\frac{13}{2}(\frac{\alpha_j}{\alpha_i})^{13}](\frac{\alpha_i}{\alpha_j})^7,$$

$$W_{67}^{2} = -\frac{33}{8192}[1-(\frac{\alpha_j}{\alpha_i})^2]^3[1+\frac{25}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{350}{99}(\frac{\alpha_j}{\alpha_i})^4+\frac{50}{11}(\frac{\alpha_j}{\alpha_i})^6+5(\frac{\alpha_j}{\alpha_i})^8+\frac{13}{3}(\frac{\alpha_j}{\alpha_i})^{10}]\ \frac{\alpha_i^{9}}{\alpha_j^{7}},$$

$$W_{67}^{3} = -\frac{2}{3315}(1+\frac{\alpha_j}{\alpha_i})^4[1-4\frac{\alpha_j}{\alpha_i}+\frac{93}{11}(\frac{\alpha_j}{\alpha_i})^2-\frac{152}{11}(\frac{\alpha_j}{\alpha_i})^3+\frac{215}{11}(\frac{\alpha_j}{\alpha_i})^4-\frac{276}{11}(\frac{\alpha_j}{\alpha_i})^5+\frac{329}{11}(\frac{\alpha_j}{\alpha_i})^6$$

$$-\frac{368}{11}(\frac{\alpha_j}{\alpha_i})^7+\frac{387}{11}(\frac{\alpha_j}{\alpha_i})^8-\frac{380}{11}(\frac{\alpha_j}{\alpha_i})^9+31(\frac{\alpha_j}{\alpha_i})^{10}-24(\frac{\alpha_j}{\alpha_i})^{11}+13(\frac{\alpha_j}{\alpha_i})^{12}-\frac{13}{4}(\frac{\alpha_j}{\alpha_i})^{13}]\ \frac{\alpha_i^{9}}{\alpha_j^{7}}$$

$$W_{67}^{4} = -\frac{11}{147456}[1-(\frac{\alpha_j}{\alpha_i})^2]^5[1+\frac{28}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{42}{11}(\frac{\alpha_j}{\alpha_i})^4+4(\frac{\alpha_j}{\alpha_i})^6+\frac{13}{5}(\frac{\alpha_j}{\alpha_i})^8]\ \frac{\alpha_i^{10}}{\alpha_j^{7}},$$

$$W_{67}^{5} = -\frac{1}{125970}(1+\frac{\alpha_j}{\alpha_i})^6[1-6\frac{\alpha_j}{\alpha_i}+\frac{193}{11}(\frac{\alpha_j}{\alpha_i})^2-\frac{388}{11}(\frac{\alpha_j}{\alpha_i})^3+\frac{5615}{99}(\frac{\alpha_j}{\alpha_i})^4-\frac{2578}{33}(\frac{\alpha_j}{\alpha_i})^5+\frac{3143}{33}(\frac{\alpha_j}{\alpha_i})^6$$

$$-\frac{10312}{2}(\frac{\alpha_j}{\alpha_i})^7+\frac{1123}{11}(\frac{\alpha_j}{\alpha_i})^8-\frac{970}{11}(\frac{\alpha_j}{\alpha_i})^9+\frac{193}{3}(\frac{\alpha_j}{\alpha_i})^{10}$$

$$-36(\frac{\alpha_j}{\alpha_i})^{11}+13(\frac{\alpha_j}{\alpha_i})^{13}-\frac{13}{6}(\frac{\alpha_j}{\alpha_i})^{14}]\ \frac{\alpha_i^{11}}{\alpha_j^{7}},$$

$$W_{67}^{6} = -\frac{11}{14745600}[1-(\frac{\alpha_j}{\alpha_i})^2]^7[1+\frac{18}{7}(\frac{\alpha_j}{\alpha_i})^2+3(\frac{\alpha_j}{\alpha_i})^4+\frac{13}{7}(\frac{\alpha_j}{\alpha_i})^6]\ \frac{\alpha_i^{12}}{\alpha_j^{7}}$$

$$W_{77}^{-15} = -73045872864/(\alpha_i^{8}\alpha_j^{8}),\quad W_{77}^{-14} = 0,$$

$$W_{77}^{-13} = 2809456650[1+(\frac{\alpha_j}{\alpha_i})^2]\ \frac{1}{\alpha_i^{6}\alpha_j^{8}},\quad W_{77}^{-12} = 0,$$

$$W_{77}^{-11} = -\frac{127702575}{2}[1+\frac{22}{13}(\frac{\alpha_j}{\alpha_i})^2+(\frac{\alpha_j}{\alpha_i})^4]\ \frac{1}{\alpha_i^{4}\alpha_j^{8}},\quad W_{77}^{-10} = 0,$$

$$W_{77}^{-9} = \frac{4729725}{4}[1+\frac{27}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{27}{13}(\frac{\alpha_j}{\alpha_i})^4+(\frac{\alpha_j}{\alpha_i})^6]\ \frac{1}{\alpha_i^{2}\alpha_j^{8}},\quad W_{77}^{-8} = 0,$$

$$W_{77}^{-7} = -\frac{675675}{32}[1+\frac{28}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{378}{143}(\frac{\alpha_j}{\alpha_i})^4+\frac{28}{13}(\frac{\alpha_j}{\alpha_i})^6+(\frac{\alpha_j}{\alpha_i})^8]\ \frac{1}{\alpha_j^{8}},\quad W_{77}^{-6} = 0,$$

$$W_{77}^{-5} = \frac{27027}{64}[1+\frac{25}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{350}{143}(\frac{\alpha_j}{\alpha_i})^4+\frac{350}{143}(\frac{\alpha_j}{\alpha_i})^6+\frac{25}{13}(\frac{\alpha_j}{\alpha_i})^8+(\frac{\alpha_j}{\alpha_i})^{10}]\ \frac{\alpha_i^{2}}{\alpha_j^{8}},\quad W_{77}^{-4} = 0,$$

$$W_{77}^{-3} = -\frac{3003}{256}[1+\frac{18}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{225}{143}(\frac{\alpha_j}{\alpha_i})^4+\frac{700}{429}(\frac{\alpha_j}{\alpha_i})^6+\frac{225}{143}(\frac{\alpha_j}{\alpha_i})^8+\frac{18}{13}(\frac{\alpha_j}{\alpha_i})^{10}+(\frac{\alpha_j}{\alpha_i})^{12}]\ \frac{\alpha_i^{4}}{\alpha_j^{8}},$$

$$W_{77}^{-2} = 0,$$

$$W_{77}^{-1} = \frac{429}{512}[1+\frac{7}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{63}{143}(\frac{\alpha_j}{\alpha_i})^4+\frac{175}{429}(\frac{\alpha_j}{\alpha_i})^6+\frac{175}{429}(\frac{\alpha_j}{\alpha_i})^8+\frac{63}{143}(\frac{\alpha_j}{\alpha_i})^{10}+\frac{7}{13}(\frac{\alpha_j}{\alpha_i})^{12}+(\frac{\alpha_j}{\alpha_i})^{14}]\frac{\alpha_i^{6}}{\alpha_j^{8}},$$

$$W_{77}^{0} = \frac{4}{15}(1+\frac{\alpha_j}{\alpha_i})[1-\frac{\alpha_j}{\alpha_i}+(\frac{\alpha_j}{\alpha_i})^2-(\frac{\alpha_j}{\alpha_i})^3+(\frac{\alpha_j}{\alpha_i})^4-(\frac{\alpha_j}{\alpha_i})^5+(\frac{\alpha_j}{\alpha_i})^6-(\frac{\alpha_j}{\alpha_i})^7+(\frac{\alpha_j}{\alpha_i})^8-(\frac{\alpha_j}{\alpha_i})^9+(\frac{\alpha_j}{\alpha_i})^{10}$$

$$-(\frac{\alpha_j}{\alpha_i})^{11}+(\frac{\alpha_j}{\alpha_i})^{12}-(\frac{\alpha_j}{\alpha_i})^{13}+(\frac{\alpha_j}{\alpha_i})^{14}]\ \frac{\alpha_i^{7}}{\alpha_j^{8}}\ ,$$

$$W_{77}^{1} = \frac{429}{8192}[1-(\frac{\alpha_j}{\alpha_i})^{2}]^{2}[1+\frac{18}{13}(\frac{\alpha_j}{\alpha_i})^{2}+\frac{225}{143}(\frac{\alpha_j}{\alpha_i})^{4}+\frac{700}{429}(\frac{\alpha_j}{\alpha_i})^{6}+\frac{225}{143}(\frac{\alpha_j}{\alpha_i})^{8}+\frac{18}{13}(\frac{\alpha_j}{\alpha_i})^{10}+(\frac{\alpha_j}{\alpha_i})^{12}]$$

$$\times(\frac{\alpha_i}{\alpha_j})^{8}\ ,$$

$$W_{77}^{2} = \frac{26}{3315}(1+\frac{\alpha_j}{\alpha_i})^{3}[1-3\frac{\alpha_j}{\alpha_i}+\frac{61}{2}(\frac{\alpha_j}{\alpha_i})^{2}-\frac{79}{2}(\frac{\alpha_j}{\alpha_i})^{3}+\frac{93}{2}(\frac{\alpha_j}{\alpha_i})^{4}-\frac{103}{2}(\frac{\alpha_j}{\alpha_i})^{5}+\frac{109}{2}(\frac{\alpha_j}{\alpha_i})^{6}-\frac{111}{2}(\frac{\alpha_j}{\alpha_i})^{7}$$

$$+\frac{109}{2}(\frac{\alpha_j}{\alpha_i})^{8}-\frac{103}{2}(\frac{\alpha_j}{\alpha_i})^{9}+\frac{93}{2}(\frac{\alpha_j}{\alpha_i})^{10}-\frac{79}{2}(\frac{\alpha_j}{\alpha_i})^{11}+\frac{61}{2}(\frac{\alpha_j}{\alpha_i})^{12}-3(\frac{\alpha_j}{\alpha_i})^{13}+(\frac{\alpha_j}{\alpha_i})^{14}]\ \frac{\alpha_i^{9}}{\alpha_j^{8}}\ ,$$

$$W_{77}^{3} = \frac{143}{147456}[1-(\frac{\alpha_j}{\alpha_i})^{2}]^{4}[1+\frac{25}{13}(\frac{\alpha_j}{\alpha_i})^{2}+\frac{350}{143}(\frac{\alpha_j}{\alpha_i})^{4}+\frac{350}{143}(\frac{\alpha_j}{\alpha_i})^{6}+\frac{25}{13}(\frac{\alpha_j}{\alpha_i})^{8}+(\frac{\alpha_j}{\alpha_i})^{10}]\ \frac{\alpha_i^{10}}{\alpha_j^{8}},$$

$$W_{77}^{4} = \frac{1}{9690}(1+\frac{\alpha_j}{\alpha_i})^{5}[1-5\frac{\alpha_j}{\alpha_i}+\frac{157}{13}(\frac{\alpha_j}{\alpha_i})^{2}-\frac{265}{13}(\frac{\alpha_j}{\alpha_i})^{3}+\frac{4063}{143}(\frac{\alpha_j}{\alpha_i})^{4}-\frac{5003}{143}(\frac{\alpha_j}{\alpha_i})^{5}+\frac{5615}{143}(\frac{\alpha_j}{\alpha_i})^{6}$$

$$-\frac{5827}{143}(\frac{\alpha_j}{\alpha_i})^{7}+\frac{5615}{143}(\frac{\alpha_j}{\alpha_i})^{8}-\frac{5003}{143}(\frac{\alpha_j}{\alpha_i})^{9}+\frac{4063}{143}(\frac{\alpha_j}{\alpha_i})^{10}-\frac{265}{13}(\frac{\alpha_j}{\alpha_i})^{11}+\frac{157}{13}(\frac{\alpha_j}{\alpha_i})^{12}$$

$$-5(\frac{\alpha_j}{\alpha_i})^{13}+(\frac{\alpha_j}{\alpha_i})^{14}]\ \frac{\alpha_i^{11}}{\alpha_j^{8}},$$

$$W_{77}^{5} = \frac{143}{14745600}[1-(\frac{\alpha_j}{\alpha_i})^{2}]^{6}[1+\frac{28}{13}(\frac{\alpha_j}{\alpha_i})^{2}+\frac{378}{143}(\frac{\alpha_j}{\alpha_i})^{4}+\frac{28}{13}(\frac{\alpha_j}{\alpha_i})^{6}+(\frac{\alpha_j}{\alpha_i})^{8}]\ \frac{\alpha_i^{12}}{\alpha_j^{8}}\ ,$$

$$W_{77}^{6} = \frac{1}{1220940}(1+\frac{\alpha_j}{\alpha_i})^{7}[1-7\frac{\alpha_j}{\alpha_i}+\frac{301}{13}(\frac{\alpha_j}{\alpha_i})^{2}-\frac{651}{13}(\frac{\alpha_j}{\alpha_i})^{3}+\frac{11823}{143}(\frac{\alpha_j}{\alpha_i})^{4}-\frac{16233}{143}(\frac{\alpha_j}{\alpha_i})^{5}$$

$$+\frac{58093}{429}(\frac{\alpha_j}{\alpha_i})^{6}-\frac{61483}{429}(\frac{\alpha_j}{\alpha_i})^{7}+\frac{58093}{429}(\frac{\alpha_j}{\alpha_i})^{8}-\frac{16233}{143}(\frac{\alpha_j}{\alpha_i})^{9}+\frac{11823}{143}(\frac{\alpha_j}{\alpha_i})^{10}-\frac{651}{13}(\frac{\alpha_j}{\alpha_i})^{11}$$

$$+\frac{301}{13}(\frac{\alpha_j}{\alpha_i})^{12}-7(\frac{\alpha_j}{\alpha_i})^{13}+(\frac{\alpha_j}{\alpha_i})^{14}]\ \frac{\alpha_i^{13}}{\alpha_j^{8}}\ .$$

For the parameter range (c), the first term on the right hand side of Eq. (A.26) gives the following expressions together with the contribution of Y_2 terms written by Eq. (A.22) divided by $\alpha_j\alpha_i$ [the expressions (A.23) ∿ (A.25) divided by $\alpha_j\alpha_i$]:

$$\sum_{m=0}^{[(L-1)/2]}[\frac{1}{2m+1}U_{qr}^{L-2m}+\frac{1}{m+1}V_{qr}^{L-2m}], \qquad \text{(A.39)}$$

$$U_{qr}^{L} = \frac{\alpha_i^{q+L}}{\alpha_j^{q+1}}\sum_{m=0}^{[(q-r+L)/2]}W_{qr}^{Lm}(\frac{d}{\alpha_i})^{2m}\ ,$$

$$V_{qr}^{L} = \alpha_j^{q}\alpha_i^{L-q-2}\sum_{m=0}^{[(L-q-r-2)/2]}Z_{qr}^{Lm}(\frac{d}{\alpha_i})^{2m}.$$

Since

$$W_{qr}^{Lm} = \frac{1}{m(4m-2)} W_{qr}^{L-2,m-1}, \quad Z_{qr}^{Lm} = \frac{1}{m(4m-2)} Z_{qr}^{L-2,m-1},$$

$$W_{qr}^{2n+1-\delta,m} = \frac{1}{2m+1}\frac{d}{\alpha_i} W_{qr}^{2n-\delta,m}, \quad m=0\sim n-[\frac{r-q}{2}]-\delta \text{ for } n=1-[\frac{r-q}{3}]\geq 1\sim 2+\delta,$$

$$Z_{qr}^{2n+1-\delta,m} = \frac{1}{2m+1}\frac{d}{\alpha_i} Z_{qr}^{2n-\delta,m}, \quad m=0\sim n-1-[\frac{q+r}{2}]-\delta \text{ for } n=1+[\frac{q+r}{2}]+\delta\sim 2+\delta,$$

$$\delta=(q+r)-2[(q+r)/2], \qquad (A.40)$$

only the concrete expressions for W_{qr}^{L0} and Z_{qr}^{L0} (and W_{qr}^{L1} for some cases) are necessary to be written down as follows:

$$W_{00}^{10} = -8\frac{d}{\alpha_i}, \quad W_{00}^{20} = -\frac{4}{3}[1+3(\frac{\alpha_j}{\alpha_i})^2], \quad W_{00}^{21} = -4,$$

$$W_{00}^{40} = -\frac{1}{15}[1+10(\frac{\alpha_j}{\alpha_i})^2+5(\frac{\alpha_j}{\alpha_i})^4], \quad W_{00}^{60} = -\frac{1}{630}[1+21(\frac{\alpha_j}{\alpha_i})^2+35(\frac{\alpha_j}{\alpha_i})^4+7(\frac{\alpha_j}{\alpha_i})^6],$$

$$Z_{00}^{20} = 4, \quad Z_{00}^{40} = \frac{2}{3}[1+(\frac{\alpha_j}{\alpha_i})^2], \quad Z_{00}^{60} = \frac{1}{30}[1+\frac{10}{3}(\frac{\alpha_j}{\alpha_i})^2+(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{01}^{10} = -\frac{8}{3}, \quad W_{01}^{30} = -\frac{4}{15}[1+5(\frac{\alpha_j}{\alpha_i})^2], \quad W_{01}^{50} = -\frac{1}{105}[1+14(\frac{\alpha_j}{\alpha_i})^2+\frac{35}{3}(\frac{\alpha_j}{\alpha_i})^4],$$

$$Z_{01}^{30} = \frac{4}{3}, \quad Z_{01}^{50} = \frac{2}{15}[1+\frac{5}{3}(\frac{\alpha_j}{\alpha_i})^2],$$

$$W_{02}^{20} = -\frac{8}{15}, \quad W_{02}^{40} = -\frac{4}{105}[1+7(\frac{\alpha_j}{\alpha_i})^2], \quad W_{02}^{60} = -\frac{1}{945}[1+18(\frac{\alpha_j}{\alpha_i})^2+21(\frac{\alpha_j}{\alpha_i})^4],$$

$$Z_{02}^{40} = \frac{4}{15}, \quad Z_{02}^{60} = \frac{2}{105}[1+\frac{7}{3}(\frac{\alpha_j}{\alpha_i})^2],$$

$$W_{03}^{30} = -\frac{8}{105}, \quad W_{03}^{50} = -\frac{4}{945}[1+9(\frac{\alpha_j}{\alpha_i})^2], \quad Z_{03}^{50} = \frac{4}{105}$$

$$W_{04}^{40} = -\frac{8}{945}, \quad W_{04}^{60} = -\frac{4}{10395}[1+11(\frac{\alpha_j}{\alpha_i})^2], \quad Z_{04}^{60} = \frac{4}{945}$$

$$W_{05}^{50} = -\frac{8}{10395},$$

$$W_{06}^{60} = -\frac{8}{135135},$$

$$W_{10}^{10} = \frac{4}{3}[1-3(\frac{\alpha_j}{\alpha_i})^2], \quad W_{10}^{11} = 4, \quad W_{10}^{30} = \frac{1}{15}[1-10(\frac{\alpha_j}{\alpha_i})^2-15(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{10}^{50} = \frac{1}{630}[1-21(\frac{\alpha_j}{\alpha_i})^2-105(\frac{\alpha_j}{\alpha_i})^4-35(\frac{\alpha_j}{\alpha_i})^6],$$

$Z_{10}^{30} = \frac{4}{3}$, $Z_{10}^{50} = \frac{2}{9}[1+ \frac{3}{5}(\frac{\alpha_j}{\alpha_i})^2]$,

$W_{11}^{10} = \frac{8}{3}\frac{d}{\alpha_i}$, $W_{11}^{20} = \frac{4}{15}[1-5(\frac{\alpha_j}{\alpha_i})^2]$, $W_{11}^{21} = \frac{4}{3}$,

$W_{11}^{40} = \frac{1}{105}[1-14(\frac{\alpha_j}{\alpha_i})^2-35(\frac{\alpha_j}{\alpha_i})^4]$, $W_{11}^{60} = \frac{1}{5670}[1-27(\frac{\alpha_j}{\alpha_i})^2-189(\frac{\alpha_j}{\alpha_i})^4-105(\frac{\alpha_j}{\alpha_i})^6]$,

$Z_{11}^{40} = \frac{4}{9}$, $Z_{11}^{60} = \frac{2}{45}[1+(\frac{\alpha_j}{\alpha_i})^2]$,

$W_{12}^{10} = \frac{8}{15}$, $W_{12}^{30} = \frac{4}{105}[1-7(\frac{\alpha_j}{\alpha_i})^2]$, $W_{12}^{50} = \frac{1}{945}[1-18(\frac{\alpha_j}{\alpha_i})^2-63(\frac{\alpha_j}{\alpha_i})^4]$,

$Z_{12}^{50} = \frac{4}{45}$,

$W_{13}^{20} = \frac{8}{105}$, $W_{13}^{40} = \frac{4}{945}[1-9(\frac{\alpha_j}{\alpha_i})^2]$, $W_{13}^{60} = \frac{1}{10395}[1-22(\frac{\alpha_j}{\alpha_i})^2-99(\frac{\alpha_j}{\alpha_i})^4]$,

$Z_{13}^{60} = \frac{4}{315}$,

$W_{14}^{30} = \frac{8}{945}$, $W_{14}^{50} = \frac{4}{10395}[1-11(\frac{\alpha_j}{\alpha_i})^2]$,

$W_{15}^{40} = \frac{8}{10395}$, $W_{15}^{50} = \frac{4}{135135}[1-13(\frac{\alpha_j}{\alpha_i})^2]$,

$W_{16}^{50} = \frac{8}{135135}$,

$W_{17}^{60} = \frac{8}{2027025}$,

$W_{20}^{10} = -4[1-(\frac{\alpha_j}{\alpha_i})^2]\frac{d}{\alpha_i}$, $W_{20}^{11} = -4\frac{d}{\alpha_i}$

$W_{20}^{20} = -\frac{1}{5}[1-\frac{10}{3}(\frac{\alpha_j}{\alpha_i})^2+5(\frac{\alpha_j}{\alpha_i})^4]$, $W_{20}^{21} = -2[1-(\frac{\alpha_j}{\alpha_i})^2]$, $W_{20}^{22} = -1$,

$W_{20}^{30} = -\frac{1}{210}[1-7(\frac{\alpha_j}{\alpha_i})^2+35(\frac{\alpha_j}{\alpha_i})^4+35(\frac{\alpha_j}{\alpha_i})^6]$,

$W_{20}^{40} = -\frac{1}{15120}[1-12(\frac{\alpha_j}{\alpha_i})^2+126(\frac{\alpha_j}{\alpha_i})^4+420(\frac{\alpha_j}{\alpha_i})^6+105(\frac{\alpha_j}{\alpha_i})^8]$,

$Z_{20}^{40} = \frac{4}{15}$, $Z_{20}^{60} = \frac{2}{45}[1+\frac{3}{7}(\frac{\alpha_j}{\alpha_i})^2]$,

$W_{30}^{10} = [1-(\frac{\alpha_j}{\alpha_i})^2]^2$, $W_{30}^{11} = 10[1-\frac{3}{5}(\frac{\alpha_j}{\alpha_i})^2]$, $W_{30}^{12} = 5$,

$W_{30}^{30} = \frac{1}{42}[1-\frac{21}{5}(\frac{\alpha_j}{\alpha_i})^2+7(\frac{\alpha_j}{\alpha_i})^4-7(\frac{\alpha_j}{\alpha_i})^6]$,

$$W_{30}^{50} = \frac{1}{3024}[1-\frac{36}{5}(\frac{\alpha_j}{\alpha_i})^2+\frac{126}{5}(\frac{\alpha_j}{\alpha_i})^4-84(\frac{\alpha_j}{\alpha_i})^6-63(\frac{\alpha_j}{\alpha_i})^8],$$

$$Z_{30}^{50} = \frac{4}{105},$$

$$W_{40}^{10} = -7[1-\frac{10}{7}(\frac{\alpha_j}{\alpha_i})^2+\frac{3}{7}(\frac{\alpha_j}{\alpha_i})^4]\frac{d}{\alpha_i}, \quad W_{40}^{11} = -\frac{70}{3}[1-\frac{3}{7}(\frac{\alpha_j}{\alpha_i})^2]\frac{d}{\alpha_i}, \quad W_{40}^{12} = -7\frac{d}{\alpha_i},$$

$$W_{40}^{20} = -\frac{1}{6}[1-(\frac{\alpha_j}{\alpha_i})^2]^3, \quad W_{40}^{21} = -\frac{7}{2}[1-\frac{10}{7}(\frac{\alpha_j}{\alpha_i})^2+\frac{3}{7}(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{40}^{22} = -\frac{35}{6}[1-\frac{3}{7}(\frac{\alpha_j}{\alpha_i})^2], \quad W_{40}^{23} = -\frac{7}{6},$$

$$W_{40}^{40} = -\frac{1}{432}[1-\frac{36}{7}(\frac{\alpha_j}{\alpha_i})^2+\frac{54}{5}(\frac{\alpha_j}{\alpha_i})^4-12(\frac{\alpha_j}{\alpha_i})^6+9(\frac{\alpha_j}{\alpha_i})^8],$$

$$W_{40}^{60} = -\frac{1}{47520}[1-\frac{55}{7}(\frac{\alpha_j}{\alpha_i})^2+\frac{198}{7}(\frac{\alpha_j}{\alpha_i})^4-66(\frac{\alpha_j}{\alpha_i})^6+165(\frac{\alpha_j}{\alpha_i})^8+99(\frac{\alpha_j}{\alpha_i})^{10}],$$

$$Z_{40}^{60} = \frac{4}{945},$$

$$W_{50}^{10} = \frac{3}{2}[1-\frac{7}{3}(\frac{\alpha_j}{\alpha_i})^2+\frac{5}{3}(\frac{\alpha_j}{\alpha_i})^4-\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^6], \quad W_{50}^{11} = \frac{63}{2}[1-\frac{10}{9}(\frac{\alpha_j}{\alpha_i})^2+\frac{5}{21}(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{50}^{12} = \frac{105}{2}[1-\frac{1}{3}(\frac{\alpha_j}{\alpha_i})^2], \quad W_{50}^{13} = \frac{21}{2},$$

$$W_{50}^{30} = \frac{1}{48}[1-(\frac{\alpha_j}{\alpha_i})^2]^4, \quad W_{50}^{50} = \frac{1}{5280}[1-\frac{55}{9}(\frac{\alpha_j}{\alpha_i})^2+\frac{110}{7}(\frac{\alpha_j}{\alpha_i})^4-22(\frac{\alpha_j}{\alpha_i})^6+\frac{55}{3}(\frac{\alpha_j}{\alpha_i})^8],$$

$$W_{60}^{10} = -\frac{33}{2}(1-\frac{21}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{35}{33}(\frac{\alpha_j}{\alpha_i})^4-\frac{5}{33}(\frac{\alpha_j}{\alpha_i})^6]\frac{d}{\alpha_i}, \quad W_{60}^{11} = -\frac{231}{2}[1-\frac{10}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{5}{33}(\frac{\alpha_j}{\alpha_i})^4]\frac{d}{\alpha_i},$$

$$W_{60}^{12} = -\frac{231}{2}[1-\frac{3}{11}(\frac{\alpha_j}{\alpha_i})^2]\frac{d}{\alpha_i}, \quad W_{60}^{13} = -\frac{33}{2}\frac{d}{\alpha_i},$$

$$W_{60}^{20} = -\frac{11}{48}[1-\frac{36}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{42}{11}(\frac{\alpha_j}{\alpha_i})^4-\frac{20}{11}(\frac{\alpha_j}{\alpha_i})^6+\frac{3}{11}(\frac{\alpha_j}{\alpha_i})^8],$$

$$W_{60}^{21} = -\frac{33}{4}[1-\frac{21}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{35}{33}(\frac{\alpha_j}{\alpha_i})^4-\frac{5}{33}(\frac{\alpha_j}{\alpha_i})^6], \quad W_{60}^{22} = -\frac{231}{8}[1-\frac{10}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{5}{33}(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{60}^{23} = -\frac{77}{4}[1-\frac{3}{11}(\frac{\alpha_j}{\alpha_i})^2], \quad W_{60}^{24} = -\frac{33}{16}$$

$$W_{60}^{40} = -\frac{1}{480}[1-(\frac{\alpha_j}{\alpha_i})^2]^5,$$

$$W_{60}^{60} = -\frac{1}{74880}[1-\frac{78}{11}(\frac{\alpha_j}{\alpha_i})^2+\frac{65}{3}(\frac{\alpha_j}{\alpha_i})^4-\frac{260}{7}(\frac{\alpha_j}{\alpha_i})^6+39(\frac{\alpha_j}{\alpha_i})^8-26(\frac{\alpha_j}{\alpha_i})^{10}+13(\frac{\alpha_j}{\alpha_i})^{12}],$$

$$W_{70}^{10} = \frac{143}{48}[1-\frac{36}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{378}{143}(\frac{\alpha_j}{\alpha_i})^4-\frac{140}{143}(\frac{\alpha_j}{\alpha_i})^6+\frac{15}{143}(\frac{\alpha_j}{\alpha_i})^8],$$

$$W_{70}^{11} = \frac{429}{4}[1-\frac{63}{39}(\frac{\alpha_j}{\alpha_i})^2+\frac{105}{143}(\frac{\alpha_j}{\alpha_i})^4-\frac{35}{429}(\frac{\alpha_j}{\alpha_i})^6]$$

$$W_{70}^{12} = \frac{3003}{8}[1-\frac{10}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{15}{143}(\frac{\alpha_j}{\alpha_i})^4], \quad W_{70}^{13} = \frac{1001}{4}[1-\frac{3}{13}(\frac{\alpha_j}{\alpha_i})^2], \quad W_{70}^{14} = \frac{429}{16},$$

$$W_{70}^{30} = \frac{13}{480}[1-\frac{55}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{90}{13}(\frac{\alpha_j}{\alpha_i})^4-\frac{70}{13}(\frac{\alpha_j}{\alpha_i})^6+\frac{25}{13}(\frac{\alpha_j}{\alpha_i})^8],$$

$$W_{70}^{50} = \frac{1}{5760}[1-(\frac{\alpha_j}{\alpha_i})^2]^6,$$

$$W_{27}^{50} = -\frac{8}{675675},$$

$$W_{37}^{40} = \frac{8}{135135}, \quad W_{37}^{60} = \frac{4}{2297295}[1-\frac{17}{5}(\frac{\alpha_j}{\alpha_i})^2],$$

$$W_{47}^{30} = -\frac{8}{19305}, \quad W_{47}^{50} = -\frac{4}{328185}[1-\frac{17}{7}(\frac{\alpha_j}{\alpha_i})^2],$$

$$W_{57}^{20} = \frac{8}{2145}, \quad W_{57}^{40} = \frac{4}{36465}[1-\frac{17}{9}(\frac{\alpha_j}{\alpha_i})^2], \quad W_{57}^{60} = \frac{1}{692835}[1-\frac{38}{9}(\frac{\alpha_j}{\alpha_i})^2+\frac{323}{63}(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{67}^{10} = -\frac{8}{195}, \quad W_{67}^{30} = -\frac{4}{3315}[1-\frac{17}{11}(\frac{\alpha_j}{\alpha_i})^2], \quad W_{67}^{50} = -\frac{1}{62985}[1-\frac{19}{11}(\frac{\alpha_j}{\alpha_i})^2-\frac{323}{99}(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{77}^{10} = \frac{8}{15}\frac{d}{\alpha_i}, \quad W_{77}^{20} = \frac{4}{255}[1-\frac{17}{13}(\frac{\alpha_j}{\alpha_i})^2], \quad W_{77}^{40} = \frac{1}{4845}[1-\frac{38}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{323}{143}(\frac{\alpha_j}{\alpha_i})^4],$$

$$W_{77}^{60} = \frac{1}{610470}[1-\frac{63}{13}(\frac{\alpha_j}{\alpha_i})^2+\frac{1197}{143}(\frac{\alpha_j}{\alpha_i})^4-\frac{2261}{429}(\frac{\alpha_j}{\alpha_i})^6]. \quad \text{(A.41)}$$

2. $F_{p\ell}(\alpha_i,\alpha_j,\xi,\mu,s;d)$

For $\alpha_j,\alpha_i > 0$ and $P_j > 0$, Eq. (3) gives the following forms[9)] depending on the parameter values $[\beta = \alpha_j(2\xi-1)-d]$:

(a) $\beta-\alpha_i > 0$;

$$\frac{4\alpha_i}{P_\ell(\mu)} F_{p\ell}(\alpha_i,\alpha_j,\xi,\mu,s;d) = \begin{cases} Z_p(\alpha_i,\beta,\mu)-(-1)^p Z_p(-\alpha_i,\beta,\mu), & \mu>0, \\ 0, & \mu<0, \end{cases} \quad \text{(A.42)}$$

(b) $\beta-\alpha_i < 0$ and $\beta+\alpha_i > 0$;

$$\frac{4\alpha_i}{P_\ell(\mu)} F_{p\ell} = \begin{cases} Z_p(\alpha_i,\beta,\mu)+T_p(\alpha_i,\beta,\mu), & \mu>0, \\ (-1)^{p+1}[Z_p(-\alpha_i,\beta,\mu)+T_p(-\alpha_i,\beta,\mu)], & \mu<0, \end{cases} \quad \text{(A.43)}$$

(c) $\beta+\alpha_i < 0$;

$$\frac{4\alpha_i}{P_\ell(\mu)} F_{p\ell} = \begin{cases} 0, & \mu>0, \\ -Z_p(\alpha_i,\beta,\mu)+(-1)^p Z_p(-\alpha_i,\beta,\mu), & \mu<0, \end{cases} \quad \text{(A.44)}$$

where

$$Z_p(\alpha,\beta,\mu) = (i)^{p+2} \sum_{n=0}^{p} \frac{(p+n)!}{(2n)!!(p-n)!} \left(\frac{\mu}{\alpha}\right)^n \exp\left(-\frac{\alpha+\beta}{\mu}\right), \quad \text{(A.45)}$$

$$T_p(\alpha,\beta,\mu) = (i)^p \sum_{r=0}^{[p/2]} (-1)^r \left(\frac{\mu}{\alpha}\right)^{p-2r} \frac{(2p-2r-1)!!}{(2r)!!} \sum_{m=0}^{p-2r} \frac{1}{m!}\left(-\frac{\beta}{\mu}\right)^m. \quad \text{(A.46)}$$

The explicit expressions for $Z_p(\alpha,\beta,\mu)\exp[(\alpha+\beta)/\mu]$ with $p=0\sim7$ are as follows:

$-1,$ for $p=0,$

$-i(1+\mu/\alpha),$ $p=1,$

$1+3\frac{\mu}{\alpha}+3\left(\frac{\mu}{\alpha}\right)^2,$ $p=2,$

$i\left[1+6\frac{\mu}{\alpha}+15\left(\frac{\mu}{\alpha}\right)^2+15\left(\frac{\mu}{\alpha}\right)^3\right],$ $p=3,$

$-\left[1+10\frac{\mu}{\alpha}+45\left(\frac{\mu}{\alpha}\right)^2+105\left(\frac{\mu}{\alpha}\right)^3+105\left(\frac{\mu}{\alpha}\right)^4\right],$ $p=4,$

$-i\left[1+15\frac{\mu}{\alpha}+105\left(\frac{\mu}{\alpha}\right)^2+420\left(\frac{\mu}{\alpha}\right)^3+945\left(\frac{\mu}{\alpha}\right)^4+945\left(\frac{\mu}{\alpha}\right)^5\right],$ $p=5,$

$1+21\frac{\mu}{\alpha}+210\left(\frac{\mu}{\alpha}\right)^2+1260\left(\frac{\mu}{\alpha}\right)^3+4725\left(\frac{\mu}{\alpha}\right)^4+10395\left(\frac{\mu}{\alpha}\right)^5+10395\left(\frac{\mu}{\alpha}\right)^6,$ $p=6,$

$$i[1+28\,\frac{\mu}{\alpha} + 378(\frac{\mu}{\alpha})^2+3150(\frac{\mu}{\alpha})^3+17325(\frac{\mu}{\alpha})^4+62370(\frac{\mu}{\alpha})^5+135135(\frac{\mu}{\alpha})^6$$

$$+135135(\frac{\mu}{\alpha})^7], \qquad p=7. \qquad (A.47)$$

The expressions for $T_p(\alpha,\beta,\mu)$ with $p=0\sim7$ are written as follows:

$$1, \qquad \text{for } p=0,$$

$$-i(\frac{\beta}{\alpha} - \frac{\mu}{\alpha}), \qquad p=1,$$

$$\frac{1}{2} - \frac{3}{2}(\frac{\beta}{\alpha})^2+3\,\frac{\beta}{\alpha}\frac{\mu}{\alpha} - 3(\frac{\mu}{\alpha})^2, \qquad p=2,$$

$$-i[\frac{\beta}{\alpha}(\frac{3}{2} - \frac{5}{2}(\frac{\beta}{\alpha})^2)-(\frac{3}{2} - \frac{15}{2}(\frac{\beta}{\alpha})^2)\frac{\mu}{\alpha} -15\,\frac{\beta}{\alpha}(\frac{\mu}{\alpha})^2+15(\frac{\mu}{\alpha})^3], \qquad p=3,$$

$$\frac{3}{8} - \frac{15}{4}(\frac{\beta}{\alpha})^2+ \frac{35}{8}(\frac{\beta}{\alpha})^4+ \frac{\beta}{\alpha}(\frac{15}{2} - \frac{35}{2}(\frac{\beta}{\alpha})^2)\frac{\mu}{\alpha} -(\frac{15}{2} - \frac{103}{2}(\frac{\beta}{\alpha})^2)(\frac{\mu}{\alpha})^2$$

$$-105\,\frac{\beta}{\alpha}(\frac{\mu}{\alpha})^3+105(\frac{\mu}{\alpha})^4, \qquad p=4,$$

$$-i[\frac{\beta}{\alpha}(\frac{15}{8} - \frac{35}{4}(\frac{\beta}{\alpha})^2+ \frac{63}{8}(\frac{\beta}{\alpha})^4)-(\frac{15}{8} - \frac{105}{4}(\frac{\beta}{\alpha})^2+ \frac{315}{8}(\frac{\beta}{\alpha})^4)\frac{\mu}{\alpha} - \frac{\beta}{\alpha}(\frac{105}{2} - \frac{315}{2}(\frac{\beta}{\alpha})^2)$$

$$\times(\frac{\mu}{\alpha})^2+(\frac{105}{2} - \frac{945}{2}(\frac{\beta}{\alpha})^2)(\frac{\mu}{\alpha})^3+945\,\frac{\beta}{\alpha}(\frac{\mu}{\alpha})^4-945(\frac{\mu}{\alpha})^5], \qquad p=5,$$

$$\frac{5}{16} - \frac{105}{16}(\frac{\beta}{\alpha})^2+ \frac{315}{16}(\frac{\beta}{\alpha})^4- \frac{231}{16}(\frac{\beta}{\alpha})^6+ \frac{\beta}{\alpha}(\frac{105}{8} - \frac{315}{4}(\frac{\beta}{\alpha})^2+ \frac{693}{8}(\frac{\beta}{\alpha})^4)\frac{\mu}{\alpha}$$

$$-(\frac{105}{8} - \frac{945}{4}(\frac{\beta}{\alpha})^2+ \frac{3465}{8}(\frac{\beta}{\alpha})^4)(\frac{\mu}{\alpha})^2- \frac{\beta}{\alpha}(\frac{945}{2} - \frac{3465}{2}(\frac{\beta}{\alpha})^2)(\frac{\mu}{\alpha})^3$$

$$+(\frac{945}{2} - \frac{10395}{2}(\frac{\beta}{\alpha})^2)(\frac{\mu}{\alpha})^4+10395\,\frac{\beta}{\alpha}(\frac{\mu}{\alpha})^5-10395(\frac{\mu}{\alpha})^6, \qquad p=6,$$

$$-i[\frac{\beta}{\alpha}(\frac{35}{16} - \frac{315}{16}(\frac{\beta}{\alpha})^2+ \frac{693}{16}(\frac{\beta}{\alpha})^4- \frac{429}{16}(\frac{\beta}{\alpha})^6)-(\frac{35}{16} - \frac{945}{16}(\frac{\beta}{\alpha})^2+ \frac{3465}{16}(\frac{\beta}{\alpha})^4- \frac{3003}{16}(\frac{\beta}{\alpha})^6)$$

$$\times\frac{\mu}{\alpha} - \frac{\beta}{\alpha}(\frac{945}{8} - \frac{3465}{4}(\frac{\beta}{\alpha})^2+ \frac{9009}{8}(\frac{\beta}{\alpha})^4)(\frac{\mu}{\alpha})^2+(\frac{945}{8} - \frac{10395}{4}(\frac{\beta}{\alpha})^2+ \frac{45045}{8}(\frac{\beta}{\alpha})^4)(\frac{\mu}{\alpha})^3$$

$$+\frac{\beta}{\alpha}(\frac{10395}{2} - \frac{45045}{2}(\frac{\beta}{\alpha})^2)(\frac{\mu}{\alpha})^4-(\frac{10395}{2} - \frac{135135}{2}(\frac{\beta}{\alpha})^2)(\frac{\mu}{\alpha})^5-135135\,\frac{\beta}{\alpha}(\frac{\mu}{\alpha})^6$$

$$+135135(\frac{\mu}{\alpha})^7], \qquad p=7. \qquad (A.48)$$

In the case where the value $(\alpha+|\beta|)/|\mu|$ is small, we have

$$Z_p(\alpha,\beta,\mu)+T_p(\alpha,\beta,\mu) =$$

$$- \sum_{q=1}^{\infty} \frac{1}{q!}(-\frac{\alpha+\beta}{\mu})^q , \qquad \text{for } p=0,$$

$$-i \sum_{q=1}^{\infty} \frac{1}{(q+1)!}(q-\frac{\beta}{\alpha})(-\frac{\alpha+\beta}{\mu})^q , \qquad p=1,$$

$$\sum_{q=1}^{\infty} \frac{1}{(q+2)!}[q^2-3\frac{\beta}{\alpha}q-1+3(\frac{\beta}{\alpha})^2](-\frac{\alpha+\beta}{\mu})^q , \qquad p=2,$$

$$i\sum_{q=1}^{\infty} \frac{1}{(q+3)!}[q^3-6\frac{\beta}{\alpha}q^2-(4-15(\frac{\beta}{\alpha})^2)+\frac{\beta}{\alpha}(9-15(\frac{\beta}{\alpha})^2)](-\frac{\alpha+\beta}{\mu})^q, \quad p=3,$$

$$-\sum_{q=1}^{\infty} \frac{1}{(q+4)!}[q^4-10\frac{\beta}{\alpha}q^3-(10-45(\frac{\beta}{\alpha})^2)q^2+\frac{\beta}{\alpha}(55-105(\frac{\beta}{\alpha})^2)q$$

$$+9-90(\frac{\beta}{\alpha})^2+105(\frac{\beta}{\alpha})^4](-\frac{\alpha+\beta}{\mu})^q, \qquad p=4,$$

$$-i\sum_{q=1}^{\infty} \frac{1}{(q+5)!}[q^5-15\frac{\beta}{\alpha}q^4-(20-105(\frac{\beta}{\alpha})^2)q^3+(195-420(\frac{\beta}{\alpha})^2)q^2+(64-735(\frac{\beta}{\alpha})^2$$

$$+945(\frac{\beta}{\alpha})^4)q-\frac{\beta}{\alpha}(225-1050(\frac{\beta}{\alpha})^2+945(\frac{\beta}{\alpha})^4)](-\frac{\alpha+\beta}{\mu})^q , \qquad p=5,$$

$$\sum_{q=1}^{\infty} \frac{1}{(q+6)!}[q^6-21\frac{\beta}{\alpha}q^5-(35-210(\frac{\beta}{\alpha})^2)q+\frac{\beta}{\alpha}(525-1260(\frac{\beta}{\alpha})^2)q^3$$

$$+(259-3360(\frac{\beta}{\alpha})^2+4725(\frac{\beta}{\alpha})^4)q^2-\frac{\beta}{\alpha}(2079-10710(\frac{\beta}{\alpha})^2+10395(\frac{\beta}{\alpha})^4)q$$

$$-225+4725(\frac{\beta}{\alpha})^2-14175(\frac{\beta}{\alpha})^4+10395(\frac{\beta}{\alpha})^6](-\frac{\alpha+\beta}{\mu})^q , \qquad p=6,$$

$$i\sum_{q=1}^{\infty} \frac{1}{(q+7)!}[q^7-28\frac{\beta}{\alpha}q^6-(56-378(\frac{\beta}{\alpha})^2)q^5+\frac{\beta}{\alpha}(1190-3150(\frac{\beta}{\alpha})^2)q^4$$

$$+(784-11340(\frac{\beta}{\alpha})^2+17325(\frac{\beta}{\alpha})^4)q^3-\frac{\beta}{\alpha}(10612-59850(\frac{\beta}{\alpha})^2+62370(\frac{\beta}{\alpha})^4)q^2$$

$$-(2304-53487(\frac{\beta}{\alpha})^2+173250(\frac{\beta}{\alpha})^4-135135(\frac{\beta}{\alpha})^6)q+\frac{\beta}{\alpha}(11025-99225(\frac{\beta}{\alpha})^2$$

$$+218295(\frac{\beta}{\alpha})^4-135135(\frac{\beta}{\alpha})^6)](-\frac{\alpha+\beta}{\mu})^q , \qquad p=7. \qquad (A.49)$$

The series expansions for the expressions (A.42) and (A.44) can be obtained by regarding the formula (A.49) as the series expansion for the function $Z_p(\alpha,\beta,\mu)$.

3. $G_{p\ell}(\alpha_i,\alpha_j,\xi,s;d)$

By the use of Eq. (8) and (A.42) ∿ (A.46), we get ($\beta=\alpha_j\xi-d$)

(a) $\beta-\alpha_i > 0$;

$$4\alpha_i G_{p\ell}(\alpha_i,\alpha_j,\xi,s;d) = U_{p\ell}(\alpha_i,\beta)-(-1)^p U_{p\ell}(-\alpha_i,\beta), \tag{A.50}$$

(b) $\beta-\alpha_i < 0$ and $\beta+\alpha_i > 0$;

$$4\alpha_i G_{p\ell} = U_{p\ell}(\alpha_i,\beta)+(-1)^{p+\ell}U_{p\ell}(\alpha_i,-\beta)+V_{p\ell}(\alpha_i,\beta), \tag{A.51}$$

(c) $\beta+\alpha_i < 0$;

$$4\alpha_i G_{p\ell} = (-1)^{\ell+1}[U_{p\ell}(-\alpha_i,-\beta)-(-1)^p U_{p\ell}(\alpha_i,-\beta)], \tag{A.52}$$

where

$$U_{p\ell}(\alpha,\beta) = (i)^{p+2}\sum_{n=0}^{p}\frac{(2p-n)!}{n!(p-n)!(2\alpha)^{p-n}}\sum_{r=0}^{[\ell/2]}(-\frac{1}{2})^{[\ell/2]-r}\frac{(2\ell+2r-1-2[\ell/2])!!}{P!([\ell/2]-r)!(\ell+2r-2[\ell/2])!}$$

$$\times\{\sum_{m=1}^{P}(P-m)![-(\alpha+\beta)]^{m+1}e^{-(\alpha+\beta)}+[-(\alpha+\beta)]^{P}E_1(\alpha+\beta)\}, \tag{A.53}$$

$$P = p-n+\ell-2[\ell/2]+2r+1,$$

$$V_{p\ell}(\alpha,\beta) = (i)^p\sum_{r=0}^{[p/2]}(-1)^r 2\frac{(2p-2r-1)!!}{(2r)!!(\alpha)^{p-2r}}\sum_{q=0}^{[(2[(p+\ell)/2]-2r-\ell)/2]\geq 0}$$

$$\times\frac{(-\beta)^{2q+p+\ell-2[(p+\ell)/2]}}{(2q+p+\ell-2[(p+\ell)/2])!}\sum_{n=0}^{[\ell/2]}(-\frac{1}{2})^{[\ell/2]-n}$$

$$\times\frac{(2\ell+2n-1-2[\ell/2])!!}{(2[(p+\ell)/2]-2[\ell/2]+2n-2r-2q+1)([\ell/2]-n)!(\ell+2n-2[\ell/2])!}. \tag{A.54}$$

The explicit expressions for $U_{p\ell}(\alpha,\beta)$ and $V_{p\ell}$ for $p=0\sim7$ and $\ell=0\sim3$ are as follows:

$$U_{00}(\alpha,\beta) = -e^{-(\alpha+\beta)}+(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{10} = -\frac{i}{2}(1-\frac{\beta}{\alpha}+\frac{1}{\alpha})e^{-(\alpha+\beta)}+\frac{i}{2}(1-\frac{\beta}{\alpha})(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{20} = -[\frac{\beta}{2\alpha}(1-\frac{\beta}{\alpha})-(1-\frac{\beta}{2\alpha})\frac{1}{\alpha}-\frac{1}{\alpha^2}]e^{-(\alpha+\beta)}+\frac{\beta}{2\alpha}(1-\frac{\beta}{\alpha})(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{30} = -i[(\frac{1}{8} - \frac{5}{8}(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha}) - (\frac{9}{8} - \frac{5}{4}\frac{\beta}{\alpha} + \frac{5}{8}(\frac{\beta}{\alpha})^2)\frac{1}{\alpha} - (\frac{15}{4} - \frac{5}{4}\frac{\beta}{\alpha})\frac{1}{\alpha^2} - \frac{15}{4\alpha^3}]e^{-(\alpha+\beta)}$$

$$+i(\frac{1}{8} - \frac{5}{8}(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})(\alpha+\beta)E_1(\alpha+\beta) \ ,$$

$$U_{40} = -[\frac{\beta}{8\alpha}(3-7(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha}) + (1-\frac{11}{8}\frac{\beta}{\alpha} + \frac{7}{4}(\frac{\beta}{\alpha})^2 - \frac{7}{8}(\frac{\beta}{\alpha})^3)\frac{1}{\alpha} + (8+\frac{21}{4}\frac{\beta}{\alpha} + \frac{7}{4}(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^2}$$

$$+(21-\frac{21}{4}\frac{\beta}{\alpha})\frac{1}{\alpha^3} + \frac{21}{\alpha^4}]e^{-(\alpha+\beta)} + \frac{\beta}{8\alpha}(3-7(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{50} = -i[(\frac{1}{16} - \frac{7}{8}(\frac{\beta}{\alpha})^2 + \frac{21}{16}(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha}) + (\frac{15}{16} - \frac{7}{8}\frac{\beta}{\alpha} + \frac{7}{4}(\frac{\beta}{\alpha})^2 - \frac{21}{8}(\frac{\beta}{\alpha})^3 + \frac{21}{16}(\frac{\beta}{\alpha})^4)\frac{1}{\alpha}$$

$$+(\frac{105}{8} - \frac{91}{8}\frac{\beta}{\alpha} + \frac{63}{8}(\frac{\beta}{\alpha})^2 - \frac{21}{8}(\frac{\beta}{\alpha})^3)\frac{1}{\alpha^2} + (\frac{525}{8} - \frac{63}{2}\frac{\beta}{\alpha} + \frac{63}{8}(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^3} + (\frac{315}{2} - \frac{63}{2}\frac{\beta}{\alpha})\frac{1}{\alpha^4}$$

$$+\frac{315}{2\alpha^5}]e^{-(\alpha+\beta)} + i(\frac{1}{16} - \frac{7}{8}(\frac{\beta}{\alpha})^2 + \frac{21}{16}(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{60} = -[\frac{\beta}{\alpha}(\frac{5}{16} - \frac{15}{8}(\frac{\beta}{\alpha})^2 + \frac{33}{16}(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha}) - (1-\frac{11}{16}\frac{\beta}{\alpha} + \frac{3}{8}(\frac{\beta}{\alpha})^2 - \frac{9}{4}(\frac{\beta}{\alpha})^3 + \frac{33}{8}(\frac{\beta}{\alpha})^4 - \frac{33}{16}(\frac{\beta}{\alpha})^5)\frac{1}{\alpha}$$

$$-(19-\frac{141}{8}\frac{\beta}{\alpha} + \frac{135}{8}(\frac{\beta}{\alpha})^2 - \frac{99}{8}(\frac{\beta}{\alpha})^3 + \frac{33}{8}(\frac{\beta}{\alpha})^4)\frac{1}{\alpha^2} - (153-\frac{801}{8}\frac{\beta}{\alpha} + \frac{99}{2}(\frac{\beta}{\alpha})^2 - \frac{99}{8}(\frac{\beta}{\alpha})^3)\frac{1}{\alpha^3}$$

$$-(648-\frac{495}{2}\frac{\beta}{\alpha} + \frac{99}{2}(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^4} - (1485-\frac{495}{2}\frac{\beta}{\alpha})\frac{1}{\alpha^5} - \frac{1485}{\alpha^6}]e^{-(\alpha+\beta)} + \frac{\beta}{\alpha}(\frac{5}{16} - \frac{15}{8}(\frac{\beta}{\alpha})^2$$

$$+\frac{33}{16}(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{70} = -i[(\frac{5}{128} - \frac{135}{128}(\frac{\beta}{\alpha})^2 + \frac{495}{128}(\frac{\beta}{\alpha})^4 - \frac{429}{128}(\frac{\beta}{\alpha})^6)(1-\frac{\beta}{\alpha}) - (\frac{133}{128} - \frac{69}{64}\frac{\beta}{\alpha} + \frac{3}{128}(\frac{\beta}{\alpha}) + \frac{33}{32}(\frac{\beta}{\alpha})^3$$

$$+\frac{363}{128}(\frac{\beta}{\alpha})^4 - \frac{429}{64}(\frac{\beta}{\alpha})^5 + \frac{429}{128}(\frac{\beta}{\alpha})^6)\frac{1}{\alpha} - (\frac{1659}{64} - \frac{1521}{64}\frac{\beta}{\alpha} + \frac{759}{32}(\frac{\beta}{\alpha})^2 - \frac{825}{32}(\frac{\beta}{\alpha})^3 + \frac{1287}{64}(\frac{\beta}{\alpha})^4$$

$$-\frac{429}{64}(\frac{\beta}{\alpha})^5)\frac{1}{\alpha^2} - (\frac{19215}{64} - \frac{3663}{16}\frac{\beta}{\alpha} + \frac{5049}{32}(\frac{\beta}{\alpha})^2 - \frac{1287}{16}(\frac{\beta}{\alpha})^3 + \frac{1287}{64}(\frac{\beta}{\alpha})^4)\frac{1}{\alpha^3} - (\frac{31185}{16}$$

$$-\frac{16533}{16}\frac{\beta}{\alpha} + \frac{6435}{16}(\frac{\beta}{\alpha})^2 - \frac{1287}{16}(\frac{\beta}{\alpha})^3)\frac{1}{\alpha^4} - (\frac{121275}{16} - \frac{19305}{8}\frac{\beta}{\alpha} + \frac{6435}{16}(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^5}$$

$$-(\frac{135135}{8} - \frac{19305}{8}\frac{\beta}{\alpha})\frac{1}{\alpha^6} - \frac{135135}{8\alpha^7}]e^{-(\alpha+\beta)} + \frac{i}{128}(5-135(\frac{\beta}{\alpha})^2 + 495(\frac{\beta}{\alpha})^4 - 429(\frac{\beta}{\alpha})^6)$$

$$\times(1-\frac{\beta}{\alpha})(\alpha+\beta)E_1(\alpha+\beta), \qquad (A.54)$$

$$V_{00}(\alpha,\beta) = 2,$$

$$V_{10} = 2i\beta/\alpha,$$

$$V_{20} = 1-3(\frac{\beta}{\alpha})^2 - \frac{2}{\alpha^2}$$

$$V_{30} = -i\,\frac{\beta}{\alpha}[3-5(\frac{\beta}{\alpha})^2 - \frac{10}{\alpha^2}],$$

$$V_{40} = \frac{3}{4} - \frac{15}{2}(\frac{\beta}{\alpha})^2 + \frac{35}{4}(\frac{\beta}{\alpha})^4 - (5-35(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^2} + \frac{42}{\alpha^4},$$

$$V_{50} = -i\,\frac{\beta}{\alpha}[\frac{15}{4} - \frac{35}{2}(\frac{\beta}{\alpha})^2 + \frac{63}{4}(\frac{\beta}{\alpha})^4 - (35-105(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^2} + \frac{378}{\alpha^4}],$$

$$V_{60} = \frac{5}{8} - \frac{105}{8}(\frac{\beta}{\alpha})^2 + \frac{315}{8}(\frac{\beta}{\alpha})^4 - \frac{231}{8}(\frac{\beta}{\alpha})^6 - (\frac{35}{4} - \frac{315}{2}(\frac{\beta}{\alpha})^2 + \frac{1155}{4}(\frac{\beta}{\alpha})^4)\frac{1}{\alpha^2}$$
$$+(189-2079(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^4} - \frac{2970}{\alpha^6},$$

$$V_{70} = -i\,\frac{\beta}{\alpha}[\frac{35}{8} - \frac{315}{8}(\frac{\beta}{\alpha})^2 + \frac{693}{8}(\frac{\beta}{\alpha})^4 - \frac{429}{8}(\frac{\beta}{\alpha})^6 - (\frac{315}{4} - \frac{1155}{2}(\frac{\beta}{\alpha})^2 + \frac{3003}{4}(\frac{\beta}{\alpha})^4)\frac{1}{\alpha^2}$$
$$+(2079-9009(\frac{\beta}{\alpha})^2)\frac{1}{\alpha^4} - \frac{38610}{\alpha^6}], \tag{A.55}$$

$$U_{01}(\alpha,\beta) = \frac{1}{2}(\alpha+\beta-1)e^{-(\alpha+\beta)} - \frac{1}{2}(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{11} = \frac{i}{6}[(2-\frac{\beta}{\alpha})(\alpha+\beta-1) - \frac{2}{\alpha}]e^{-(\alpha+\beta)} - \frac{i}{6}(2-\frac{\beta}{\alpha})(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{21} = -[\frac{1}{8}(1-\frac{\beta}{\alpha})^2(\alpha+\beta-1) - \frac{1}{4\alpha}(3-\frac{\beta}{\alpha}) - \frac{3}{4\alpha^2}]e^{-(\alpha+\beta)} + \frac{1}{8}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{31} = i[\frac{\beta}{8\alpha}(1-\frac{\beta}{\alpha})^2(\alpha+\beta-1) + \frac{1}{4\alpha}(4-3\frac{\beta}{\alpha} + (\frac{\beta}{\alpha})^2) + \frac{3}{4\alpha^2}(4-\frac{\beta}{\alpha}) + \frac{3}{\alpha^3}]e^{-(\alpha+\beta)}$$
$$-i\,\frac{\beta}{8\alpha}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{41} = -[\frac{1}{48}(1-\frac{\beta}{\alpha})^2(1-7(\frac{\beta}{\alpha})^2)(\alpha+\beta-1) + \frac{1}{24\alpha}(25-27\frac{\beta}{\alpha} + 21(\frac{\beta}{\alpha})^2 - 7(\frac{\beta}{\alpha})^3)$$
$$+\frac{1}{8\alpha^2}(55-28\frac{\beta}{\alpha} + 7(\frac{\beta}{\alpha})^2) + \frac{35}{2\alpha^3}(1-\frac{1}{5}\frac{\beta}{\alpha}) + \frac{35}{2\alpha^4}]e^{-(\alpha+\beta)}$$
$$+\frac{1}{48}(1-\frac{\beta}{\alpha})^2(1-7(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{51} = i[\frac{1}{16}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-3(\frac{\beta}{\alpha})^2)(\alpha+\beta-1) - \frac{1}{8\alpha}(8-9\frac{\beta}{\alpha} + 11(\frac{\beta}{\alpha})^2 - 9(\frac{\beta}{\alpha})^3 + 3(\frac{\beta}{\alpha})^4)$$
$$-\frac{1}{8\alpha^2}(96-69\frac{\beta}{\alpha} + 36(\frac{\beta}{\alpha})^2 - 9(\frac{\beta}{\alpha})^3) - \frac{1}{\alpha^3}(57-\frac{45}{2}\frac{\beta}{\alpha} + \frac{9}{2}(\frac{\beta}{\alpha})^2) - \frac{45}{2\alpha^4}(6-\frac{\beta}{\alpha}) - \frac{135}{\alpha^5}]e^{-(\alpha+\beta)}$$
$$-\frac{i}{16}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-3(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{61} = -[\frac{1}{128}(1-\frac{\beta}{\alpha})^2(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)(\alpha+\beta-1)-\frac{1}{64\alpha}(63-61\frac{\beta}{\alpha}+78(\frac{\beta}{\alpha})^2$$

$$-114(\frac{\beta}{\alpha})^3+99(\frac{\beta}{\alpha})^4-33(\frac{\beta}{\alpha})^5)-\frac{1}{64\alpha^2}(1155-972\frac{\beta}{\alpha}+738(\frac{\beta}{\alpha})^2-396(\frac{\beta}{\alpha})^3+99(\frac{\beta}{\alpha})^4)$$

$$-\frac{1}{16\alpha^3}(2205-1233\frac{\beta}{\alpha}+495(\frac{\beta}{\alpha})^2-99(\frac{\beta}{\alpha})^3)-\frac{1}{16\alpha^4}(9135-2970\frac{\beta}{\alpha}+495(\frac{\beta}{\alpha})^2)$$

$$-\frac{1485}{8\alpha^5}(7-\frac{\beta}{\alpha})-\frac{10395}{8\alpha^6}]e^{-(\alpha+\beta)}+\frac{1}{128}(1-\frac{\beta}{\alpha})^2(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{71} = i[\frac{5}{128}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4)(\alpha+\beta-1)$$

$$+\frac{1}{\alpha}(1-\frac{59}{64}\frac{\beta}{\alpha}+\frac{49}{64}(\frac{\beta}{\alpha})^2-\frac{121}{96}(\frac{\beta}{\alpha})^3+\frac{77}{32}(\frac{\beta}{\alpha})^4-\frac{143}{64}(\frac{\beta}{\alpha})^5+\frac{143}{192}(\frac{\beta}{\alpha})^6)$$

$$+\frac{1}{\alpha^2}(25-\frac{1423}{64}\frac{\beta}{\alpha}+\frac{319}{16}(\frac{\beta}{\alpha})^2-\frac{517}{32}(\frac{\beta}{\alpha})^3+\frac{143}{16}(\frac{\beta}{\alpha})^4-\frac{143}{64}(\frac{\beta}{\alpha})^5)$$

$$+\frac{1}{\alpha^3}(278-\frac{3025}{16}\frac{\beta}{\alpha}+\frac{1749}{16}(\frac{\beta}{\alpha})^2-\frac{715}{16}(\frac{\beta}{\alpha})^3+\frac{143}{16}(\frac{\beta}{\alpha})^4)$$

$$+\frac{1}{\alpha^4}(1760-\frac{13035}{16}\frac{\beta}{\alpha}+\frac{2145}{8}(\frac{\beta}{\alpha})^2-\frac{715}{16}(\frac{\beta}{\alpha})^3)+\frac{1}{\alpha^5}(6765-\frac{15015}{8}\frac{\beta}{\alpha}+\frac{2145}{8}(\frac{\beta}{\alpha})^2)$$

$$+\frac{15015}{8\alpha^6}(8-\frac{\beta}{\alpha})+\frac{15015}{\alpha^7}]e^{-(\alpha+\beta)}-i\frac{5}{128}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4)(\alpha+\beta)^2E_1(\alpha+\beta),$$

(A.56)

$$V_{01}(\alpha,\beta) = 0,$$

$$V_{11} = \frac{2}{3\alpha}i,$$

$$V_{21} = \frac{2}{\alpha}\frac{\beta}{\alpha},$$

$$V_{31} = \frac{i}{\alpha}[1-5(\frac{\beta}{\alpha})^2-\frac{6}{\alpha^2}],$$

$$V_{41} = \frac{1}{\alpha}\frac{\beta}{\alpha}[5-\frac{35}{3}(\frac{\beta}{\alpha})^2-\frac{42}{\alpha^2}],$$

$$V_{51} = \frac{i}{\alpha}[\frac{5}{4}-\frac{35}{2}(\frac{\beta}{\alpha})^2+\frac{105}{4}(\frac{\beta}{\alpha})^4-\frac{21}{\alpha^2}(1-9(\frac{\beta}{\alpha})^2)+\frac{270}{\alpha^4}],$$

$$V_{61} = \frac{1}{\alpha}\frac{\beta}{\alpha}[\frac{35}{4}-\frac{105}{2}(\frac{\beta}{\alpha})^2+\frac{231}{4}(\frac{\beta}{\alpha})^4-\frac{1}{\alpha^2}(189-693(\frac{\beta}{\alpha})^2)+\frac{2970}{\alpha^4}],$$

$$V_{71} = \frac{i}{\alpha}\Big[\frac{35}{24} - \frac{315}{8}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{1155}{8}\Big(\frac{\beta}{\alpha}\Big)^4 - \frac{1001}{8}\Big(\frac{\beta}{\alpha}\Big)^6 - \frac{1}{\alpha^2}\Big(\frac{189}{4} - \frac{2079}{2}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{9009}{4}\Big(\frac{\beta}{\alpha}\Big)^4\Big)$$

$$+ \frac{1}{\alpha^4}\Big(1485 - 19305\Big(\frac{\beta}{\alpha}\Big)^2\Big) - \frac{30030}{\alpha^6}\Big] , \qquad \text{(A.57)}$$

$$U_{02} = -\frac{1}{4}(\alpha+\beta-1)(\alpha+\beta)e^{-(\alpha+\beta)} + \Big(\frac{1}{4}(\alpha+\beta)^2 - \frac{1}{2}\Big)(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{12} = -i\Big[\frac{1}{16}\Big(3-\frac{\beta}{\alpha}\Big)(\alpha+\beta-1)(\alpha+\beta) + \frac{1}{8}\Big(1+\frac{\beta}{\alpha}\Big) + \frac{1}{8\alpha}\Big]e^{-(\alpha+\beta)}$$

$$+ i\Big[\frac{1}{16}\Big(3-\frac{\beta}{\alpha}\Big)(\alpha+\beta)^2 - \frac{1}{4}\Big(1-\frac{\beta}{\alpha}\Big)\Big](\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{22} = \Big[\Big(\frac{1}{10} - \frac{9}{80}\frac{\beta}{\alpha} + \frac{3}{80}\Big(\frac{\beta}{\alpha}\Big)^2\Big)(\alpha+\beta-1)(\alpha+\beta) + \Big(\frac{1}{5} + \frac{1}{40}\frac{\beta}{\alpha} - \frac{7}{40}\Big(\frac{\beta}{\alpha}\Big)^2\Big) + \frac{1}{\alpha}\Big(\frac{2}{5} + \frac{1}{40}\frac{\beta}{\alpha}\Big)$$

$$+ \frac{2}{5\alpha^2}\Big]e^{-(\alpha+\beta)} - \Big[\Big(\frac{1}{10} - \frac{9}{80}\frac{\beta}{\alpha} + \frac{3}{80}\Big(\frac{\beta}{\alpha}\Big)^2\Big)(\alpha+\beta)^2 + \frac{1}{4}\frac{\beta}{\alpha}\Big(1-\frac{\beta}{\alpha}\Big)\Big](\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{32} = i\Big[\frac{1}{32}\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta-1)(\alpha+\beta) + \Big(\frac{1}{8} - \frac{1}{4}\frac{\beta}{\alpha}\Big)\Big(1-\Big(\frac{\beta}{\alpha}\Big)^2\Big) + \frac{1}{\alpha}\Big(\frac{3}{4} - \frac{1}{8}\frac{\beta}{\alpha}\Big(1+\frac{\beta}{\alpha}\Big)\Big) + \frac{1}{\alpha^2}\Big(\frac{15}{8} - \frac{1}{8}\frac{\beta}{\alpha}\Big)$$

$$+ \frac{15}{8\alpha^3}\Big]e^{-(\alpha+\beta)} - i\Big[\frac{1}{32}\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta)^2 + \frac{1}{16}\Big(1-5\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)\Big](\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{42} = \Big[\frac{1}{32}\frac{\beta}{\alpha}\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta-1)(\alpha+\beta) + \frac{\beta}{\alpha}\Big(\frac{1}{4} - \frac{3}{8}\frac{\beta}{\alpha}\Big)\Big(1-\Big(\frac{\beta}{\alpha}\Big)^2\Big) - \frac{1}{\alpha}\Big(1 - \frac{5}{8}\frac{\beta}{\alpha} - \frac{1}{8}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{1}{4}\Big(\frac{\beta}{\alpha}\Big)^3\Big)$$

$$- \frac{1}{\alpha^2}\Big(5 - \frac{9}{8}\frac{\beta}{\alpha} - \frac{1}{8}\Big(\frac{\beta}{\alpha}\Big)^2\Big) - \frac{1}{\alpha^3}\Big(12 - \frac{9}{8}\frac{\beta}{\alpha}\Big) - \frac{12}{\alpha^4}\Big]e^{-(\alpha+\beta)}$$

$$- \frac{\beta}{\alpha}\Big[\frac{1}{32}\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta)^2 + \Big(\frac{3}{16} - \frac{7}{16}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)\Big](\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{52} = i\Big[\Big(\frac{1}{256} - \frac{9}{256}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta-1)(\alpha+\beta) + \frac{1}{128}\Big(5 - 7\frac{\beta}{\alpha} - 57\Big(\frac{\beta}{\alpha}\Big)^2 + 75\Big(\frac{\beta}{\alpha}\Big)^3\Big)\Big(1-\Big(\frac{\beta}{\alpha}\Big)^2\Big)$$

$$- \frac{1}{\alpha}\Big(\frac{135}{128} - \frac{37}{32}\frac{\beta}{\alpha} + \frac{37}{64}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{15}{32}\Big(\frac{\beta}{\alpha}\Big)^3 - \frac{57}{128}\Big(\frac{\beta}{\alpha}\Big)^4\Big) - \frac{1}{\alpha^2}\Big(\frac{315}{32} - \frac{139}{32}\frac{\beta}{\alpha} + \frac{9}{32}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{15}{32}\Big(\frac{\beta}{\alpha}\Big)^3\Big)$$

$$- \frac{1}{\alpha^3}\Big(\frac{1365}{32} - \frac{153}{16}\frac{\beta}{\alpha} + \frac{9}{32}\Big(\frac{\beta}{\alpha}\Big)^2\Big) - \frac{1}{\alpha^4}\Big(\frac{1575}{16} - \frac{153}{16}\frac{\beta}{\alpha}\Big) - \frac{1575}{16\alpha^5}\Big]e^{-(\alpha+\beta)}$$

$$- i\Big[\Big(\frac{1}{256} - \frac{9}{256}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta)^2 + \Big(\frac{1}{32} - \frac{7}{16}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{21}{32}\Big(\frac{\beta}{\alpha}\Big)^4\Big)\Big(1-\frac{\beta}{\alpha}\Big)\Big](\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{62} = \Big[\frac{\beta}{\alpha}\Big(\frac{3}{256} - \frac{11}{256}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta-1)(\alpha+\beta) + \frac{1}{128}\frac{\beta}{\alpha}\Big(23 - 29\frac{\beta}{\alpha} - 99\Big(\frac{\beta}{\alpha}\Big)^2 + 121\Big(\frac{\beta}{\alpha}\Big)^3\Big)\Big(1-\Big(\frac{\beta}{\alpha}\Big)^2\Big)$$

$$+ \frac{1}{\alpha}\Big(1 - \frac{157}{128}\frac{\beta}{\alpha} + \frac{51}{32}\Big(\frac{\beta}{\alpha}\Big)^2 - \frac{39}{64}\Big(\frac{\beta}{\alpha}\Big)^3 - \frac{33}{32}\Big(\frac{\beta}{\alpha}\Big)^4 + \frac{99}{128}\Big(\frac{\beta}{\alpha}\Big)^5\Big)$$

$$+ \frac{1}{\alpha^2}\Big(16 - \frac{333}{32}\frac{\beta}{\alpha} + \frac{117}{32}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{33}{32}\Big(\frac{\beta}{\alpha}\Big)^3\Big(1-\frac{\beta}{\alpha}\Big)\Big) + \frac{1}{\alpha^3}\Big(111 - \frac{1323}{32}\frac{\beta}{\alpha} + \frac{99}{16}\Big(\frac{\beta}{\alpha}\Big)^2 + \frac{33}{32}\Big(\frac{\beta}{\alpha}\Big)^3\Big)$$

$$+\frac{1}{\alpha^4}(441-\frac{1485}{16}\frac{\beta}{\alpha}+\frac{99}{16}(\frac{\beta}{\alpha})^2)+\frac{1}{\alpha^5}(990-\frac{1485}{16}\frac{\beta}{\alpha})+\frac{990}{\alpha^6}]e^{-(\alpha+\beta)}$$

$$-\frac{\beta}{\alpha}[(\frac{3}{256}-\frac{11}{256}(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})^3(\alpha+\beta)^2+(\frac{5}{32}-\frac{15}{16}(\frac{\beta}{\alpha})^2+\frac{33}{32}(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})]$$

$$\times(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_{72}=i[\frac{1}{2560}(3-66(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})^3(\alpha+\beta-1)(\alpha+\beta)+(\frac{7}{320}-\frac{17}{640}\frac{\beta}{\alpha}-\frac{11}{20}(\frac{\beta}{\alpha})^2+\frac{209}{320}(\frac{\beta}{\alpha})^3$$

$$+\frac{429}{320}(\frac{\beta}{\alpha})^4-\frac{1001}{640}(\frac{\beta}{\alpha})^5)(1-(\frac{\beta}{\alpha})^2)+\frac{1}{\alpha}(\frac{623}{640}-\frac{597}{640}\frac{\beta}{\alpha}+\frac{51}{32}(\frac{\beta}{\alpha})^2-\frac{165}{64}(\frac{\beta}{\alpha})^3+\frac{99}{128}(\frac{\beta}{\alpha})^4$$

$$+\frac{1287}{640}(\frac{\beta}{\alpha})^5-\frac{429}{320}(\frac{\beta}{\alpha})^6)+\frac{1}{\alpha^2}(\frac{14763}{640}-\frac{2337}{128}\frac{\beta}{\alpha}+\frac{759}{64}(\frac{\beta}{\alpha})^2-\frac{165}{64}(\frac{\beta}{\alpha})^3-\frac{429}{128}(\frac{\beta}{\alpha})^4+\frac{1287}{640}(\frac{\beta}{\alpha})^5)$$

$$+\frac{1}{\alpha^3}(\frac{30303}{128}-\frac{1947}{16}\frac{\beta}{\alpha}+\frac{2475}{64}(\frac{\beta}{\alpha})^2-\frac{429}{128}(\frac{\beta}{\alpha})^4)+\frac{1}{\alpha^4}(\frac{22869}{16}-\frac{1881}{4}\frac{\beta}{\alpha}+\frac{1287}{16}(\frac{\beta}{\alpha})^2)$$

$$+\frac{1}{\alpha^5}(\frac{21483}{4}-\frac{16731}{16}\frac{\beta}{\alpha}+\frac{1287}{16}(\frac{\beta}{\alpha})^2)+\frac{1}{\alpha^6}(\frac{189189}{16}-\frac{16731}{16}\frac{\beta}{\alpha})+\frac{189189}{16\alpha^7}]e^{-(\alpha+\beta)}$$

$$-i[\frac{1}{2560}(3-66(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})^3(\alpha+\beta)^2+\frac{1}{256}(5-135(\frac{\beta}{\alpha})^2+495(\frac{\beta}{\alpha})^4-429(\frac{\beta}{\alpha})^6)$$

$$\times(1-\frac{\beta}{\alpha})](\alpha+\beta)E_1(\alpha+\beta), \quad (A.58)$$

$$V_{02}=V_{12}=0,$$

$$V_{22}=-\frac{4}{5\alpha^2}$$

$$V_{32}=4\frac{\beta}{\alpha^3}i$$

$$V_{42}=-\frac{1}{\alpha^2}[2-14(\frac{\beta}{\alpha})^2-\frac{24}{\alpha^2}],$$

$$V_{52}=i\frac{\beta}{\alpha^3}[14-42(\frac{\beta}{\alpha})^2-\frac{216}{\alpha^2}],$$

$$V_{62}=-\frac{1}{\alpha^2}[\frac{7}{2}-63(\frac{\beta}{\alpha})^2+\frac{231}{2}(\frac{\beta}{\alpha})^4-\frac{1}{\alpha^2}(108-1188(\frac{\beta}{\alpha})^2)+\frac{1980}{\alpha^4}],$$

$$V_{72}=i\frac{\beta}{\alpha^3}[\frac{63}{2}-231(\frac{\beta}{\alpha})^2+\frac{3003}{10}(\frac{\beta}{\alpha})^4-\frac{1}{\alpha^2}(1188-5148(\frac{\beta}{\alpha})^2)+\frac{25740}{\alpha^4}],$$

(A.59)

$$U_{03} = [\frac{5}{48}(\alpha+\beta-1)(\alpha+\beta)^2 - \frac{13}{24}(\alpha+\beta) + \frac{1}{8}]e^{-(\alpha+\beta)} - [\frac{5}{48}(\alpha+\beta)^2 - \frac{3}{4}](\alpha+\beta)^2 E_1(\alpha+\beta),$$

$$U_{13} = i[(\frac{1}{12} - \frac{1}{48}\frac{\beta}{\alpha})(\alpha+\beta-1)(\alpha+\beta)^2 - (\frac{1}{3} - \frac{5}{24}\frac{\beta}{\alpha})(\alpha+\beta) - \frac{1}{8}\frac{\beta}{\alpha}]e^{-(\alpha+\beta)}$$

$$-i[(\frac{1}{12} - \frac{1}{48}\frac{\beta}{\alpha})(\alpha+\beta)^2 - \frac{1}{2} + \frac{1}{4}\frac{\beta}{\alpha}](\alpha+\beta)^2 E_1(\alpha+\beta),$$

$$U_{23} = -[(\frac{5}{96} - \frac{1}{24}\frac{\beta}{\alpha} + \frac{1}{96}(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2(\alpha+\beta-1) - (\frac{1}{12} - \frac{7}{24}\frac{\beta}{\alpha} + \frac{1}{6}(\frac{\beta}{\alpha})^2)(\alpha+\beta)$$

$$-\frac{1}{8} - \frac{1}{8}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha}) - \frac{1}{8\alpha}(1+\frac{\beta}{\alpha}) - \frac{1}{8\alpha^2}]e^{-(\alpha+\beta)}$$

$$+[(\frac{5}{96} - \frac{1}{24}\frac{\beta}{\alpha} + \frac{1}{96}(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2 - \frac{3}{16}(1-\frac{\beta}{\alpha})^2](\alpha+\beta)^2 E_1(\alpha+\beta),$$

$$U_{33} = -i[(\frac{1}{42} - \frac{29}{672}\frac{\beta}{\alpha} + \frac{5}{168}(\frac{\beta}{\alpha})^2 - \frac{5}{672}(\frac{\beta}{\alpha})^3)(\alpha+\beta)^2(\alpha+\beta-1) + (\frac{1}{21} + \frac{17}{168}\frac{\beta}{\alpha} - \frac{53}{168}(\frac{\beta}{\alpha})^2 + \frac{29}{168}(\frac{\beta}{\alpha})^3)$$

$$\times(\alpha+\beta) - \frac{1}{7} + \frac{1}{14}\frac{\beta}{\alpha} + \frac{11}{56}(\frac{\beta}{\alpha})^2 - \frac{1}{7}(\frac{\beta}{\alpha})^3 - \frac{1}{\alpha}(1+\frac{\beta}{\alpha})(\frac{3}{7} - \frac{11}{56}\frac{\beta}{\alpha}) - \frac{1}{\alpha^2}(\frac{6}{7} + \frac{13}{56}\frac{\beta}{\alpha}) - \frac{6}{7\alpha^3}]$$

$$\times e^{-(\alpha+\beta)} + i[(\frac{1}{42} - \frac{29}{672}\frac{\beta}{\alpha} + \frac{5}{168}(\frac{\beta}{\alpha})^2 - \frac{5}{672}(\frac{\beta}{\alpha})^3)(\alpha+\beta)^2 + \frac{3}{16}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2](\alpha+\beta)^2 E_1(\alpha+\beta),$$

$$U_{43} = [\frac{5}{768}(1-\frac{\beta}{\alpha})^4(\alpha+\beta)^2(\alpha+\beta-1) + (\frac{17}{384} - \frac{5}{192}\frac{\beta}{\alpha} - \frac{79}{384}(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})^2(\alpha+\beta)$$

$$+(\frac{9}{128} - \frac{5}{64}\frac{\beta}{\alpha} - \frac{23}{128}(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})^2 - \frac{1}{\alpha}(1+\frac{\beta}{\alpha})(\frac{25}{32} - \frac{13}{16}\frac{\beta}{\alpha} + \frac{9}{32}(\frac{\beta}{\alpha})^2)$$

$$-\frac{1}{\alpha^2}(\frac{95}{32} + \frac{9}{16}\frac{\beta}{\alpha} - \frac{17}{32}(\frac{\beta}{\alpha})^2) - \frac{1}{\alpha^3}(\frac{105}{16} + \frac{9}{16}\frac{\beta}{\alpha}) - \frac{105}{16\alpha^4}]e^{-(\alpha+\beta)}$$

$$-[\frac{5}{768}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2 + \frac{1}{32} - \frac{7}{32}(\frac{\beta}{\alpha})^2](1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2 E_1(\alpha+\beta),$$

$$U_{53} = -i[\frac{5}{768}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^4(\alpha+\beta)^2(\alpha+\beta-1) + \frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\frac{41}{384} - \frac{5}{192}\frac{\beta}{\alpha} + \frac{103}{384}(\frac{\beta}{\alpha})^2)(\alpha+\beta)$$

$$-\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\frac{17}{128} - \frac{5}{64}\frac{\beta}{\alpha} - \frac{31}{128}(\frac{\beta}{\alpha})^2) + \frac{1}{\alpha}(1+\frac{\beta}{\alpha})(1 - \frac{53}{32}\frac{\beta}{\alpha} + \frac{21}{16}(\frac{\beta}{\alpha})^2 - \frac{13}{32}(\frac{\beta}{\alpha})^3)$$

$$+\frac{1}{\alpha^2}[7 - \frac{11}{32}\frac{\beta}{\alpha} - \frac{33}{16}(\frac{\beta}{\alpha})^2 + \frac{29}{32}(\frac{\beta}{\alpha})^3) + \frac{1}{\alpha^3}(27 + \frac{15}{16}\frac{\beta}{\alpha} - \frac{33}{16}(\frac{\beta}{\alpha})^2) + \frac{1}{\alpha^4}(60 + \frac{15}{16}\frac{\beta}{\alpha})$$

$$+\frac{60}{\alpha^5}]e^{-(\alpha+\beta)} + i[\frac{5}{768}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2 + \frac{3}{32} - \frac{9}{32}(\frac{\beta}{\alpha})^2]\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2 E_1(\alpha+\beta),$$

$$U_{63} = [\frac{1}{1536}(1-\frac{\beta}{\alpha})^4(1-11(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2(\alpha+\beta-1) + (1-\frac{\beta}{\alpha})^2(\frac{5}{384} - \frac{1}{384}\frac{\beta}{\alpha} - \frac{43}{192}(\frac{\beta}{\alpha})^2 + \frac{11}{384}(\frac{\beta}{\alpha})^3$$

$$+\frac{143}{384}(\frac{\beta}{\alpha})^4)(\alpha+\beta) - (1-\frac{\beta}{\alpha})^2(\frac{1}{64} - \frac{1}{128}\frac{\beta}{\alpha} - \frac{1}{4}(\frac{\beta}{\alpha})^2 + \frac{11}{128}(\frac{\beta}{\alpha})^3 + \frac{11}{32}(\frac{\beta}{\alpha})^4)$$

$$+\frac{1}{\alpha}(1+\frac{\beta}{\alpha})(\frac{133}{128} - \frac{35}{16}\frac{\beta}{\alpha} + \frac{183}{64}(\frac{\beta}{\alpha})^2 - \frac{33}{16}(\frac{\beta}{\alpha})^3 + \frac{77}{128}(\frac{\beta}{\alpha})^4) + \frac{1}{\alpha^2}(\frac{1645}{128} - \frac{17}{4}\frac{\beta}{\alpha} - \frac{177}{64}(\frac{\beta}{\alpha})^2$$

$$+\frac{33}{8}(\frac{\beta}{\alpha})^3-\frac{187}{128}(\frac{\beta}{\alpha})^4)+\frac{1}{\alpha^3}(\frac{315}{4}-\frac{123}{16}\frac{\beta}{\alpha}-\frac{165}{16}(\frac{\beta}{\alpha})^2+\frac{33}{8}(\frac{\beta}{\alpha})^3)$$

$$+\frac{1}{\alpha^4}(\frac{4725}{16}-\frac{165}{16}\frac{\beta}{\alpha}(1+\frac{\beta}{\alpha}))+\frac{1}{\alpha^5}(\frac{10395}{16}-\frac{165}{16}\frac{\beta}{\alpha})+\frac{10395}{16\alpha^6}]e^{-(\alpha+\beta)}$$

$$-[\frac{1}{1536}(1-\frac{\beta}{\alpha})^2(1-11(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2+\frac{3}{256}-\frac{27}{128}(\frac{\beta}{\alpha})^2+\frac{99}{256}(\frac{\beta}{\alpha})^4](1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2E_1(\alpha+\beta),$$

$$U_{73} = -i[\frac{1}{1536}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^4(3-13(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2(\alpha+\beta-1)+\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\frac{1}{16}-\frac{1}{128}\frac{\beta}{\alpha}-\frac{85}{192}(\frac{\beta}{\alpha})^2$$

$$+\frac{13}{384}(\frac{\beta}{\alpha})^3+\frac{13}{24}(\frac{\beta}{\alpha})^4)(\alpha+\beta)-\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\frac{9}{128}-\frac{3}{128}\frac{\beta}{\alpha}-\frac{15}{32}(\frac{\beta}{\alpha})^2+\frac{13}{128}(\frac{\beta}{\alpha})^3+\frac{65}{128}(\frac{\beta}{\alpha})^4)$$

$$-\frac{1}{\alpha}(1+\frac{\beta}{\alpha})(1-\frac{277}{128}\frac{\beta}{\alpha}+\frac{15}{4}(\frac{\beta}{\alpha})^2-\frac{299}{64}(\frac{\beta}{\alpha})^3+\frac{13}{4}(\frac{\beta}{\alpha})^4-\frac{117}{128}(\frac{\beta}{\alpha})^5)$$

$$-\frac{1}{\alpha^2}(20-\frac{1461}{128}\frac{\beta}{\alpha}+\frac{19}{16}(\frac{\beta}{\alpha})^2+\frac{461}{64}(\frac{\beta}{\alpha})^3-\frac{117}{16}(\frac{\beta}{\alpha})^4+\frac{299}{128}(\frac{\beta}{\alpha})^5)$$

$$-\frac{1}{\alpha^3}(183-\frac{765}{16}\frac{\beta}{\alpha}-\frac{153}{8}(\frac{\beta}{\alpha})^2+\frac{195}{8}(\frac{\beta}{\alpha})^3-\frac{117}{16}(\frac{\beta}{\alpha})^4)-\frac{1}{\alpha^4}(1035-\frac{1065}{8}\frac{\beta}{\alpha}-\frac{975}{16}(\frac{\beta}{\alpha})^2$$

$$+\frac{195}{8}(\frac{\beta}{\alpha})^3)-\frac{1}{\alpha^5}(3765-\frac{4095}{16}\frac{\beta}{\alpha}-\frac{975}{16}(\frac{\beta}{\alpha})^2)-\frac{1}{\alpha^6}(8190-\frac{4095}{16}\frac{\beta}{\alpha})-\frac{8190}{\alpha^7}]e^{-(\alpha+\beta)}$$

$$+i[\frac{1}{1536}(1-\frac{\beta}{\alpha})^2(3-13(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2+\frac{15}{256}-\frac{55}{128}(\frac{\beta}{\alpha})^2+\frac{143}{256}(\frac{\beta}{\alpha})^4]\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)E_1(\alpha+\beta),$$

(A.60)

$$V_{03} = V_{13} = V_{23} = 0,$$

$$V_{33} = -i\frac{12}{7\alpha^3},$$

$$V_{43} = -12\frac{\beta}{\alpha^4},$$

$$V_{53} = -\frac{i}{\alpha^3}[6-54(\frac{\beta}{\alpha})^2-\frac{120}{\alpha^2}],$$

$$V_{63} = -\frac{\beta}{\alpha^4}[54-198(\frac{\beta}{\alpha})^2-\frac{1320}{\alpha^2}],$$

$$V_{73} = -\frac{i}{\alpha^3}[\frac{27}{2}-297(\frac{\beta}{\alpha})^2+\frac{1287}{2}(\frac{\beta}{\alpha})^4-\frac{1}{\alpha^2}(660-8580(\frac{\beta}{\alpha})^2)+\frac{16380}{\alpha^4}].$$

(A.61)

For small values of $\alpha+|\beta|$, we obtain the following series expansions for $U_{p\ell}(\alpha,\beta)$ with $p=0\sim7$ and $\ell=0\sim3$:

$$U_{00}(\alpha,\beta) = \alpha(1-\gamma)-(\alpha+\beta)\ln(\alpha+\beta)-\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+1)!}(\alpha+\beta)^{m+1},$$

$$U_{10} = -i\frac{\beta}{2}-\frac{i}{2}(1-\frac{\beta}{\alpha})(\alpha+\beta)\ln(\alpha+\beta)-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)!}(m+1-\frac{\beta}{\alpha})(\alpha+\beta)^{m+1},$$

$$U_{20} = \alpha(\frac{1}{3}-\frac{1}{2}(\frac{\beta}{\alpha})^2)-\frac{\beta}{2\alpha}(1-\frac{\beta}{\alpha})(\alpha+\beta)\ln(\alpha+\beta)+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+3)!}[m^2+(2-3\frac{\beta}{\alpha})m$$
$$-3\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})](\alpha+\beta)^{m+1},$$

$$U_{30} = -i\beta(\frac{13}{24}-\frac{5}{8}(\frac{\beta}{\alpha})^2)-\frac{i}{8}(1-5(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})(\alpha+\beta)\ln(\alpha+\beta)+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+4)!}$$
$$\times[m^3+(3-6\frac{\beta}{\alpha})m^2-(1+12\frac{\beta}{\alpha}-15(\frac{\beta}{\alpha})^2)m-(3-15(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})](\alpha+\beta)^{m+1},$$

$$U_{40} = \alpha(\frac{2}{15}-\frac{23}{24}(\frac{\beta}{\alpha})^2+\frac{7}{8}(\frac{\beta}{\alpha})^4)-\frac{\beta}{\alpha}(\frac{3}{8}-\frac{7}{8}(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})(\alpha+\beta)\ln(\alpha+\beta)-\sum_{m=1}^{\infty}(-1)^m$$
$$\times\frac{1}{m(m+5)!}[m^4+(4-10\frac{\beta}{\alpha})m^3-(4+30\frac{\beta}{\alpha}-45(\frac{\beta}{\alpha})^2)m^2-(16-25\frac{\beta}{\alpha}-90(\frac{\beta}{\alpha})^2+105(\frac{\beta}{\alpha})^3)$$
$$\times m+\frac{\beta}{\alpha}(45-105(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})](\alpha+\beta)^{m+1},$$

$$U_{50} = -i\beta(\frac{113}{240}-\frac{7}{4}(\frac{\beta}{\alpha})^2+\frac{21}{16}(\frac{\beta}{\alpha})^4)-\frac{i}{16}(1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})(\alpha+\beta)\ln(\alpha+\beta)$$
$$-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+6)!}[m^5+(5-15\frac{\beta}{\alpha})m^4-(10+60\frac{\beta}{\alpha}-105(\frac{\beta}{\alpha})^2)m^3-(50-105\frac{\beta}{\alpha}$$
$$-315(\frac{\beta}{\alpha})^2+420(\frac{\beta}{\alpha})^3)m^2+(9+330\frac{\beta}{\alpha}-420(\frac{\beta}{\alpha})^2-840(\frac{\beta}{\alpha})^3+945(\frac{\beta}{\alpha})^4)m$$
$$+45(1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})](\alpha+\beta)^{m+1},$$

$$U_{60} = \alpha(\frac{8}{105}-\frac{103}{80}(\frac{\beta}{\alpha})^2+\frac{13}{4}(\frac{\beta}{\alpha})^4-\frac{33}{16}(\frac{\beta}{\alpha})^6)-\frac{\beta}{\alpha}(\frac{5}{16}-\frac{15}{8}(\frac{\beta}{\alpha})^2+\frac{33}{16}(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})(\alpha+\beta)$$
$$\times\ln(\alpha+\beta)+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+7)!}[m^6+(6-21\frac{\beta}{\alpha})m^5-(20+105\frac{\beta}{\alpha}-210(\frac{\beta}{\alpha})^2)m^4$$
$$-(120-315\frac{\beta}{\alpha}-840(\frac{\beta}{\alpha})^2+1260(\frac{\beta}{\alpha})^3)m^3+(64+1365\frac{\beta}{\alpha}-2100(\frac{\beta}{\alpha})^2-3780(\frac{\beta}{\alpha})^3$$
$$+4725(\frac{\beta}{\alpha})^4)m^2+(384-609\frac{\beta}{\alpha}-5880(\frac{\beta}{\alpha})^2+6930(\frac{\beta}{\alpha})^3+9450(\frac{\beta}{\alpha})^4-10395(\frac{\beta}{\alpha})^5)m$$
$$-\frac{\beta}{\alpha}(1575-9450(\frac{\beta}{\alpha})^2+10395(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})](\alpha+\beta)^{m+1},.$$

$$U_{70} = -i\beta\left(\frac{1873}{4480} - \frac{2039}{640}\left(\frac{\beta}{\alpha}\right)^2 + \frac{781}{128}\left(\frac{\beta}{\alpha}\right)^4 - \frac{429}{128}\left(\frac{\beta}{\alpha}\right)^6\right) - \frac{i}{128}\left(5-135\left(\frac{\beta}{\alpha}\right)^2+495\left(\frac{\beta}{\alpha}\right)^4-429\left(\frac{\beta}{\alpha}\right)^6\right)$$

$$\times\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\ln(\alpha+\beta)+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+8)!}\left[m^7+\left(7-28\frac{\beta}{\alpha}\right)m^6-\left(35+168\frac{\beta}{\alpha}\right.\right.$$

$$\left.-378\left(\frac{\beta}{\alpha}\right)^2\right)m^5-\left(245-770\frac{\beta}{\alpha}-1890\left(\frac{\beta}{\alpha}\right)^2+3150\left(\frac{\beta}{\alpha}\right)^3\right)m^4+\left(259+4200\frac{\beta}{\alpha}-7560\left(\frac{\beta}{\alpha}\right)^2\right.$$

$$\left.-12600\left(\frac{\beta}{\alpha}\right)^3+17325\left(\frac{\beta}{\alpha}\right)^4\right)m^3+\left(1813-3892\frac{\beta}{\alpha}-30240\left(\frac{\beta}{\alpha}\right)^2+40950\left(\frac{\beta}{\alpha}\right)^3+51975\left(\frac{\beta}{\alpha}\right)^4\right.$$

$$\left.-62370\left(\frac{\beta}{\alpha}\right)^5\right)m^2-\left(225+2079\frac{\beta}{\alpha}-21357\left(\frac{\beta}{\alpha}\right)^2-107100\left(\frac{\beta}{\alpha}\right)^3+121275\left(\frac{\beta}{\alpha}\right)^4+124740\left(\frac{\beta}{\alpha}\right)^5\right.$$

$$\left.\left.-135135\left(\frac{\beta}{\alpha}\right)^6\right)m-315\left(5-135\left(\frac{\beta}{\alpha}\right)^2+495\left(\frac{\beta}{\alpha}\right)^4-429\left(\frac{\beta}{\alpha}\right)^6\right)\left(1-\frac{\beta}{\alpha}\right)\right](\alpha+\beta)^{m+1},$$

(A.62)

$$U_{01} = -\frac{1}{2}+(\alpha+\beta)+\left[\frac{1}{2}(\gamma+\ln(\alpha+\beta))-\frac{3}{4}\right](\alpha+\beta)^2+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)!}(\alpha+\beta)^{m+2},$$

$$U_{11} = -\frac{i}{3\alpha}+\frac{i}{2}\frac{\beta}{\alpha}+\frac{i}{2}\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)+i\left[\left(\frac{1}{3}-\frac{1}{6}\frac{\beta}{\alpha}\right)(\gamma+\ln(\alpha+\beta))-\frac{4}{9}+\frac{11}{36}\frac{\beta}{\alpha}\right](\alpha+\beta)^2$$

$$+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+3)!}\left(m+2-\frac{\beta}{\alpha}\right)(\alpha+\beta)^{m+2},$$

$$U_{21} = \frac{3}{4\alpha^2}-\frac{\beta}{\alpha^2}-\frac{1}{4}+\frac{3}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{1}{2}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)-\left[\frac{1}{8}\left(1-\frac{\beta}{\alpha}\right)^2\ln(\alpha+\beta)+\frac{3}{32}-\frac{19}{48}\frac{\beta}{\alpha}\right.$$

$$\left.+\frac{25}{96}\left(\frac{\beta}{\alpha}\right)^2\right](\alpha+\beta)^2-\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+4)!}\left[m^2+\left(4-3\frac{\beta}{\alpha}\right)m+3\left(1-\frac{\beta}{\alpha}\right)^2\right](\alpha+\beta)^{m+2},$$

$$U_{31} = \frac{3}{\alpha^3}i-\frac{15}{4}\frac{\beta}{\alpha^3}i-\frac{i}{2\alpha}\left(1-5\left(\frac{\beta}{\alpha}\right)^2\right)+\frac{i}{4}\frac{\beta}{\alpha}\left(3-5\left(\frac{\beta}{\alpha}\right)^2\right)+\frac{i}{8}\left(1-\frac{\beta}{\alpha}\right)\left(1-5\left(\frac{\beta}{\alpha}\right)^2\right)(\alpha+\beta)$$

$$+i\left[\frac{1}{8}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^2\ln(\alpha+\beta)+\frac{1}{15}+\frac{41}{480}\frac{\beta}{\alpha}-\frac{107}{240}\left(\frac{\beta}{\alpha}\right)^2+\frac{137}{480}\left(\frac{\beta}{\alpha}\right)^3\right](\alpha+\beta)^2$$

$$-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+5)!}\left[m^3+6\left(1-\frac{\beta}{\alpha}\right)m^2+\left(8-24\frac{\beta}{\alpha}+15\left(\frac{\beta}{\alpha}\right)^2\right)m-15\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^2\right](\alpha+\beta)^{m+2},$$

$$U_{41} = -\frac{35}{2\alpha^4}+21\frac{\beta}{\alpha^4}+\frac{15}{8\alpha^2}\left(1-7\left(\frac{\beta}{\alpha}\right)^2\right)-\frac{5}{2}\frac{\beta}{\alpha^2}\left(1-\frac{7}{3}\left(\frac{\beta}{\alpha}\right)^2\right)-\frac{3}{16}+\frac{15}{8}\left(\frac{\beta}{\alpha}\right)^2-\frac{35}{16}\left(\frac{\beta}{\alpha}\right)^4$$

$$+\frac{1}{8}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)\left(3-7\left(\frac{\beta}{\alpha}\right)^2\right)(\alpha+\beta)-\left[\frac{1}{48\alpha}\left(1-\frac{\beta}{\alpha}\right)^2\left(1-7\left(\frac{\beta}{\alpha}\right)^2\right)\ln(\alpha+\beta)+\frac{23}{576}-\frac{277}{1440}\frac{\beta}{\alpha}\right.$$

$$\left.-\frac{9}{160}\left(\frac{\beta}{\alpha}\right)^2+\frac{91}{160}\left(\frac{\beta}{\alpha}\right)^3-\frac{343}{960}\left(\frac{\beta}{\alpha}\right)^4\right](\alpha+\beta)^2+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+6)!}\left[m^4+\left(8-10\frac{\beta}{\alpha}\right)m^3\right.$$

$$\left.+\left(14-60\frac{\beta}{\alpha}+45\left(\frac{\beta}{\alpha}\right)^2\right)m^2+\left(8+65\frac{\beta}{\alpha}-180\left(\frac{\beta}{\alpha}\right)^2+105\left(\frac{\beta}{\alpha}\right)^3\right)m+15\left(1-\frac{\beta}{\alpha}\right)^2\left(1-7\left(\frac{\beta}{\alpha}\right)^2\right)\right](\alpha+\beta)^{m+2},$$

$$U_{51} = -\frac{135}{\alpha^5}i+\frac{315}{2}\frac{\beta}{\alpha^5}i+\frac{21}{2\alpha^3}i(1-9(\frac{\beta}{\alpha})^2)-\frac{105}{8}\frac{\beta}{\alpha^3}i(1-3(\frac{\beta}{\alpha})^2)-\frac{5}{8\alpha}i(1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4)$$

$$+\frac{i}{16}\frac{\beta}{\alpha}(15-70(\frac{\beta}{\alpha})^2+63(\frac{\beta}{\alpha})^4)+\frac{i}{16}(1-\frac{\beta}{\alpha})(1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4)(\alpha+\beta)$$

$$+i[\frac{1}{16}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-3(\frac{\beta}{\alpha})^2)\ln(\alpha+\beta)-\frac{2}{105}-\frac{689}{6720}\frac{\beta}{\alpha}+\frac{1439}{3360}(\frac{\beta}{\alpha})^2-\frac{31}{3360}(\frac{\beta}{\alpha})^3-\frac{879}{1120}(\frac{\beta}{\alpha})^4$$

$$+\frac{1089}{2240}(\frac{\beta}{\alpha})^5](\alpha+\beta)^2+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+7)!}[m^5+(10-15\frac{\beta}{\alpha})m^4+(20-120\frac{\beta}{\alpha}+105(\frac{\beta}{\alpha})^2)m^3$$

$$-(40+165\frac{\beta}{\alpha}+630(\frac{\beta}{\alpha})^2-420(\frac{\beta}{\alpha})^3)m^2-(96-300\frac{\beta}{\alpha}-525(\frac{\beta}{\alpha})^2+1680(\frac{\beta}{\alpha})^3-945(\frac{\beta}{\alpha})^4)m$$

$$+315\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-3(\frac{\beta}{\alpha})^2)](\alpha+\beta)^{m+2},$$

$$U_{61} = \frac{10395}{8\alpha^6}-1485\frac{\beta}{\alpha^6}-\frac{315}{4\alpha^4}(1-11(\frac{\beta}{\alpha})^2)+\frac{63}{2}\frac{\beta}{\alpha^4}(3-11(\frac{\beta}{\alpha})^2)+\frac{105}{32\alpha^2}(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)$$

$$-\frac{7}{8}\frac{\beta}{\alpha^2}(5-30(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)-\frac{5}{32}+\frac{105}{32}(\frac{\beta}{\alpha})^2-\frac{315}{32}(\frac{\beta}{\alpha})^4+\frac{231}{32}(\frac{\beta}{\alpha})^6+\frac{1}{16}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})$$

$$\times(5-30(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)(\alpha+\beta)-[\frac{1}{128}(1-\frac{\beta}{\alpha})^2(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)\ln(\alpha+\beta)$$

$$+\frac{19}{1024}-\frac{5671}{53760}\frac{\beta}{\alpha}-\frac{20891}{107520}(\frac{\beta}{\alpha})^2+\frac{7829}{8960}(\frac{\beta}{\alpha})^3-\frac{1077}{7168}(\frac{\beta}{\alpha})^4-\frac{20493}{17920}(\frac{\beta}{\alpha})^5+\frac{25113}{35840}(\frac{\beta}{\alpha})^6](\alpha+\beta)^2$$

$$-\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+8)!}[m^6+(12-21\frac{\beta}{\alpha})m^5+(25-210\frac{\beta}{\alpha}+210(\frac{\beta}{\alpha})^2)m^4-(120+315\frac{\beta}{\alpha}-1680(\frac{\beta}{\alpha})^2$$

$$+1260(\frac{\beta}{\alpha})^3)m^3-(341-1470\frac{\beta}{\alpha}-1680(\frac{\beta}{\alpha})^2+7560(\frac{\beta}{\alpha})^3-4725(\frac{\beta}{\alpha})^4)m^2$$

$$+(108+2541\frac{\beta}{\alpha}-6720(\frac{\beta}{\alpha})^2-4410(\frac{\beta}{\alpha})^3+18900(\frac{\beta}{\alpha})^4-10395(\frac{\beta}{\alpha})^5)m$$

$$+(1-\frac{\beta}{\alpha})^2(315-5670(\frac{\beta}{\alpha})^2+10395(\frac{\beta}{\alpha})^4]](\alpha+\beta)^{m+2},$$

$$U_{71} = \frac{15015}{\alpha^7}i-\frac{135135}{8}\frac{\beta}{\alpha^7}i-\frac{1485}{2\alpha^5}i(1-13(\frac{\beta}{\alpha})^2)+\frac{1155}{4}\frac{\beta}{\alpha^5}i(3-13(\frac{\beta}{\alpha})^2)$$

$$+\frac{63}{8\alpha^3}i(3-66(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4)-\frac{63}{32}\frac{\beta}{\alpha^3}i(15-110(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4)-\frac{7}{48\alpha}i(5-135(\frac{\beta}{\alpha})^2$$

$$+495(\frac{\beta}{\alpha})^4-429(\frac{\beta}{\alpha})^6)+\frac{i}{32}\frac{\beta}{\alpha}(35-315(\frac{\beta}{\alpha})^2+693(\frac{\beta}{\alpha})^4-429(\frac{\beta}{\alpha})^6)+\frac{i}{128}(1-\frac{\beta}{\alpha})$$

$$\times(5-135(\frac{\beta}{\alpha})^2+495(\frac{\beta}{\alpha})^4-429(\frac{\beta}{\alpha})^6)(\alpha+\beta)+i[\frac{5}{128}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4)\ln(\alpha+\beta)$$

$$-\frac{1024}{120960}-\frac{80951}{967680}\frac{\beta}{\alpha}+\frac{58913}{161280}(\frac{\beta}{\alpha})^2+\frac{61531}{193536}(\frac{\beta}{\alpha})^3-\frac{83039}{48384}(\frac{\beta}{\alpha})^4+\frac{143341}{322560}(\frac{\beta}{\alpha})^5+\frac{839267}{483840}(\frac{\beta}{\alpha})^6$$

$$-\frac{1019447}{967680}(\frac{\beta}{\alpha})^7](\alpha+\beta)^2-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+9)!}[m^7+(14-28\frac{\beta}{\alpha})m^6+(28-336\frac{\beta}{\alpha}+378(\frac{\beta}{\alpha})^2)m^5$$

$$-(280+490\frac{\beta}{\alpha}-3780(\frac{\beta}{\alpha})^2+3150(\frac{\beta}{\alpha})^3)m^4-(896-5040\frac{\beta}{\alpha}-3780(\frac{\beta}{\alpha})^2+25200(\frac{\beta}{\alpha})^3$$

$$-17325(\frac{\beta}{\alpha})^4)m^3+(896+11228\frac{\beta}{\alpha}-37800(\frac{\beta}{\alpha})^2-15750(\frac{\beta}{\alpha})^3+103950(\frac{\beta}{\alpha})^4-62370(\frac{\beta}{\alpha})^5)m^2$$

$$+(3072-9744\frac{\beta}{\alpha}-52353(\frac{\beta}{\alpha})^2+138600(\frac{\beta}{\alpha})^3+34650(\frac{\beta}{\alpha})^4-249480(\frac{\beta}{\alpha})^5+135135(\frac{\beta}{\alpha})^6)m$$

$$+\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(14175-103950(\frac{\beta}{\alpha})^2+135135(\frac{\beta}{\alpha})^4)](\alpha+\beta)^{m+2}, \qquad (A.63)$$

$$U_{02}=\frac{1}{2}[1-\frac{1}{2}(\alpha+\beta)^2](\alpha+\beta)[\gamma+\ell n(\alpha+\beta)]+\frac{1}{4}[1-4(\alpha+\beta)+2(\alpha+\beta)^2](\alpha+\beta)$$

$$-\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+2)!}(\alpha+\beta)^{m+3},$$

$$U_{12}=\frac{i}{4}[1-\frac{\beta}{\alpha}-\frac{1}{4}(3-\frac{\beta}{\alpha})(\alpha+\beta)^2](\alpha+\beta)[\gamma+\ell n(\alpha+\beta)]-\frac{i}{8\alpha}+i[\frac{1}{4}-(\frac{2}{3}-\frac{1}{3}\frac{\beta}{\alpha})(\alpha+\beta)$$

$$+(\frac{23}{64}-\frac{9}{64}\frac{\beta}{\alpha})(\alpha+\beta)^2](\alpha+\beta)-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+2)!(m+4)}(m+3-\frac{\beta}{\alpha})(\alpha+\beta)^{m+3},$$

$$U_{22}=[\frac{1}{4}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})+\frac{1}{10}(1-\frac{9}{8}\frac{\beta}{\alpha}+\frac{3}{8}(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2](\alpha+\beta)[\gamma+\ell n(\alpha+\beta)]+\frac{2}{5\alpha^2}-\frac{3}{8}\frac{\beta}{\alpha^2}$$

$$-[\frac{1}{6}(1-\frac{\beta}{\alpha}-\frac{1}{2}(\frac{\beta}{\alpha})^2)-\frac{1}{4}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)+\frac{1}{100}(17-\frac{381}{16}\frac{\beta}{\alpha}+\frac{147}{16}(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2](\alpha+\beta)$$

$$+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+2)!(m+4)(m+5)}[m^2+(6-3\frac{\beta}{\alpha})m+8-9\frac{\beta}{\alpha}+3(\frac{\beta}{\alpha})^2](\alpha+\beta)^{m+3},$$

$$U_{32}=i[\frac{1}{16}(1-5(\frac{\beta}{\alpha})^2)+\frac{1}{32}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2](1-\frac{\beta}{\alpha})(\alpha+\beta)\ell n(\alpha+\beta)+\frac{15}{8\alpha^3}i-2\frac{\beta}{\alpha^3}i$$

$$-\frac{3}{16\alpha}i(1-5(\frac{\beta}{\alpha})^2)-i[\frac{1}{64}-\frac{55}{192}\frac{\beta}{\alpha}+\frac{25}{192}(\frac{\beta}{\alpha})^2+\frac{35}{192}(\frac{\beta}{\alpha})^3+\frac{1}{4}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)$$

$$+\frac{1}{128}(\frac{13}{3}-\frac{109}{5}\frac{\beta}{\alpha}+\frac{137}{5}(\frac{\beta}{\alpha})^2-\frac{157}{15}(\frac{\beta}{\alpha})^3)(\alpha+\beta)^2](\alpha+\beta)+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+2)!}$$

$$\times\frac{1}{(m+4)(m+5)(m+6)}[m^3+(9-6\frac{\beta}{\alpha})m^2+(23-36\frac{\beta}{\alpha}+15(\frac{\beta}{\alpha})^2)m+15(1-\frac{\beta}{\alpha})^3](\alpha+\beta)^{m+3},$$

$$U_{42}=[\frac{1}{16}(3-7(\frac{\beta}{\alpha})^2)+\frac{1}{32}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2]\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})(\alpha+\beta)\ell n(\alpha+\beta)-\frac{12}{\alpha^4}+\frac{105}{8}\frac{\beta}{\alpha^4}$$

$$+\frac{1}{\alpha^2}(1-7(\frac{\beta}{\alpha})^2)-\frac{15}{16}\frac{\beta}{\alpha^2}(1-7(\frac{\beta}{\alpha})^2)-[\frac{1}{15}+\frac{41}{960}\frac{\beta}{\alpha}-\frac{167}{320}(\frac{\beta}{\alpha})^2+\frac{91}{960}(\frac{\beta}{\alpha})^3+\frac{329}{960}(\frac{\beta}{\alpha})^4$$

$$-\frac{1}{24}(1-7(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})^2(\alpha+\beta)+\frac{1}{70}(1+\frac{299}{192}\frac{\beta}{\alpha}-\frac{2397}{192}(\frac{\beta}{\alpha})^2+\frac{3057}{192}(\frac{\beta}{\alpha})^3$$

$$-\frac{1159}{192}(\frac{\beta}{\alpha})^4)(\alpha+\beta)^2]-\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+2)!(m+4)\ldots(m+6)(m+7)}[m^4+(12-10\frac{\beta}{\alpha})m^3$$

$$+(44-90\frac{\beta}{\alpha}+45(\frac{\beta}{\alpha})^2)m^2+(48-215\frac{\beta}{\alpha}+270(\frac{\beta}{\alpha})^2-105(\frac{\beta}{\alpha})^3)m-105\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^3](\alpha+\beta)^{m+3},$$

$$U_{52}=\frac{i}{32}[1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4+(\frac{1}{8}-\frac{9}{8}(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})^2(\alpha+\beta)^2](1-\frac{\beta}{\alpha})(\alpha+\beta)\ell n(\alpha+\beta)-\frac{1575}{16\alpha^5}i$$

$$+108\frac{\beta}{\alpha^5}i+\frac{105}{16\alpha^3}i(1-9(\frac{\beta}{\alpha})^2)-7\frac{\beta}{\alpha^3}i(1-3(\frac{\beta}{\alpha})^2)-\frac{15}{64\alpha}i(1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4)$$

$$-\frac{i}{8}[\frac{3}{16}-\frac{497}{240}\frac{\beta}{\alpha}-\frac{119}{120}(\frac{\beta}{\alpha})^2+\frac{959}{120}(\frac{\beta}{\alpha})^3-\frac{21}{80}(\frac{\beta}{\alpha})^4-\frac{399}{80}(\frac{\beta}{\alpha})^5+(1-3(\frac{\beta}{\alpha})^2)\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(\alpha+\beta)$$

$$+\frac{1}{128}(\frac{53}{6}-\frac{3623}{70}\frac{\beta}{\alpha}+\frac{377}{35}(\frac{\beta}{\alpha})^2+\frac{21419}{105}(\frac{\beta}{\alpha})^3-\frac{19287}{70}(\frac{\beta}{\alpha})^4+\frac{7269}{70}(\frac{\beta}{\alpha})^5)(\alpha+\beta)^2]$$

$$-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+2)!(m+4)\ldots(m+7)(m+8)}[m^5+15(1-\frac{\beta}{\alpha})m^4+(70-180\frac{\beta}{\alpha}+105(\frac{\beta}{\alpha})^2)m^3$$

$$+(90-615\frac{\beta}{\alpha}+945(\frac{\beta}{\alpha})^2-420(\frac{\beta}{\alpha})^3)m^2-(71+450\frac{\beta}{\alpha}-2100(\frac{\beta}{\alpha})^2+2520(\frac{\beta}{\alpha})^3-945(\frac{\beta}{\alpha})^4)m$$

$$-105(1-9(\frac{\beta}{\alpha})^2)(1-\frac{\beta}{\alpha})^3](\alpha+\beta)^{m+3},$$

$$U_{62}=\frac{1}{32}[5-30(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4+\frac{1}{8}(1-\frac{\beta}{\alpha})^2(3-11(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2]\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})(\alpha+\beta)\ell n(\alpha+\beta)$$

$$+\frac{990}{\alpha^6}-\frac{17325}{16}\frac{\beta}{\alpha^6}-\frac{54}{\alpha^4}(1-11(\frac{\beta}{\alpha})^2)+\frac{315}{16}\frac{\beta}{\alpha^4}(3-11(\frac{\beta}{\alpha})^2)+\frac{7}{4\alpha^2}(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)$$

$$-\frac{21}{64}\frac{\beta}{\alpha^2}(5-30(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)-[\frac{4}{105}+\frac{1483}{13440}\frac{\beta}{\alpha}-\frac{2027}{2688}(\frac{\beta}{\alpha})^2-\frac{151}{448}(\frac{\beta}{\alpha})^3+\frac{879}{448}(\frac{\beta}{\alpha})^4$$

$$+\frac{429}{4480}(\frac{\beta}{\alpha})^5-\frac{5049}{4480}(\frac{\beta}{\alpha})^6-\frac{1}{64}(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)(1-\frac{\beta}{\alpha})^2(\alpha+\beta)+\frac{1}{105}(\frac{1}{3}+\frac{4357}{2048}\frac{\beta}{\alpha}$$

$$-\frac{24887}{2048}(\frac{\beta}{\alpha})^2+\frac{23531}{3072}(\frac{\beta}{\alpha})^3+\frac{5029}{2048}(\frac{\beta}{\alpha})^4-\frac{73799}{20480}(\frac{\beta}{\alpha})^5+\frac{83039}{6144}(\frac{\beta}{\alpha})^6)(\alpha+\beta)^2]$$

$$+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+2)!(m+4)\ldots(m+8)(m+9)}[m^6+(18-21\frac{\beta}{\alpha})m^5+(100-315\frac{\beta}{\alpha}+210(\frac{\beta}{\alpha})^2)m^4$$

$$+(120-1365\frac{\beta}{\alpha}+2520(\frac{\beta}{\alpha})^2-1260(\frac{\beta}{\alpha})^3)m^3-(416+945\frac{\beta}{\alpha}-7980(\frac{\beta}{\alpha})^2+11340(\frac{\beta}{\alpha})^3-4725(\frac{\beta}{\alpha})^4)m^2$$

$$-(768-3591\frac{\beta}{\alpha}-2520(\frac{\beta}{\alpha})^2+23310(\frac{\beta}{\alpha})^3-28350(\frac{\beta}{\alpha})^4+10395(\frac{\beta}{\alpha})^5)m$$

$$+945(3-11(\frac{\beta}{\alpha})^2)\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^3](\alpha+\beta)^{m+3},$$

$$U_{72} = \frac{i}{256}[5-135(\frac{\beta}{\alpha})^2+495(\frac{\beta}{\alpha})^4-429(\frac{\beta}{\alpha})^6+(\frac{3}{10} - \frac{33}{5}(\frac{\beta}{\alpha})^2+ \frac{143}{10}(\frac{\beta}{\alpha})^4)(1- \frac{\beta}{\alpha})^2(\alpha+\beta)^2](1- \frac{\beta}{\alpha})$$

$$\times(\alpha+\beta)\ell n(\alpha+\beta)+ \frac{189189}{16\alpha^7} i-12870i \frac{\beta}{\alpha^7} - \frac{17325}{32\alpha^5} i(1-13(\frac{\beta}{\alpha})^2)+ \frac{\beta}{\alpha^5} i(594-2574(\frac{\beta}{\alpha})^2)$$

$$+ \frac{315}{64\alpha^3} i(3-66(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4)- \frac{21}{4} \frac{\beta}{\alpha^3} i(3-22(\frac{\beta}{\alpha})^2+ \frac{143}{5}(\frac{\beta}{\alpha})^4)- \frac{7i}{128\alpha}(5-135(\frac{\beta}{\alpha})^2$$

$$+495(\frac{\beta}{\alpha})^4-429(\frac{\beta}{\alpha})^6)-i[\frac{1}{2048}(43- \frac{16489}{35} \frac{\beta}{\alpha} - \frac{27051}{35}(\frac{\beta}{\alpha})^2+ \frac{28247}{7}(\frac{\beta}{\alpha})^3+ \frac{12199}{7}(\frac{\beta}{\alpha})^4$$

$$- \frac{55935}{7}(\frac{\beta}{\alpha})^5- \frac{26169}{35}(\frac{\beta}{\alpha})^6+\frac{146289}{35}(\frac{\beta}{\alpha})^7)+ \frac{1}{32}(\frac{5}{2} - \frac{55}{3}(\frac{\beta}{\alpha})^2+ \frac{143}{6}(\frac{\beta}{\alpha})^4)\frac{\beta}{\alpha}(1- \frac{\beta}{\alpha})^2(\alpha+\beta)$$

$$+ \frac{1}{20480} (\frac{317}{5} - \frac{134273}{315} \frac{\beta}{\alpha} - \frac{59683}{105}(\frac{\beta}{\alpha})^2+ \frac{98053}{21}(\frac{\beta}{\alpha})^3- \frac{242891}{63}(\frac{\beta}{\alpha})^4- \frac{611281}{105}(\frac{\beta}{\alpha})^5$$

$$+ \frac{995423}{105}(\frac{\beta}{\alpha})^6- \frac{1115543}{315}(\frac{\beta}{\alpha})^7)(\alpha+\beta)^2]+i \sum_{m=1}^{\infty} (-1)^m \frac{1}{m(m+2)(m+2)!(m+4)\ldots(m+9)(m+10)}$$

$$\times[m^7+(21-28 \frac{\beta}{\alpha})m^6+(133-504 \frac{\beta}{\alpha} +378(\frac{\beta}{\alpha})^2)m^5+(105-2590 \frac{\beta}{\alpha} +5670(\frac{\beta}{\alpha})^2-3150(\frac{\beta}{\alpha})^3)m^4$$

$$-(1421+840 \frac{\beta}{\alpha} -22680(\frac{\beta}{\alpha})^2+37800(\frac{\beta}{\alpha})^3-17325(\frac{\beta}{\alpha})^4)m^3-(2961-19628 \frac{\beta}{\alpha} +110250(\frac{\beta}{\alpha})^2$$

$$-155925(\frac{\beta}{\alpha})^4+62370(\frac{\beta}{\alpha})^5)m^2+(1287+24024 \frac{\beta}{\alpha} -99603(\frac{\beta}{\alpha})^2+18900(\frac{\beta}{\alpha})^3+294525(\frac{\beta}{\alpha})^4$$

$$-374220(\frac{\beta}{\alpha})^5+135135(\frac{\beta}{\alpha})^6)m+945(3-66(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4)(1- \frac{\beta}{\alpha})^3](\alpha+\beta)^{m+3}, \tag{A.64}$$

$$U_{03} = - \frac{1}{4}[3- \frac{5}{12}(\alpha+\beta)^2](\alpha+\beta)^2[\gamma+\ell n(\alpha+\beta)]+ \frac{1}{8} - \frac{2}{3}(\alpha+\beta)+ \frac{1}{2}(\alpha+\beta)^2+ \frac{2}{3}(\alpha+\beta)^3$$

$$- \frac{143}{576}(\alpha+\beta)^4+ \sum_{m=1}^{\infty} (-1)^m \frac{1}{m(m+2)(m+4)!}(m+5)(\alpha+\beta)^{m+4},$$

$$U_{13} = - \frac{i}{2}[1- \frac{1}{2} \frac{\beta}{\alpha} -(\frac{1}{6} - \frac{1}{24} \frac{\beta}{\alpha})(\alpha+\beta)^2](\alpha+\beta)^2[\gamma+\ell n(\alpha+\beta)]- \frac{i}{8} \frac{\beta}{\alpha} - \frac{i}{3}(1- \frac{\beta}{\alpha})(\alpha+\beta)$$

$$+ \frac{i}{4}(1- \frac{\beta}{\alpha})(\alpha+\beta)^2+ \frac{i}{2}(1- \frac{1}{3} \frac{\beta}{\alpha})(\alpha+\beta)^3- \frac{i}{36}(7- \frac{31}{16} \frac{\beta}{\alpha})(\alpha+\beta)^4$$

$$+i \sum_{m=1}^{\infty} (-1)^m \frac{1}{m(m+2)(m+4)!}(m+4- \frac{\beta}{\alpha})(\alpha+\beta)^{m+4},$$

$$U_{23} = [\frac{3}{16}(1- \frac{\beta}{\alpha})^2- \frac{1}{96}(5-4 \frac{\beta}{\alpha} +(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2](\alpha+\beta)^2[\gamma+\ell n(\alpha+\beta)]+ \frac{1}{8}$$

$$+ \frac{1}{16}[1-3(\frac{\beta}{\alpha})^2]- \frac{\beta}{3\alpha}(1- \frac{\beta}{\alpha})(\alpha+\beta)+ \frac{1}{64}(1+18 \frac{\beta}{\alpha} -15(\frac{\beta}{\alpha})^2](\alpha+\beta)^2$$

$$-\frac{1}{10}\Big[\frac{8}{3}-3\frac{\beta}{\alpha}+\Big(\frac{\beta}{\alpha}\Big)^2\Big](\alpha+\beta)^3+\frac{1}{48}\Big[\frac{133}{24}-5\frac{\beta}{\alpha}+\frac{11}{8}\Big(\frac{\beta}{\alpha}\Big)^2\Big](\alpha+\beta)^4$$

$$-\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+4)!(m+6)}\Big[m^2+\Big(8-3\frac{\beta}{\alpha}\Big)m+15-12\frac{\beta}{\alpha}+3\Big(\frac{\beta}{\alpha}\Big)^2\Big](\alpha+\beta)^{m+4},$$

$$U_{33}=-i\Big[\frac{3}{16}\frac{\beta}{\alpha}\Big(1-\frac{\beta}{\alpha}\Big)^2+\frac{1}{42}\Big(1-\frac{29}{16}\frac{\beta}{\alpha}+\frac{5}{4}\Big(\frac{\beta}{\alpha}\Big)^2-\frac{5}{16}\Big(\frac{\beta}{\alpha}\Big)^3\Big)(\alpha+\beta)^2\Big](\alpha+\beta)^2[\gamma+\ln(\alpha+\beta)]$$

$$+\frac{6}{7\alpha^3}i-\frac{5\beta}{8\alpha^3}i-\frac{i}{16}\frac{\beta}{\alpha}\Big[3-5\Big(\frac{\beta}{\alpha}\Big)^2\Big]-\frac{i}{12}\Big[1-5\Big(\frac{\beta}{\alpha}\Big)^2\Big]\Big(1-\frac{\beta}{\alpha}\Big)(\alpha+\beta)+\frac{i}{10}\Big[1-\frac{9}{32}\frac{\beta}{\alpha}-\frac{57}{16}\Big(\frac{\beta}{\alpha}\Big)^2$$

$$+\frac{87}{32}\Big(\frac{\beta}{\alpha}\Big)^3\Big](\alpha+\beta)^2-\frac{i}{12}\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta)^3+\frac{i}{588}\Big[\frac{83}{3}-\frac{1901}{32}\frac{\beta}{\alpha}+\frac{185}{4}\Big(\frac{\beta}{\alpha}\Big)^2-\frac{405}{32}\Big(\frac{\beta}{\alpha}\Big)^3\Big](\alpha+\beta)^4$$

$$-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+4)!(m+6)(m+7)}\Big[m^3+\Big(12-6\frac{\beta}{\alpha}\Big)m^2+\Big(44-48\frac{\beta}{\alpha}+15\Big(\frac{\beta}{\alpha}\Big)^2\Big)m$$

$$+48-87\frac{\beta}{\alpha}+60\Big(\frac{\beta}{\alpha}\Big)^2-15\Big(\frac{\beta}{\alpha}\Big)^3\Big](\alpha+\beta)^{m+4},$$

$$U_{43}=\frac{1}{32}\Big(1-\frac{\beta}{\alpha}\Big)^2\Big[1-7\Big(\frac{\beta}{\alpha}\Big)^2+\frac{5}{24}\Big(1-\frac{\beta}{\alpha}\Big)^2(\alpha+\beta)^2\Big](\alpha+\beta)^2\ln(\alpha+\beta)-\frac{105}{16\alpha^4}+6\frac{\beta}{\alpha^4}$$

$$+\frac{5}{16\alpha^2}\Big[1-7\Big(\frac{\beta}{\alpha}\Big)^2\Big]+\frac{1}{64}\Big[3-30\Big(\frac{\beta}{\alpha}\Big)^2+35\Big(\frac{\beta}{\alpha}\Big)^4\Big]-\frac{1}{4}\frac{\beta}{\alpha}\Big[1-\frac{7}{3}\Big(\frac{\beta}{\alpha}\Big)^2\Big]\Big(1-\frac{\beta}{\alpha}\Big)(\alpha+\beta)$$

$$-\frac{1}{192}\Big[\frac{13}{2}-\frac{227}{5}\frac{\beta}{\alpha}+\frac{69}{5}\Big(\frac{\beta}{\alpha}\Big)^2+\frac{469}{5}\Big(\frac{\beta}{\alpha}\Big)^3-\frac{679}{10}\Big(\frac{\beta}{\alpha}\Big)^4\Big](\alpha+\beta)^2-\frac{1}{12}\frac{\beta}{\alpha}\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta)^3$$

$$-\frac{1}{512}\Big[\frac{161}{36}-\frac{1685}{63}\frac{\beta}{\alpha}+\frac{685}{14}\Big(\frac{\beta}{\alpha}\Big)^2-\frac{775}{21}\Big(\frac{\beta}{\alpha}\Big)^3+\frac{845}{84}\Big(\frac{\beta}{\alpha}\Big)^4\Big](\alpha+\beta)^4$$

$$+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+4)!(m+6)(m+7)(m+8)}\Big[m^4+\Big(16-10\frac{\beta}{\alpha}\Big)m^3+\Big(86-120\frac{\beta}{\alpha}+45\Big(\frac{\beta}{\alpha}\Big)^2\Big)m^2$$

$$+\Big(176-425\frac{\beta}{\alpha}+360\Big(\frac{\beta}{\alpha}\Big)^2-105\Big(\frac{\beta}{\alpha}\Big)^3\Big)+105\Big(1-\frac{\beta}{\alpha}\Big)^4\Big](\alpha+\beta)^{m+4},$$

$$U_{53}=-\frac{i}{32}\frac{\beta}{\alpha}\Big(1-\frac{\beta}{\alpha}\Big)^2\Big[3-9\Big(\frac{\beta}{\alpha}\Big)^2+\frac{5}{24}\Big(1-\frac{\beta}{\alpha}\Big)^2(\alpha+\beta)^2\Big](\alpha+\beta)^2\ln(\alpha+\beta)-\frac{60}{\alpha^5}i+\frac{945\beta}{16\alpha^5}i$$

$$+\frac{3i}{\alpha^3}\Big[1-9\Big(\frac{\beta}{\alpha}\Big)^2\Big]-\frac{35\beta}{16\alpha^3}i\Big[1-3\Big(\frac{\beta}{\alpha}\Big)^2\Big]-\frac{i}{64}\frac{\beta}{\alpha}\Big[15-70\Big(\frac{\beta}{\alpha}\Big)^2+63\Big(\frac{\beta}{\alpha}\Big)^4\Big]$$

$$-\frac{i}{24}\Big[1-14\Big(\frac{\beta}{\alpha}\Big)^2+21\Big(\frac{\beta}{\alpha}\Big)^4\Big]\Big(1-\frac{\beta}{\alpha}\Big)(\alpha+\beta)+\frac{i}{35}\Big[1+\frac{339}{128}\frac{\beta}{\alpha}-\frac{1089}{64}\Big(\frac{\beta}{\alpha}\Big)^2+\frac{381}{64}\Big(\frac{\beta}{\alpha}\Big)^3$$

$$+\frac{1587}{64}\Big(\frac{\beta}{\alpha}\Big)^4-\frac{2217}{128}\Big(\frac{\beta}{\alpha}\Big)^5\Big](\alpha+\beta)^2-\frac{i}{96}\Big[1-9\Big(\frac{\beta}{\alpha}\Big)^2\Big]\Big(1-\frac{\beta}{\alpha}\Big)^3(\alpha+\beta)^3+\frac{i}{378}\Big[1+\frac{1805}{1024}\frac{\beta}{\alpha}$$

$$-\frac{5015}{256}\Big(\frac{\beta}{\alpha}\Big)^2+\frac{19175}{512}\Big(\frac{\beta}{\alpha}\Big)^3-\frac{7255}{256}\Big(\frac{\beta}{\alpha}\Big)^4+\frac{7885}{1024}\Big(\frac{\beta}{\alpha}\Big)^5\Big](\alpha+\beta)^4+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)(m+4)!}$$

$$\times\frac{1}{(m+6)(m+7)\dots(m+9)}\Big[m^5+\Big(20-15\frac{\beta}{\alpha}\Big)m^4+\Big(140-240\frac{\beta}{\alpha}+105\Big(\frac{\beta}{\alpha}\Big)^2\Big)m^3+\Big(400-1245\frac{\beta}{\alpha}$$

$$+1260(\frac{\beta}{\alpha})^2-420(\frac{\beta}{\alpha})^3)m^2+(384-2280\ \frac{\beta}{\alpha}\ +4305(\frac{\beta}{\alpha})^2-3360(\frac{\beta}{\alpha})^3+945(\frac{\beta}{\alpha})^4)$$

$$-945\ \frac{\beta}{\alpha}(1-\ \frac{\beta}{\alpha})^4](\alpha+\beta)^{m+4},$$

$$U_{63}\ =\ \frac{1}{256}(1-\ \frac{\beta}{\alpha})^2[3-54(\frac{\beta}{\alpha})^2+99(\frac{\beta}{\alpha})^4+\ \frac{1}{6}(1-\ \frac{\beta}{\alpha})^2(1-11(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2](\alpha+\beta)^2\ln(\alpha+\beta)$$

$$+\ \frac{10395}{16\alpha^6}\ -660\ \frac{\beta}{\alpha^5}\ -\ \frac{945}{32\alpha^4}[1-11(\frac{\beta}{\alpha})^2]+9\ \frac{\beta}{\alpha^4}[3-11(\frac{\beta}{\alpha})^2]+\ \frac{35}{64\alpha^2}[1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4]$$

$$+\ \frac{1}{128}[5-105(\frac{\beta}{\alpha})^2+315(\frac{\beta}{\alpha})^4-231(\frac{\beta}{\alpha})^6]-\ \frac{1}{24}\ \frac{\beta}{\alpha}[5-30(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4](1-\ \frac{\beta}{\alpha})(\alpha+\beta)$$

$$-\ \frac{1}{512}[\frac{37}{4}\ -\ \frac{4971}{70}\ \frac{\beta}{\alpha}\ -\ \frac{8991}{140}(\frac{\beta}{\alpha})^2+\ \frac{17187}{35}(\frac{\beta}{\alpha})^3-\ \frac{5331}{28}(\frac{\beta}{\alpha})^4-\ \frac{38379}{10}(\frac{\beta}{\alpha})^5+\ \frac{52239}{140}(\frac{\beta}{\alpha})^6](\alpha+\beta)^2$$

$$-\ \frac{1}{96}\ \frac{\beta}{\alpha}[3-11(\frac{\beta}{\alpha})^2](1-\ \frac{\beta}{\alpha})^3(\alpha+\beta)^3-\ \frac{1}{9216}[\frac{293}{20}\ -\ \frac{10643}{105}\ \frac{\beta}{\alpha}\ +\ \frac{6199}{84}(\frac{\beta}{\alpha})^2+\ \frac{9638}{21}(\frac{\beta}{\alpha})^3$$

$$-\ \frac{85621}{84}(\frac{\beta}{\alpha})^4+\ \frac{82577}{105}(\frac{\beta}{\alpha})^5-\ \frac{89507}{420}(\frac{\beta}{\alpha})^6](\alpha+\beta)^4-\ \sum_{m=1}^{\infty}(-1)^m\ \frac{1}{m(m+2)(m+4)!}$$

$$\times\ \frac{1}{(m+6)(m+7)\ldots(m+10)}[m^6+(24-21\ \frac{\beta}{\alpha})m^5+(205-420\ \frac{\beta}{\alpha}\ +210(\frac{\beta}{\alpha})^2)m^4+(720-2835\ \frac{\beta}{\alpha}$$

$$+3360(\frac{\beta}{\alpha})^2-1260(\frac{\beta}{\alpha})^3)m^3+(739-7140\ \frac{\beta}{\alpha}\ +16800(\frac{\beta}{\alpha})^2-15120(\frac{\beta}{\alpha})^3+4725(\frac{\beta}{\alpha})^4)m^2$$

$$-(744+3759\ \frac{\beta}{\alpha}\ -26880(\frac{\beta}{\alpha})^2+49770(\frac{\beta}{\alpha})^3-37800(\frac{\beta}{\alpha})^4+10395(\frac{\beta}{\alpha})^5)m$$

$$-945(1-11(\frac{\beta}{\alpha})^2)(1-\ \frac{\beta}{\alpha})^4](\alpha+\beta)^{m+4},$$

$$U_{73}\ =\ -\ \frac{i}{256}\ \frac{\beta}{\alpha}(1-\ \frac{\beta}{\alpha})^2[15-110(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4+\ \frac{1}{6}(1-\ \frac{\beta}{\alpha})^2(3-13(\frac{\beta}{\alpha})^2)(\alpha+\beta)^2](\alpha+\beta)^2\ln(\alpha+\beta)$$

$$+\ \frac{8190}{\alpha^7}\ i-\ \frac{1351351\beta}{16\alpha^7}\ i-\ \frac{330i}{\alpha^5}[1-13(\frac{\beta}{\alpha})^2]+\ \frac{3465\beta}{32\alpha^5}i[3-13(\frac{\beta}{\alpha})^2]+\ \frac{9i}{4\alpha^3}[3-66(\frac{\beta}{\alpha})^2$$

$$+143(\frac{\beta}{\alpha})^4]-\ \frac{21\beta}{64\alpha^3}\ i[15-110(\frac{\beta}{\alpha})^2+143(\frac{\beta}{\alpha})^4]-\ \frac{i}{128}\ \frac{\beta}{\alpha}[35-315(\frac{\beta}{\alpha})^2+693(\frac{\beta}{\alpha})^4$$

$$-429(\frac{\beta}{\alpha})^6]-\ \frac{i}{64}[\frac{5}{3}\ -45(\frac{\beta}{\alpha})^2+165(\frac{\beta}{\alpha})^4-143(\frac{\beta}{\alpha})^6](1-\ \frac{\beta}{\alpha})(\alpha+\beta)+\ \frac{i}{21}[\frac{4}{15}\ +\ \frac{49451}{30720}\ \frac{\beta}{\alpha}$$

$$-\ \frac{48413}{5120}(\frac{\beta}{\alpha})^2-\ \frac{21631}{6144}(\frac{\beta}{\alpha})^3+\ \frac{59939}{1536}(\frac{\beta}{\alpha})^4-\ \frac{166441}{10240}(\frac{\beta}{\alpha})^5-\ \frac{538967}{15360}(\frac{\beta}{\alpha})^6+\ \frac{719147}{30720}(\frac{\beta}{\alpha})^7](\alpha+\beta)^2$$

$$-\ \frac{i}{320}[1-22(\frac{\beta}{\alpha})^2+\ \frac{143}{3}(\frac{\beta}{\alpha})^4](1-\ \frac{\beta}{\alpha})^3(\alpha+\beta)^3+\ \frac{i}{693}[\frac{1}{3}\ +\ \frac{148849}{61440}\ \frac{\beta}{\alpha}\ -\ \frac{90473}{5120}(\frac{\beta}{\alpha})^2$$

$$+\ \frac{266747}{12288}(\frac{\beta}{\alpha})^3+\ \frac{48137}{1536}(\frac{\beta}{\alpha})^4-\ \frac{371431}{4096}(\frac{\beta}{\alpha})^5+\ \frac{1106261}{15360}(\frac{\beta}{\alpha})^6-\ \frac{1196351}{61440}(\frac{\beta}{\alpha})^7](\alpha+\beta)^4$$

$$-i\sum_{m=1}^{\infty}(-1)^m \frac{1}{m(m+2)(m+4)!(m+6)(m+7)\ldots(m+11)}[m^7+28(1-\frac{\beta}{\alpha})m^6+(280-672\frac{\beta}{\alpha}$$

$$+378(\frac{\beta}{\alpha})^2)m^5+(1120-5530\frac{\beta}{\alpha}+7560(\frac{\beta}{\alpha})^2-3150(\frac{\beta}{\alpha})^3)m^4+(784-16800\frac{\beta}{\alpha}$$

$$+49140(\frac{\beta}{\alpha})^2-50400(\frac{\beta}{\alpha})^3+17325(\frac{\beta}{\alpha})^4)m^3-(4928+3892\frac{\beta}{\alpha}-105840(\frac{\beta}{\alpha})^2+242550(\frac{\beta}{\alpha})^3$$

$$-207900(\frac{\beta}{\alpha})^4+62370(\frac{\beta}{\alpha})^5)m^2-(7680-47712\frac{\beta}{\alpha}+6993(\frac{\beta}{\alpha})^2+327600(\frac{\beta}{\alpha})^3$$

$$-658350(\frac{\beta}{\alpha})^4+498960(\frac{\beta}{\alpha})^5-135135(\frac{\beta}{\alpha})^6)m+10395\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^4(3-13(\frac{\beta}{\alpha})^2)](\alpha+\beta)^{m+4}. \quad \text{(A.65)}$$

The series expansion for the expression (A.51) can easily be obtained from (A.62) ∿ (A.65) by regarding $V_{p\ell}(\alpha,\beta) = 0$ for all p and ℓ.

4. $C_{qa}{}^{\ell}(\alpha_j,s;d)$

The definition (11) leads to

$$C_{qa}{}^{\ell}(\alpha_j,s;d) = (4/P_j)\int_0^1 d\mu\mu F_{q\ell}(\alpha_j,\alpha_1/2,1,\mu,s;d), \quad \text{(A.66)}$$

which gives, from Eqs. (A.42) ∿ (A.46), the following forms ($\beta=\alpha_1/2-d$):

(a) $\alpha_1/2-\alpha_j>d$;

$$\alpha_j P_j C_{qa}{}^{\ell}(\alpha_j,s;d) = W_{q\ell}(\alpha_j,\beta)-(-1)^q W_{q\ell}(-\alpha_j,\beta), \quad \text{(A.67)}$$

(b) $\alpha_1/2+\alpha_j>d>\alpha_1/2-\alpha_j$;

$$\alpha_j P_j C_{qa}{}^{\ell} = W_{q\ell}(\alpha_j,\beta)+S_{q\ell}(\alpha_j,\beta), \quad \text{(A.68)}$$

(c) $d>\alpha_1/2+\alpha_j$;

$$C_{qa}{}^{\ell} = 0,$$

where

$$W_{q\ell}(\alpha,\beta) = (i)^{q+2}\sum_{n=0}^{[\ell/2]}(-\frac{1}{2})^{[\ell/2]-n}\frac{(2\ell+2n-1-2[\ell/2])!!}{([\ell/2]-n)!(\ell+2n-2[\ell/2])!}$$

$$\times \sum_{m=0}^{q} \frac{(2q-m)!}{m!(q-m)!Q!(2\alpha)^{q-m}} \left[\sum_{p=1}^{Q} (Q-p)!(-(\alpha+\beta))^{p-1} \exp(-(\alpha+\beta)) + (-(\alpha+\beta))^{Q} E_1(\alpha+\beta) \right], \quad \text{(A.69)}$$

$$S_{q\ell}(\alpha,\beta) = (i)^q \sum_{n=0}^{[\ell/2]} \left(-\frac{1}{2}\right)^{[\ell/2]-n} \frac{(2\ell+2n-1-2[\ell/2])!!}{([\ell/2]-n)!(\ell+2n-2[\ell/2])!}$$

$$\times \sum_{r=0}^{[q/2]} (-1)^r \frac{(2q-2r-1)!!}{(2r)!!(\alpha)^{q-2r}} \sum_{m=0}^{q-2r} \frac{1}{m!(Q-2r)}(-\beta)^m, \quad \text{(A.70)}$$

$Q = q-m+\ell-2[\ell/2]+2n+2.$

The expression $2W_{q\ell}(\alpha,\beta)/(i)^{q+2}$ is written in the following form by using (A.11):

$$2W_{q\ell}(\alpha,\beta)/(i)^{q+2}$$

$$= \begin{cases} [\text{(A.11) with } L=0 \text{ for } r=q], & \text{for } \ell=0, \\ [\text{(A.11) with } L=1 \text{ for } r=q], & \text{for } \ell=1, \\ (-1/2)[\text{(A.11) with } L=0 \text{ for } r=q]+(3/2)[\text{(A.11) with } L=2 \text{ for } r=q], & \text{for } \ell=2, \\ (-3/2)[\text{(A.11) with } L=1 \text{ for } r=q]+(5/2)[\text{(A.11) with } L=3 \text{ for } r=q], & \text{for } \ell=3, \end{cases} \quad \text{(A.71)}$$

in which $x=-(\alpha+\beta)$ and $\alpha_i = -\alpha$.

The explicit expressions for $S_{q\ell}(\alpha,\beta)$ are written as follows for $q=0\sim7$ and $\ell=0\sim3$:

$$S_{00} = 1/2 ,$$

$$S_{10} = \frac{i}{\alpha}\left(\frac{1}{3} - \frac{\beta}{2}\right) ,$$

$$S_{20} = \frac{1}{4} - \frac{1}{\alpha^2}\left(\frac{3}{4} - \beta + \frac{3}{4}\beta^2\right),$$

$$S_{30} = \frac{i}{\alpha}\left(\frac{1}{2} - \frac{3}{4}\beta\right) - \frac{i}{\alpha^3}\left(3 - \frac{15}{4}\beta + \frac{5}{2}\beta^2 - \frac{5}{4}\beta^3\right),$$

$$S_{40} = \frac{3}{16} - \frac{1}{\alpha^2}\left(\frac{15}{8} - \frac{5}{2}\beta + \frac{15}{8}\beta^2\right) + \frac{1}{\alpha^4}\left(\frac{35}{2} - 21\beta + \frac{105}{8}\beta^2 - \frac{35}{6}\beta^3 + \frac{35}{16}\beta^4\right),$$

$$S_{50} = \frac{i}{\alpha}\left(\frac{5}{8} - \frac{15}{16}\beta\right) - \frac{i}{\alpha^3}\left(\frac{21}{2} - \frac{105}{8}\beta + \frac{35}{4}\beta^2 - \frac{35}{8}\beta^3\right)$$

$$+ \frac{i}{\alpha^5}\left(135 - \frac{315}{2}\beta + \frac{189}{2}\beta^2 - \frac{315}{8}\beta^3 + \frac{105}{8}\beta^4 - \frac{63}{16}\beta^5\right) ,$$

$$S_{60} = \frac{5}{32} - \frac{1}{\alpha^2}(\frac{105}{32} - \frac{35}{8}\beta + \frac{105}{32}\beta^2) + \frac{1}{\alpha^4}(\frac{315}{4} - \frac{189}{2}\beta + \frac{945}{16}\beta^2 - \frac{105}{4}\beta^3 + \frac{315}{32}\beta^4)$$

$$- \frac{1}{\alpha^6}(\frac{10395}{8} - 1485\beta + \frac{3465}{4}\beta^2 - \frac{693}{2}\beta^3 + \frac{3465}{32}\beta^4 - \frac{231}{8}\beta^5 + \frac{231}{32}\beta^6),$$

$$S_{70} = \frac{i}{\alpha}(\frac{35}{48} - \frac{35}{32}\beta) - \frac{i}{\alpha^3}(\frac{189}{8} - \frac{945}{32}\beta + \frac{315}{16}\beta^2 - \frac{315}{32}\beta^3) + \frac{i}{\alpha^5}(\frac{1485}{2} - \frac{3465}{4}\beta + \frac{2079}{4}\beta^2$$

$$- \frac{3465}{16}\beta^3 + \frac{1155}{16}\beta^4 - \frac{693}{32}\beta^5) - \frac{i}{\alpha^7}(15015 - \frac{135135}{8}\beta + \frac{19305}{2}\beta^2 - \frac{15015}{4}\beta^3$$

$$+ \frac{9009}{8}\beta^4 - \frac{9009}{32}\beta^5 + \frac{1001}{16}\beta^6 - \frac{429}{32}\beta^7), \qquad \text{(A.72)}$$

$$S_{00} = 1/3,$$

$$S_{11} = \frac{i}{\alpha}(\frac{1}{4} - \frac{\beta}{3}),$$

$$S_{21} = \frac{1}{6} - \frac{1}{\alpha^2}(\frac{3}{5} - \frac{3}{4}\beta + \frac{1}{2}\beta^2),$$

$$S_{31} = \frac{i}{\alpha}(\frac{3}{8} - \frac{\beta}{2}) - \frac{i}{\alpha^3}(\frac{5}{2} - 3\beta + \frac{15}{8}\beta^2 - \frac{5}{6}\beta^3),$$

$$S_{41} = \frac{1}{8} - \frac{1}{\alpha^2}(\frac{3}{2} - \frac{15}{8}\beta + \frac{5}{4}\beta^2) + \frac{1}{\alpha^4}(15 - \frac{35}{2}\beta + \frac{21}{2}\beta^2 - \frac{35}{8}\beta^3 + \frac{35}{24}\beta^4),$$

$$S_{51} = \frac{i}{\alpha}(\frac{15}{32} - \frac{5}{8}\beta) - \frac{i}{\alpha^3}(\frac{35}{4} - \frac{21}{2}\beta + \frac{105}{16}\beta^2 - \frac{35}{12}\beta^3)$$

$$+ \frac{i}{\alpha^5}(\frac{945}{8} - 135\beta + \frac{315}{4}\beta^2 - \frac{63}{2}\beta^3 + \frac{315}{32}\beta^4 - \frac{21}{8}\beta^5),$$

$$S_{61} = \frac{5}{48} - \frac{1}{\alpha^2}(\frac{21}{8} - \frac{105}{32}\beta + \frac{35}{16}\beta^2) + \frac{1}{\alpha^4}(\frac{135}{2} - \frac{315}{4}\beta + \frac{189}{4}\beta^2 - \frac{315}{16}\beta^3 + \frac{105}{16}\beta^4)$$

$$- \frac{1}{\alpha^6}(1155 - \frac{10395}{8}\beta + \frac{1485}{2}\beta^2 - \frac{1155}{4}\beta^3 + \frac{693}{8}\beta^4 - \frac{693}{32}\beta^5 + \frac{77}{16}\beta^6),$$

$$S_{71} = \frac{i}{\alpha}(\frac{35}{64} - \frac{35}{48}\beta) - \frac{i}{\alpha^3}(\frac{315}{16} - \frac{189}{8}\beta + \frac{945}{64}\beta^2 - \frac{105}{16}\beta^3) + \frac{i}{\alpha^5}(\frac{10395}{16} - \frac{1485}{2}\beta + \frac{3465}{8}\beta^2$$

$$- \frac{693}{4}\beta^3 + \frac{3465}{64}\beta^4 - \frac{231}{16}\beta^5) - \frac{i}{\alpha^7}(\frac{27027}{2} - 15015\beta + \frac{135135}{16}\beta^2 - \frac{6435}{2}\beta^3 + \frac{15015}{16}\beta^4$$

$$- \frac{9009}{40}\beta^5 + \frac{3003}{64}\beta^6 - \frac{143}{16}\beta^7), \qquad \text{(A.73)}$$

$$S_{02} = -S_{00}/2 + 3/8,$$

$$S_{12} = -\frac{1}{2}S_{10} + \frac{i}{\alpha}(\frac{3}{10} - \frac{3}{8}\beta),$$

$$S_{22} = -\frac{1}{2}S_{20}+\frac{3}{16}-\frac{1}{\alpha^2}(\frac{3}{4}-\frac{9}{10}\beta+\frac{9}{16}\beta^2),$$

$$S_{32} = -\frac{1}{2}S_{30}+\frac{i}{\alpha}(\frac{9}{20}-\frac{9}{16}\beta)-\frac{i}{\alpha^3}(\frac{45}{14}-\frac{15}{4}\beta+\frac{9}{4}\beta^2-\frac{15}{16}\beta^3),$$

$$S_{42} = -\frac{1}{2}S_{40}+\frac{9}{64}-\frac{1}{\alpha^2}(\frac{15}{8}-\frac{9}{4}\beta+\frac{45}{32}\beta^2)+\frac{1}{\alpha^4}(\frac{315}{16}-\frac{45}{2}\beta+\frac{105}{8}\beta^2-\frac{21}{4}\beta^3+\frac{105}{64}\beta^4),$$

$$S_{52} = -\frac{1}{2}S_{50}+\frac{i}{\alpha}(\frac{9}{16}-\frac{45}{64}\beta)-\frac{i}{\alpha^3}(\frac{45}{4}-\frac{105}{8}\beta+\frac{63}{8}\beta^2-\frac{105}{32}\beta^3)$$

$$+\frac{i}{\alpha^5}(\frac{315}{2}-\frac{2835}{16}\beta+\frac{405}{4}\beta^2-\frac{315}{8}\beta^3+\frac{189}{16}\beta^4-\frac{189}{64}\beta^5),$$

$$S_{62} = -\frac{1}{2}S_{60}+\frac{15}{128}-\frac{1}{\alpha^2}(\frac{105}{32}-\frac{63}{16}\beta+\frac{315}{128}\beta^2)+\frac{1}{\alpha^4}(\frac{2835}{32}-\frac{405}{4}\beta+\frac{945}{16}\beta^2-\frac{189}{8}\beta^3+\frac{945}{128}\beta^4)$$

$$-\frac{1}{\alpha^6}(\frac{6237}{4}-\frac{3465}{2}\beta+\frac{31185}{32}\beta^2-\frac{1485}{4}\beta^3+\frac{3465}{32}\beta^4-\frac{2079}{80}\beta^5+\frac{693}{128}\beta^6),$$

$$S_{72} = -\frac{1}{2}S_{70}+\frac{i}{\alpha}(\frac{21}{32}-\frac{105}{128}\beta)-\frac{i}{\alpha^3}(\frac{405}{16}-\frac{945}{32}\beta+\frac{567}{32}\beta^2-\frac{945}{128}\beta^3)+\frac{i}{\alpha^5}(\frac{3465}{4}$$

$$-\frac{31185}{32}\beta+\frac{4455}{8}\beta^2-\frac{3465}{16}\beta^3+\frac{2079}{32}\beta^4-\frac{2079}{128}\beta^5)-\frac{i}{\alpha^7}(\frac{36855}{2}-\frac{81081}{4}\beta$$

$$+\frac{45045}{4}\beta^2-\frac{135135}{32}\beta^3+\frac{19305}{16}\beta^4-\frac{9009}{32}\beta^5+\frac{9009}{160}\beta^6-\frac{1287}{128}\beta^7), \quad (A.74)$$

$$S_{03} = (-3/2)S_{01}+1/2,$$

$$S_{13} = -\frac{3}{2}S_{11}+\frac{i}{\alpha}(\frac{5}{12}-\frac{\beta}{2})$$

$$S_{23} = -\frac{3}{2}S_{21}+\frac{1}{4}-\frac{1}{\alpha^2}(\frac{15}{14}-\frac{5}{4}\beta+\frac{3}{4}\beta^2),$$

$$S_{33} = -\frac{3}{2}S_{31}+\frac{i}{\alpha}(\frac{5}{8}-\frac{3}{4}\beta)-\frac{i}{\alpha^3}(\frac{75}{16}-\frac{75}{14}\beta+\frac{25}{8}\beta^2-\frac{5}{4}\beta^3),$$

$$S_{43} = -\frac{3}{2}S_{41}+\frac{3}{16}-\frac{1}{\alpha^2}(\frac{75}{28}-\frac{25}{8}\beta+\frac{15}{8}\beta^2)+\frac{1}{\alpha^4}(\frac{175}{6}-\frac{525}{16}\beta+\frac{75}{4}\beta^2-\frac{175}{24}\beta^3+\frac{35}{16}\beta^4),$$

$$S_{53} = -\frac{3}{2}S_{51}+\frac{i}{\alpha}(\frac{25}{32}-\frac{15}{16}\beta)-\frac{i}{\alpha^3}(\frac{525}{32}-\frac{75}{4}\beta+\frac{175}{16}\beta^2-\frac{35}{8}\beta^3)$$

$$+\frac{i}{\alpha^5}(\frac{945}{4}-\frac{525}{2}\beta+\frac{4725}{32}\beta^2-\frac{225}{4}\beta^3+\frac{525}{32}\beta^4-\frac{63}{16}\beta^5),$$

$$S_{63} = -\frac{3}{2}S_{61}+\frac{5}{32}-\frac{1}{\alpha^2}(\frac{75}{16}-\frac{175}{32}\beta+\frac{105}{32}\beta^2)+\frac{1}{\alpha^4}(\frac{525}{4}-\frac{4725}{32}\beta+\frac{675}{8}\beta^2-\frac{525}{16}\beta^3+\frac{315}{32}\beta^4)$$

$$-\frac{1}{\alpha^6}(\frac{4725}{2}-\frac{10395}{4}\beta+\frac{5775}{4}\beta^2-\frac{17325}{32}\beta^3+\frac{2475}{16}\beta^4-\frac{1155}{32}\beta^5+\frac{231}{32}\beta^6),$$

$$S_{73} = -\frac{3}{2}S_{71}+\frac{i}{\alpha}(\frac{175}{192}-\frac{35}{32}\beta)-\frac{i}{\alpha^3}(\frac{4725}{128}-\frac{675}{16}\beta+\frac{1575}{64}\beta^2-\frac{315}{32}\beta^3)+\frac{i}{\alpha^5}(\frac{10395}{8}$$

$$-\frac{5775}{4}\beta+\frac{51975}{64}\beta^2-\frac{2475}{8}\beta^3+\frac{5775}{64}\beta^4-\frac{693}{32}\beta^5)-\frac{i}{\alpha^7}(\frac{225225}{8}-\frac{61425}{2}\beta$$

$$+\frac{135135}{8}\beta^2-\frac{25025}{4}\beta^3+\frac{225225}{128}\beta^4-\frac{6435}{16}\beta^5+\frac{5005}{64}\beta^6-\frac{429}{32}\beta^7). \quad (A.75)$$

For small values of $\alpha+|\beta|$, the series expansions for

$$C_{q\ell}(\alpha,\beta) = 2(i)^{-q}[W_{q\ell}(\alpha,\beta)+S_{q\ell}(\alpha,\beta)]/(\alpha+\beta) \quad (A.76)$$

with $q=0\sim7$ and $\ell=0\sim3$ are written as follows ($x=\alpha+\beta$):

$$C_{00} = 2+(\gamma+\ell n x)x-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}+\sum_{n=1}^{\infty}\frac{n+2}{(n+1)!}(-x)^n ,$$

$$C_{10}= 1-\frac{\beta}{\alpha}+\frac{2}{3}(1-\frac{1}{2}\frac{\beta}{\alpha})[(\gamma+\ell n x)x-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+2)!}[\frac{3}{2}(1-\frac{1}{2}\frac{\beta}{\alpha})(n^2+\frac{16}{9}n)+2(1-\frac{\beta}{\alpha})](-x)^n,$$

$$C_{20} = -\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})+\frac{1}{4}(1-\frac{\beta}{\alpha})^2[(\gamma+\ell n x)x-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+3)!}\{\frac{1}{4}(1-\frac{\beta}{\alpha})^2(n^3+7n^2)+[\frac{5}{2}-9\frac{\beta}{\alpha}+\frac{9}{2}(\frac{\beta}{\alpha})^2]n-6\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})\}(-x)^n,$$

$$C_{30} = -\frac{1}{4}(1-\frac{\beta}{\alpha})[1-5(\frac{\beta}{\alpha})^2]-\frac{1}{4}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2[(\gamma+\ell n x)x-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}]$$

$$-\sum_{n=1}^{\infty}\frac{1}{(n+4)!}\{\frac{1}{4}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(n^4+11n^3)+[2+\frac{23}{2}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2]n^2$$

$$+[8+12\frac{\beta}{\alpha}-48(\frac{\beta}{\alpha})^2+24(\frac{\beta}{\alpha})^3]n+6(1-\frac{\beta}{\alpha})[1-5(\frac{\beta}{\alpha})^2]\}(-x)^n,$$

$$C_{40} = \frac{3}{4}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})[1-\frac{7}{3}(\frac{\beta}{\alpha})^2]-\frac{1}{24}(1-\frac{\beta}{\alpha})^2[1-7(\frac{\beta}{\alpha})^2][(\gamma+\ell n x)x-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}]$$

$$-\sum_{n=1}^{\infty}\frac{1}{(n+5)!}\{\frac{1}{24}(1-\frac{\beta}{\alpha})^2[1-7(\frac{\beta}{\alpha})^2](n^5+16n^4)+[\frac{149}{24}-\frac{101}{12}\frac{\beta}{\alpha}(1+3\frac{\beta}{\alpha}$$

$$-7(\frac{\beta}{\alpha})^2+\frac{7}{2}(\frac{\beta}{\alpha})^3)]n^3+[\frac{283}{12}-\frac{283}{6}\frac{\beta}{\alpha}-\frac{163}{2}(\frac{\beta}{\alpha})^2(1-\frac{7}{3}\frac{\beta}{\alpha}+\frac{7}{6}(\frac{\beta}{\alpha})^2)]n^2$$

$$+[27-130\frac{\beta}{\alpha}-60(\frac{\beta}{\alpha})^2+350(\frac{\beta}{\alpha})^3(1-\frac{1}{2}\frac{\beta}{\alpha})]n-90\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})[1-\frac{7}{3}(\frac{\beta}{\alpha})^2]\}(-x)^n,$$

$$C_{50}=\frac{1}{8}(1-\frac{\beta}{\alpha})(1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4)+\frac{1}{8}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-3(\frac{\beta}{\alpha})^2)[(\gamma+\ell nx)x-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+6)!}\{\frac{1}{8}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2(1-3(\frac{\beta}{\alpha})^2)(n^6-22n^5)-[2-\frac{197}{8}\frac{\beta}{\alpha}(1-2\frac{\beta}{\alpha}-2(\frac{\beta}{\alpha})^2$$

$$+6(\frac{\beta}{\alpha})^3-3(\frac{\beta}{\alpha})^4)]n^4-[12-\frac{293}{2}\frac{\beta}{\alpha}+233(\frac{\beta}{\alpha})^2(1+\frac{\beta}{\alpha}-3(\frac{\beta}{\alpha})^2+\frac{3}{2}(\frac{\beta}{\alpha})^3)]n^3$$

$$+[8+\frac{939}{2}\frac{\beta}{\alpha}-849(\frac{\beta}{\alpha})^2-639(\frac{\beta}{\alpha})^3(1-3\frac{\beta}{\alpha}+\frac{3}{2}(\frac{\beta}{\alpha})^2)]n^2+[108+480\frac{\beta}{\alpha}$$

$$-1920(\frac{\beta}{\alpha})^2-240(\frac{\beta}{\alpha})^3+3240(\frac{\beta}{\alpha})^4(1-\frac{1}{2}\frac{\beta}{\alpha})]n+90(1-\frac{\beta}{\alpha})[1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4]\}(-x)^n,$$

$$C_{60}=-\frac{5}{8}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})[1-6(\frac{\beta}{\alpha})^2+\frac{33}{5}(\frac{\beta}{\alpha})^4]+\frac{1}{64}(1-\frac{\beta}{\alpha})^2[1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4][(\gamma+\ell nx)x$$

$$-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+7)!}\{\frac{1}{64}(1-\frac{\beta}{\alpha})^2[1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4](n^7+29n^6)$$

$$+[\frac{223}{64}-\frac{351}{32}\frac{\beta}{\alpha}(1+\frac{17}{2}\frac{\beta}{\alpha}-18(\frac{\beta}{\alpha})^2-\frac{15}{2}(\frac{\beta}{\alpha})^3+33(\frac{\beta}{\alpha})^4-\frac{33}{2}(\frac{\beta}{\alpha})^5)]n^5$$

$$+[\frac{1415}{64}-\frac{967}{32}\frac{\beta}{\alpha}-\frac{39287}{64}(\frac{\beta}{\alpha})^2(1-\frac{36}{17}\frac{\beta}{\alpha}-\frac{15}{17}(\frac{\beta}{\alpha})^2+\frac{66}{17}(\frac{\beta}{\alpha})^3-\frac{33}{17}(\frac{\beta}{\alpha})^4)]n^4$$

$$+[\frac{1343}{8}-\frac{127}{4}\frac{\beta}{\alpha}-\frac{22655}{8}(\frac{\beta}{\alpha})^2+\frac{10215}{2}(\frac{\beta}{\alpha})^3(1+\frac{5}{12}\frac{\beta}{\alpha}-\frac{11}{6}(\frac{\beta}{\alpha})^2+\frac{11}{12}(\frac{\beta}{\alpha})^3)]n^3$$

$$+[\frac{9809}{16}-\frac{8577}{8}\frac{\beta}{\alpha}-\frac{128001}{16}(\frac{\beta}{\alpha})^2+\frac{60057}{4}(\frac{\beta}{\alpha})^3+\frac{83295}{16}(\frac{\beta}{\alpha})^4(1-\frac{22}{5}\frac{\beta}{\alpha}+\frac{11}{5}(\frac{\beta}{\alpha})^2)]n^2$$

$$+[\frac{2757}{4}-\frac{8421}{2}\frac{\beta}{\alpha}-\frac{29085}{4}(\frac{\beta}{\alpha})^2+29925(\frac{\beta}{\alpha})^3-\frac{4725}{4}(\frac{\beta}{\alpha})^4-\frac{72765}{2}(\frac{\beta}{\alpha})^5(1-\frac{1}{2}\frac{\beta}{\alpha})]n$$

$$-3150\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})[1-6(\frac{\beta}{\alpha})^2+\frac{33}{5}(\frac{\beta}{\alpha})^4]\}(-x)^n,$$

$$C_{70}=-\frac{5}{64}(1-\frac{\beta}{\alpha})[1-27(\frac{\beta}{\alpha})^2+99(\frac{\beta}{\alpha})^4-\frac{429}{5}(\frac{\beta}{\alpha})^6]-\frac{5}{64}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2[1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4]$$

$$\times[(\gamma+\ell nx)x-\sum_{n=2}^{\infty}\frac{(-x)^n}{(n-1)(n-1)!}]-\sum_{n=1}^{\infty}\frac{1}{(n+8)!}\{\frac{5}{64}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2[1-\frac{22}{3}(\frac{\beta}{\alpha})^2$$

$$+\frac{143}{15}(\frac{\beta}{\alpha})^4](n^8+37n^7)+[2+\frac{2915}{64}\frac{\beta}{\alpha}(1-2\frac{\beta}{\alpha}-\frac{19}{3}(\frac{\beta}{\alpha})^2+\frac{44}{3}(\frac{\beta}{\alpha})^3+\frac{11}{5}(\frac{\beta}{\alpha})^4$$

$$-\frac{286}{15}(\frac{\beta}{\alpha})^5+\frac{143}{15}(\frac{\beta}{\alpha})^6)]n^6+[16+\frac{22011}{64}\frac{\beta}{\alpha}-\frac{25595}{32}(\frac{\beta}{\alpha})^2(1+\frac{19}{6}\frac{\beta}{\alpha}$$

$$-\frac{22}{3}(\frac{\beta}{\alpha})^2-\frac{11}{10}(\frac{\beta}{\alpha})^3+\frac{143}{15}(\frac{\beta}{\alpha})^4-\frac{143}{30}(\frac{\beta}{\alpha})^5)]n^5-[54-\frac{7047}{4}\frac{\beta}{\alpha}+\frac{7103}{2}(\frac{\beta}{\alpha})^2$$

$$+\frac{163685}{12}\left(\frac{\beta}{\alpha}\right)^3\left(1-\frac{44}{19}\frac{\beta}{\alpha}-\frac{33}{95}\left(\frac{\beta}{\alpha}\right)^2+\frac{286}{95}\left(\frac{\beta}{\alpha}\right)^3-\frac{143}{95}\left(\frac{\beta}{\alpha}\right)^4\right)]n^4-[544$$

$$-\frac{136933}{16}\frac{\beta}{\alpha}+\frac{82277}{8}\left(\frac{\beta}{\alpha}\right)^2+\frac{2555135}{48}\left(\frac{\beta}{\alpha}\right)^3-\frac{1304215}{12}\left(\frac{\beta}{\alpha}\right)^4\left(1+\frac{3}{20}\frac{\beta}{\alpha}-\frac{13}{10}\left(\frac{\beta}{\alpha}\right)^2\right.$$

$$\left.+\frac{13}{20}\left(\frac{\beta}{\alpha}\right)^3\right)]n^3-[26-\frac{104747}{4}\frac{\beta}{\alpha}+\frac{87723}{2}\left(\frac{\beta}{\alpha}\right)^2+\frac{547515}{4}\left(\frac{\beta}{\alpha}\right)^3-278685\left(\frac{\beta}{\alpha}\right)^4$$

$$-\frac{146421}{4}\left(\frac{\beta}{\alpha}\right)^5\left(1-\frac{26}{3}\frac{\beta}{\alpha}+\frac{13}{3}\left(\frac{\beta}{\alpha}\right)^2\right)]n^2+[3600+26964\frac{\beta}{\alpha}-121464\left(\frac{\beta}{\alpha}\right)^2$$

$$-109200\left(\frac{\beta}{\alpha}\right)^3+508200\left(\frac{\beta}{\alpha}\right)^4-69300\left(\frac{\beta}{\alpha}\right)^5-480480\left(\frac{\beta}{\alpha}\right)^6\left(1-\frac{1}{2}\frac{\beta}{\alpha}\right)]n$$

$$+3150\left(1-\frac{\beta}{\alpha}\right)[1-27\left(\frac{\beta}{\alpha}\right)^2+99\left(\frac{\beta}{\alpha}\right)^4-\frac{429}{5}\left(\frac{\beta}{\alpha}\right)^6]\}(-x)^n,$$

$$C_{01}=\frac{1}{2}(1-x)-\frac{1}{6}[(\gamma+\ell nx)x^2+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+2)!}\left(\frac{1}{6}n^2+\frac{2}{3}n+1\right)(-x)^{n+1},$$

$$C_{11}=\frac{1}{4}\left(1-\frac{\beta}{\alpha}\right)-\frac{1}{3}\left(1-\frac{1}{2}\frac{\beta}{\alpha}\right)x-\frac{1}{8}\left(1-\frac{1}{3}\frac{\beta}{\alpha}\right)[(\gamma+\ell nx)x^2+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1)(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+3)!}[\frac{1}{8}\left(1-\frac{1}{3}\frac{\beta}{\alpha}\right)(n^3+7n^2+18n)+2-\frac{\beta}{\alpha}](-x)^{n+1},$$

$$C_{21}=-\frac{1}{4}\left(1-\frac{\beta}{\alpha}\right)[\frac{\beta}{\alpha}+\frac{1}{2}\left(1-\frac{\beta}{\alpha}\right)x]+\frac{1}{15}[1-\frac{9}{8}\frac{\beta}{\alpha}+\frac{3}{8}\left(\frac{\beta}{\alpha}\right)^2][(\gamma+\ell nx)x^2+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1)(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+4)!}\{\frac{1}{15}[1-\frac{9}{8}\frac{\beta}{\alpha}+\frac{3}{8}\left(\frac{\beta}{\alpha}\right)^2](n^4+11n^3+46n^2)+[\frac{27}{5}-\frac{36}{5}\frac{\beta}{\alpha}\left(1-\frac{1}{3}\frac{\beta}{\alpha}\right)]n$$

$$+3\left(1-\frac{\beta}{\alpha}\right)^2\}(-x)^{n+1},$$

$$C_{31}=-\frac{1}{16}\left(1-\frac{\beta}{\alpha}\right)[1-5\left(\frac{\beta}{\alpha}\right)^2-2\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)x]-\frac{1}{48}\left(1-\frac{\beta}{\alpha}\right)^3[(\gamma+\ell nx)x^2+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+5)!}\{\frac{1}{48}\left(1-\frac{\beta}{\alpha}\right)^3(n^5+16n^4+101n^3)+[\frac{139}{24}-\frac{163}{8}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}+\frac{1}{3}\left(\frac{\beta}{\alpha}\right)^2\right)]n^2$$

$$+[\frac{11}{2}-\frac{63}{2}\frac{\beta}{\alpha}+\frac{75}{2}\left(\frac{\beta}{\alpha}\right)^2\left(1-\frac{1}{3}\frac{\beta}{\alpha}\right)]n-15\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^2\}(-x)^{n+1},$$

$$C_{41}=\frac{3}{16}\left(1-\frac{\beta}{\alpha}\right)[\frac{\beta}{\alpha}\left(1-\frac{7}{3}\left(\frac{\beta}{\alpha}\right)^2\right)+\frac{1}{9}\left(1-\frac{\beta}{\alpha}\right)\left(1-7\left(\frac{\beta}{\alpha}\right)^2\right)x]+\frac{1}{48}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^3[(\gamma+\ell nx)x^2$$

$$+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1)(n-1)!}]-\sum_{n=1}^{\infty}\frac{1}{(n+6)!}\{\frac{1}{48}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^3(n^6+22n^5+197n^4)+[1+\frac{233}{12}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^3]n^3$$

$$+[9+\frac{173}{4}\frac{\beta}{\alpha}-\frac{639}{4}\left(\frac{\beta}{\alpha}\right)^2\left(1-\frac{\beta}{\alpha}+\frac{1}{3}\left(\frac{\beta}{\alpha}\right)^2\right)]n^2+[23+20\frac{\beta}{\alpha}-225\left(\frac{\beta}{\alpha}\right)^2+270\left(\frac{\beta}{\alpha}\right)^3\left(1-\frac{1}{3}\frac{\beta}{\alpha}\right)]n$$

$$+15\left(1-\frac{\beta}{\alpha}\right)^2[1-7\left(\frac{\beta}{\alpha}\right)^2]\}(1-x)^{n+1},$$

$$C_{51} = \frac{1}{32}(1-\frac{\beta}{\alpha})[(1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4)-2\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})(1-3(\frac{\beta}{\alpha})^2)x]+\frac{1}{384}(1-\frac{\beta}{\alpha})^3[1-9(\frac{\beta}{\alpha})^2]$$

$$[(\gamma+\ell n x)x^2+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1)(n-1)!}]-\sum_{n=1}^{\infty}\frac{1}{(n+7)!}\{\frac{1}{384}(1-\frac{\beta}{\alpha})^3[1-9(\frac{\beta}{\alpha})^2](n^7+29n^6+351n^5)$$

$$+[\frac{2695}{384}-\frac{2311}{128}\frac{\beta}{\alpha}(1+2\frac{\beta}{\alpha}-\frac{26}{3}(\frac{\beta}{\alpha})^2+9(\frac{\beta}{\alpha})^3-3(\frac{\beta}{\alpha})^4)]n^4+[\frac{1663}{48}-\frac{1375}{16}\frac{\beta}{\alpha}$$

$$-\frac{1135}{8}(\frac{\beta}{\alpha})^2(1-\frac{13}{3}\frac{\beta}{\alpha}+\frac{9}{2}(\frac{\beta}{\alpha})^2-\frac{3}{2}(\frac{\beta}{\alpha})^3)]n^3+[\frac{2843}{32}-\frac{9873}{32}\frac{\beta}{\alpha}-\frac{3873}{16}(\frac{\beta}{\alpha})^2+\frac{24063}{16}(\frac{\beta}{\alpha})^3$$

$$\times(1-\frac{27}{26}\frac{\beta}{\alpha}+\frac{9}{26}(\frac{\beta}{\alpha})^2)]n^2+[\frac{663}{8}-\frac{4605}{8}\frac{\beta}{\alpha}+\frac{735}{4}(\frac{\beta}{\alpha})^2+\frac{7875}{4}(\frac{\beta}{\alpha})^3-\frac{19845}{8}(\frac{\beta}{\alpha})^4(1-\frac{1}{3}\frac{\beta}{\alpha})]n$$

$$-315\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2[1-3(\frac{\beta}{\alpha})^2]\}(1-x)^{n+1},$$

$$C_{61} = -\frac{5}{32}(1-\frac{\beta}{\alpha})[\frac{\beta}{\alpha}(1-6(\frac{\beta}{\alpha})^2+\frac{33}{5}(\frac{\beta}{\alpha})^4)+\frac{1}{20}(1-\frac{\beta}{\alpha})(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)x]$$

$$-\frac{1}{128}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^3[1-\frac{11}{3}(\frac{\beta}{\alpha})^2][(\gamma+\ell n x)x^2+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+8)!}$$

$$\times\{\frac{1}{128}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^3[1-\frac{11}{3}(\frac{\beta}{\alpha})^2](n^8+37n^7+583n^6)-[1-\frac{5119}{128}\frac{\beta}{\alpha}(1-3\frac{\beta}{\alpha}-\frac{2}{3}(\frac{\beta}{\alpha})^2+10(\frac{\beta}{\alpha})^3$$

$$-11(\frac{\beta}{\alpha})^4+\frac{11}{3}(\frac{\beta}{\alpha})^5)]n^5-[13-\frac{1891}{8}\frac{\beta}{\alpha}+\frac{5169}{8}(\frac{\beta}{\alpha})^2(1+\frac{2}{9}\frac{\beta}{\alpha}-\frac{10}{3}(\frac{\beta}{\alpha})^2$$

$$+\frac{11}{3}(\frac{\beta}{\alpha})^3-\frac{11}{9}(\frac{\beta}{\alpha})^4)]n^4-[38-\frac{31105}{32}\frac{\beta}{\alpha}+\frac{77859}{32}(\frac{\beta}{\alpha})^2+\frac{23713}{48}(\frac{\beta}{\alpha})^3(1-15\frac{\beta}{\alpha}$$

$$+\frac{33}{2}(\frac{\beta}{\alpha})^2-\frac{11}{2}(\frac{\beta}{\alpha})^3)]n^3+[82+\frac{17679}{8}\frac{\beta}{\alpha}-\frac{55053}{8}(\frac{\beta}{\alpha})^2+\frac{603}{4}(\frac{\beta}{\alpha})^3+\frac{66555}{4}(\frac{\beta}{\alpha})^4$$

$$\times(1-\frac{11}{10}\frac{\beta}{\alpha}+\frac{11}{30}(\frac{\beta}{\alpha})^2)]n^2+[423+1596\frac{\beta}{\alpha}-11130(\frac{\beta}{\alpha})^2+7140(\frac{\beta}{\alpha})^3+20475(\frac{\beta}{\alpha})^4$$

$$-27720(\frac{\beta}{\alpha})^5(1-\frac{1}{3}\frac{\beta}{\alpha})]n+315(1-\frac{\beta}{\alpha})^2[1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4]\}(-x)^{n+1},$$

$$C_{71} = -\frac{5}{256}(1-\frac{\beta}{\alpha})[1-27(\frac{\beta}{\alpha})^2+99(\frac{\beta}{\alpha})^4-\frac{429}{5}(\frac{\beta}{\alpha})^6-2\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})(1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4)x]$$

$$-\frac{1}{1280}(1-\frac{\beta}{\alpha})^3[1-22(\frac{\beta}{\alpha})^2+\frac{143}{3}(\frac{\beta}{\alpha})^4][(\gamma+\ell n x)x^2+\sum_{n=2}^{\infty}\frac{(-x)^{n+1}}{(n-1)(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+9)!}\{\frac{1}{1280}(1-\frac{\beta}{\alpha})^3[1-22(\frac{\beta}{\alpha})^2+\frac{143}{3}(\frac{\beta}{\alpha})^4](n^9+46n^8+916n^7)$$

$$+[\frac{4543}{640}-\frac{15549}{640}\frac{\beta}{\alpha}(1+\frac{19}{3}\frac{\beta}{\alpha}-\frac{65}{3}(\frac{\beta}{\alpha})^2+\frac{55}{9}(\frac{\beta}{\alpha})^3+\frac{121}{3}(\frac{\beta}{\alpha})^4-\frac{143}{3}(\frac{\beta}{\alpha})^5+\frac{143}{9}(\frac{\beta}{\alpha})^6)]n^6$$

$$+[\frac{54439}{1280}-\frac{185077}{1280}\frac{\beta}{\alpha}-\frac{1399141}{1280}(\frac{\beta}{\alpha})^2(1-\frac{65}{19}\frac{\beta}{\alpha}+\frac{55}{57}(\frac{\beta}{\alpha})^2+\frac{121}{19}(\frac{\beta}{\alpha})^3-\frac{143}{19}(\frac{\beta}{\alpha})^4+\frac{143}{57}(\frac{\beta}{\alpha})^5)]n^5$$

$$+[\frac{71981}{320}-\frac{140743}{320}\frac{\beta}{\alpha}-\frac{1750039}{320}(\frac{\beta}{\alpha})^2+\frac{1114633}{64}(\frac{\beta}{\alpha})^3(1-\frac{11}{39}\frac{\beta}{\alpha}-\frac{121}{65}(\frac{\beta}{\alpha})^2+\frac{11}{5}(\frac{\beta}{\alpha})^3$$

$$-\frac{11}{15}(\frac{\beta}{\alpha})^4)]n^4+[\frac{342501}{320}-\frac{526703}{320}\frac{\beta}{\alpha}-\frac{6397119}{320}(\frac{\beta}{\alpha})^2+\frac{3668193}{64}(\frac{\beta}{\alpha})^3-\frac{977757}{64}(\frac{\beta}{\alpha})^4(1+\frac{33}{5}\frac{\beta}{\alpha}$$

$$-\frac{39}{5}(\frac{\beta}{\alpha})^2+\frac{13}{5}(\frac{\beta}{\alpha})^3)]n^3+[\frac{230599}{80}-\frac{754757}{80}\frac{\beta}{\alpha}-\frac{3294261}{80}(\frac{\beta}{\alpha})^2+\frac{2273067}{16}(\frac{\beta}{\alpha})^3$$

$$-\frac{790383}{16}(\frac{\beta}{\alpha})^4-\frac{16935039}{80}(\frac{\beta}{\alpha})^5(1-\frac{13}{11}\frac{\beta}{\alpha}+\frac{13}{33}(\frac{\beta}{\alpha})^2)]n^2+[\frac{5577}{2}-\frac{46137}{2}\frac{\beta}{\alpha}-\frac{37233}{2}(\frac{\beta}{\alpha})^2$$

$$+\frac{419895}{2}(\frac{\beta}{\alpha})^3-\frac{336105}{2}(\frac{\beta}{\alpha})^4-\frac{492723}{2}(\frac{\beta}{\alpha})^5+\frac{729729}{2}(\frac{\beta}{\alpha})^6(1-\frac{1}{3}\frac{\beta}{\alpha})]n$$

$$-14175\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2[1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4]\}(-x)^{n+1}\ ,$$

$$C_{02}=\frac{1}{2}-\frac{3}{8}x+\frac{1}{4}x^2+\frac{1}{16}[(\gamma+\ell nx)x^3-\sum_{m=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+3)!}\frac{1}{16}(n^3+7n^2+18n+24)(-x)^{n+2},$$

$$C_{12}=\frac{1}{4}(1-\frac{\beta}{\alpha})-\frac{1}{4}(1-\frac{1}{2}\frac{\beta}{\alpha})x+\frac{3}{16}(1-\frac{1}{3}\frac{\beta}{\alpha})x^2+\frac{1}{20}(1-\frac{1}{4}\frac{\beta}{\alpha})[(\gamma+\ell nx)x^3-\sum_{n=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+4)!}[\frac{1}{20}(1-\frac{1}{4}\frac{\beta}{\alpha})(n^4+11n^3+46n^2+96n)+\frac{9}{2}(1-\frac{1}{3}\frac{\beta}{\alpha})](-x)^{n+2},$$

$$C_{22}=-\frac{1}{4}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})-\frac{3}{32}(1-\frac{\beta}{\alpha})^2x+\frac{1}{10}[1-\frac{9}{8}\frac{\beta}{\alpha}+\frac{3}{8}(\frac{\beta}{\alpha})^2]x^2+\frac{1}{32}[1-\frac{4}{5}\frac{\beta}{\alpha}+\frac{1}{5}(\frac{\beta}{\alpha})^2][(\gamma+\ell nx)x^3$$

$$-\sum_{n=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+5)!}\{\frac{1}{32}[1-\frac{4}{5}\frac{\beta}{\alpha}+\frac{1}{5}(\frac{\beta}{\alpha})^2](n^5+16n^4+101n^3+326n^2)$$

$$+[\frac{69}{4}-15\frac{\beta}{\alpha}+\frac{15}{4}(\frac{\beta}{\alpha})^2]n+12-\frac{27}{2}\frac{\beta}{\alpha}(1-\frac{1}{3}\frac{\beta}{\alpha})\}(-x)^{n+2}$$

$$C_{32}=-\frac{1}{16}(1-\frac{\beta}{\alpha})[1-5(\frac{\beta}{\alpha})^2-\frac{3}{2}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})x-\frac{1}{2}(1-\frac{\beta}{\alpha})^2x^2]+\frac{1}{70}[1-\frac{29}{16}\frac{\beta}{\alpha}+\frac{5}{4}(\frac{\beta}{\alpha})^2-\frac{5}{16}(\frac{\beta}{\alpha})^3]$$

$$\times[(\gamma+\ell nx)x^3-\sum_{n=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+6)!}\{\frac{1}{70}[1-\frac{29}{16}\frac{\beta}{\alpha}+\frac{5}{4}(\frac{\beta}{\alpha})^2-\frac{5}{16}(\frac{\beta}{\alpha})^3]$$

$$\times(n^6+22n^5+197n^4+932n^3)+[\frac{2451}{70}-\frac{18531}{280}\frac{\beta}{\alpha}(1-\frac{20}{29}\frac{\beta}{\alpha}+\frac{5}{29}(\frac{\beta}{\alpha})^2)]n^2$$

$$+[\frac{327}{7}-\frac{720}{7}\frac{\beta}{\alpha}(1-\frac{3}{4}\frac{\beta}{\alpha}+\frac{3}{16}(\frac{\beta}{\alpha})^2)]n+\frac{45}{2}(1-\frac{\beta^3}{\alpha})\}(-x)^{n+2},$$

$$C_{42}=\frac{3}{16}(1-\frac{\beta}{\alpha})[\frac{\beta}{\alpha}(1-\frac{7}{3}(\frac{\beta}{\alpha})^2)+\frac{1}{12}(1-\frac{\beta}{\alpha})(1-7(\frac{\beta}{\alpha})^2)x+\frac{1}{6}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^2x^2+\frac{1}{256}(1-\frac{\beta}{\alpha})^4[(\gamma+\ell nx)x^3$$

$$-\sum_{n=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+7)!}\{\frac{1}{256}(1-\frac{\beta}{\alpha})^4(n^7+29n^6+351n^5+2311n^4)$$

$$+[\frac{1087}{32}-\frac{1135}{8}\frac{\beta}{\alpha}(1-\frac{3}{2}\frac{\beta}{\alpha}+(\frac{\beta}{\alpha})^2-\frac{1}{4}(\frac{\beta}{\alpha})^3)]n^3+[\frac{4305}{64}-\frac{5313}{16}\frac{\beta}{\alpha}+\frac{16659}{32}(\frac{\beta}{\alpha})^2$$

$$\times(1-\frac{2}{3}\frac{\beta}{\alpha}+\frac{1}{6}(\frac{\beta}{\alpha})^2)]n^2+[\frac{837}{16}-\frac{1605}{4}\frac{\beta}{\alpha}+\frac{6075}{8}(\frac{\beta}{\alpha})^2-\frac{2205}{4}(\frac{\beta}{\alpha})^3(1-\frac{1}{4}\frac{\beta}{\alpha})]n$$

$$-\frac{315}{2}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^3\}(-x)^{n+2},$$

$$C_{52}=\frac{1}{32}(1-\frac{\beta}{\alpha})[1-14(\frac{\beta}{\alpha})^2+21(\frac{\beta}{\alpha})^4-\frac{3}{2}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})(1-3(\frac{\beta}{\alpha})^2)x-\frac{1}{8}(1-\frac{\beta}{\alpha})^2(1-9(\frac{\beta}{\alpha})^2)x^2]$$

$$-\frac{1}{256}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^4[(\gamma+\ln x)x^3-\sum_{n=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]-\sum_{n=1}^{\infty}\frac{1}{(n+8)!}\{\frac{1}{256}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^4(n^8+37n^7$$

$$+583n^6+5119n^5)+[\frac{3}{2}+\frac{1723}{16}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^4]n^4+[24+\frac{22273}{64}\frac{\beta}{\alpha}-\frac{23713}{16}(\frac{\beta}{\alpha})^2(1-\frac{3}{2}\frac{\beta}{\alpha}+(\frac{\beta}{\alpha})^2$$

$$-\frac{1}{4}(\frac{\beta}{\alpha})^3)]n^3+[129+\frac{8631}{16}\frac{\beta}{\alpha}-\frac{12681}{4}(\frac{\beta}{\alpha})^2+\frac{39933}{8}(\frac{\beta}{\alpha})^3(1-\frac{2}{3}\frac{\beta}{\alpha}+\frac{1}{6}(\frac{\beta}{\alpha})^3)]n^2$$

$$+[264+45\frac{\beta}{\alpha}-3465(\frac{\beta}{\alpha})^2+6930(\frac{\beta}{\alpha})^3(1-\frac{8}{11}\frac{\beta}{\alpha}+\frac{2}{11}(\frac{\beta}{\alpha})^2)]n+\frac{315}{2}(1-\frac{\beta}{\alpha})^3$$

$$\times[1-9(\frac{\beta}{\alpha})^2]\}(-x)^{n+2},$$

$$C_{62}=-\frac{5}{32}(1-\frac{\beta}{\alpha})[\frac{\beta}{\alpha}(1-6(\frac{\beta}{\alpha})^4+\frac{33}{5}(\frac{\beta}{\alpha})\quad)+\frac{3}{80}(1-\frac{\beta}{\alpha})(1-18(\frac{\beta}{\alpha})^2+33(\frac{\beta}{\alpha})^4)x$$

$$-\frac{3}{40}(1-\frac{\beta}{\alpha})^2\frac{\beta}{\alpha}(1-\frac{11}{3}(\frac{\beta}{\alpha})^2)x^2]-\frac{1}{2560}(1-\frac{\beta}{\alpha})^4[1-11(\frac{\beta}{\alpha})^2][(\gamma+\ln x)x^3$$

$$-\sum_{n=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]-\sum_{n=1}^{\infty}\frac{1}{(n+9)!}\{\frac{1}{2560}(1-\frac{\beta}{\alpha})^4[1-11(\frac{\beta}{\alpha})^2](n^9+46n^8+916n^7+10366n^6)$$

$$+[\frac{77479}{2560}-\frac{73639}{640}\frac{\beta}{\alpha}(1+\frac{5}{4}\frac{\beta}{\alpha}-10(\frac{\beta}{\alpha})^2+\frac{65}{4}(\frac{\beta}{\alpha})^3-11(\frac{\beta}{\alpha})^4+\frac{11}{4}(\frac{\beta}{\alpha})^5)]n^5$$

$$+[\frac{103981}{640}-\frac{90781}{160}\frac{\beta}{\alpha}-\frac{85741}{128}(\frac{\beta}{\alpha})^2(1-8\frac{\beta}{\alpha}+13(\frac{\beta}{\alpha})^2-\frac{44}{5}(\frac{\beta}{\alpha})^3+\frac{11}{5}(\frac{\beta}{\alpha})^4)]n^4$$

$$+[\frac{380901}{640}-\frac{347301}{160}\frac{\beta}{\alpha}-\frac{226341}{128}(\frac{\beta}{\alpha})^2+\frac{266661}{16}(\frac{\beta}{\alpha})^3(1-\frac{13}{8}\frac{\beta}{\alpha}+\frac{11}{10}(\frac{\beta}{\alpha})^2-\frac{11}{40}(\frac{\beta}{\alpha})^4)]n^3$$

$$+[\frac{197319}{160}-\frac{242019}{40}\frac{\beta}{\alpha}-\frac{8919}{32}(\frac{\beta}{\alpha})^2+\frac{132399}{4}(\frac{\beta}{\alpha})^3-\frac{1819467}{32}(\frac{\beta}{\alpha})^4(1-\frac{44}{65}\frac{\beta}{\alpha}+\frac{11}{65}(\frac{\beta}{\alpha})^2)]n^2$$

$$+[\frac{4041}{4}-9072\frac{\beta}{\alpha}+\frac{38745}{4}(\frac{\beta}{\alpha})^2+32130(\frac{\beta}{\alpha})^3-\frac{303345}{4}(\frac{\beta}{\alpha})^4+56133(\frac{\beta}{\alpha})^5(1-\frac{1}{4}\frac{\beta}{\alpha})]n$$

$$-\frac{8505}{2}(1-\frac{\beta}{\alpha})^3\frac{\beta}{\alpha}[1-\frac{11}{3}(\frac{\beta}{\alpha})^2]\}(-x)^{n+2},$$

$$C_{72}=-\frac{5}{256}(1-\frac{\beta}{\alpha})[1-27(\frac{\beta}{\alpha})^2+99(\frac{\beta}{\alpha})^4-\frac{429}{5}(\frac{\beta}{\alpha})^6-\frac{3}{2}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})(1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4)x$$

$$-\frac{3}{50}(1-\frac{\beta}{\alpha})^2(1-22(\frac{\beta}{\alpha})^2+\frac{143}{3}(\frac{\beta}{\alpha})^4)x^2]+\frac{3}{2560}(1-\frac{\beta}{\alpha})^4\frac{\beta}{\alpha}[1-\frac{13}{3}(\frac{\beta}{\alpha})^2][(\gamma+\ln x)x^3$$

$$-\sum_{n=2}^{\infty}\frac{(-x)^{n+2}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+10)!}\{\frac{3}{2560}(1-\frac{\beta}{\alpha})^4\frac{\beta}{\alpha}[1-\frac{13}{3}(\frac{\beta}{\alpha})^2](n^{10}+56n^9+1376n^8$$

$$+19526n^7)-[\frac{3}{2}-\frac{531897}{2560}\frac{\beta}{\alpha}(1-4\frac{\beta}{\alpha}+\frac{5}{3}(\frac{\beta}{\alpha})^2+\frac{40}{3}(\frac{\beta}{\alpha})^3-25(\frac{\beta}{\alpha})^4+\frac{52}{3}(\frac{\beta}{\alpha})^5-\frac{13}{3}(\frac{\beta}{\alpha})^6)]n^6$$

$$-[33-\frac{1672791}{1280}\frac{\beta}{\alpha}+\frac{1619031}{320}(\frac{\beta}{\alpha})^2(1-\frac{5}{12}\frac{\beta}{\alpha}-\frac{10}{3}(\frac{\beta}{\alpha})^2+\frac{25}{4}(\frac{\beta}{\alpha})^3-\frac{13}{3}(\frac{\beta}{\alpha})^4+\frac{13}{12}(\frac{\beta}{\alpha})^5)]n^5$$

$$-[\frac{465}{2}-\frac{3882933}{640}\frac{\beta}{\alpha}+\frac{3462933}{160}(\frac{\beta}{\alpha})^2-\frac{1124071}{128}(\frac{\beta}{\alpha})^3(1+8\frac{\beta}{\alpha}-15(\frac{\beta}{\alpha})^2+\frac{52}{5}(\frac{\beta}{\alpha})^3-\frac{13}{5}(\frac{\beta}{\alpha})^4)]n^4$$

$$-[390-\frac{6338229}{320}\frac{\beta}{\alpha}+\frac{5565429}{80}(\frac{\beta}{\alpha})^2-\frac{1915623}{64}(\frac{\beta}{\alpha})^3-\frac{1613223}{8}(\frac{\beta}{\alpha})^4(1-\frac{15}{8}\frac{\beta}{\alpha}+\frac{13}{10}(\frac{\beta}{\alpha})^2$$

$$-\frac{13}{40}(\frac{\beta}{\alpha})^3)]n^3+[\frac{3483}{2}+\frac{576201}{16}\frac{\beta}{\alpha}-\frac{653481}{4}(\frac{\beta}{\alpha})^2+\frac{1784655}{16}(\frac{\beta}{\alpha})^3+374940(\frac{\beta}{\alpha})^4$$

$$-\frac{12027825}{16}(\frac{\beta}{\alpha})^5(1-\frac{52}{75}\frac{\beta}{\alpha}+\frac{13}{75}(\frac{\beta}{\alpha})^2)]n^2+[6183+19026\frac{\beta}{\alpha}-213192(\frac{\beta}{\alpha})^2+297675(\frac{\beta}{\alpha})^3$$

$$+307125(\frac{\beta}{\alpha})^4-969570(\frac{\beta}{\alpha})^5+737100(\frac{\beta}{\alpha})^6(1-\frac{1}{4}\frac{\beta}{\alpha})]n+\frac{8505}{2}(1-\frac{\beta}{\alpha})^3[1-22(\frac{\beta}{\alpha})^2$$

$$+\frac{143}{3}(\frac{\beta}{\alpha})^4]\}(-x)^{n+2}\ ,$$

$$C_{03}=\frac{5}{8}-\frac{5}{12}x+\frac{5}{24}x^2-\frac{5}{48}x^3-\frac{1}{48}[(\gamma+\ell nx)x^4+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+4)!}\frac{1}{48}(n^4+11n^3$$

$$+46n^2+96n+120)(-x)^{n+3},$$

$$C_{13}=\frac{5}{16}(1-\frac{\beta}{\alpha})-\frac{5}{18}(1-\frac{1}{2}\frac{\beta}{\alpha})x+\frac{5}{32}(1-\frac{1}{3}\frac{\beta}{\alpha})x^2-\frac{1}{12}(1-\frac{1}{4}\frac{\beta}{\alpha})x^3-\frac{5}{288}(1-\frac{1}{5}\frac{\beta}{\alpha})[(\gamma+\ell nx)x^4$$

$$+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+5)!}[\frac{5}{288}(1-\frac{1}{5}\frac{\beta}{\alpha})(n^5+16n^4+101n^3+326n^2+600n)$$

$$+10(1-\frac{1}{4}\frac{\beta}{\alpha})](-x)^{n+3},$$

$$C_{23}=-\frac{5}{16}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})-\frac{5}{48}(1-\frac{\beta}{\alpha})^2x+\frac{1}{12}[1-\frac{9}{8}\frac{\beta}{\alpha}+\frac{3}{8}(\frac{\beta}{\alpha})^2]x^2-\frac{5}{96}[1-\frac{4}{5}\frac{\beta}{\alpha}+\frac{1}{5}(\frac{\beta}{\alpha})\ ]x^3$$

$$-\frac{1}{84}[1-\frac{5}{8}\frac{\beta}{\alpha}+\frac{1}{8}(\frac{\beta}{\alpha})^2][(\gamma+\ell nx)x^4+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}]+\sum_{n=1}^{\infty}\frac{1}{(n+6)!}\{\frac{1}{84}[1-\frac{5}{8}\frac{\beta}{\alpha}$$

$$+\frac{1}{8}(\frac{\beta}{\alpha})^2](n^6+22n^5+197n^4+932n^3+2556n^2)+[\frac{685}{14}-\frac{225}{7}\frac{\beta}{\alpha}(1-\frac{1}{5}\frac{\beta}{\alpha})]n$$

$$+\frac{75}{2}[1-\frac{4}{5}\frac{\beta}{\alpha}+\frac{1}{5}(\frac{\beta}{\alpha})^2]\}(-x)^{n+3},$$

$$C_{33}=-\frac{5}{64}(1-\frac{\beta}{\alpha})[1-5(\frac{\beta}{\alpha})^2]+\frac{5}{48}(1-\frac{\beta}{\alpha})^2\frac{\beta}{\alpha}x+\frac{5}{192}(1-\frac{\beta}{\alpha})^3x^2-\frac{1}{42}[1-\frac{29}{16}\frac{\beta}{\alpha}+\frac{5}{4}(\frac{\beta}{\alpha})^2-\frac{5}{16}(\frac{\beta}{\alpha})^3]x^3$$

$$-\frac{5}{768}\left[1-\frac{47}{35}\frac{\beta}{\alpha}+\frac{5}{7}\left(\frac{\beta}{\alpha}\right)^2-\frac{1}{7}\left(\frac{\beta}{\alpha}\right)^3\right]\left[(\gamma+\ln x)x^4+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}\right]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+7)!}\left\{\frac{5}{768}\left[1-\frac{47}{35}\frac{\beta}{\alpha}+\frac{5}{7}\left(\frac{\beta}{\alpha}\right)^2-\frac{1}{7}\left(\frac{\beta}{\alpha}\right)^3\right](n^7+29n^6+351n^5+2311n^4+9080n^3)\right.$$

$$+\left[\frac{9095}{64}-\frac{86997}{448}\frac{\beta}{\alpha}\left(1-\frac{25}{47}\frac{\beta}{\alpha}+\frac{5}{47}\left(\frac{\beta}{\alpha}\right)^2\right)\right]n^2+\left[\frac{3155}{16}-\frac{4695}{16}\frac{\beta}{\alpha}+\frac{2625}{16}\left(\frac{\beta}{\alpha}\right)^2\left(1-\frac{1}{5}\frac{\beta}{\alpha}\right)\right]n$$

$$\left.+120\left[1-\frac{29}{16}\frac{\beta}{\alpha}+\frac{5}{4}\left(\frac{\beta}{\alpha}\right)^2-\frac{5}{16}\left(\frac{\beta}{\alpha}\right)^3\right]\right\}(-x)^{n+3},$$

$$C_{43}=\frac{15}{64}\left(1-\frac{\beta}{\alpha}\right)\left[\frac{\beta}{\alpha}\left(1-\frac{7}{3}\left(\frac{\beta}{\alpha}\right)^2\right)+\frac{2}{27}\left(1-\frac{\beta}{\alpha}\right)\left(1-7\left(\frac{\beta}{\alpha}\right)^2\right)x-\frac{1}{9}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^2x^2-\frac{1}{36}\left(1-\frac{\beta}{\alpha}\right)^3x^3\right]$$

$$-\frac{1}{378}\left[1-\frac{325}{128}\frac{\beta}{\alpha}+\frac{345}{128}\left(\frac{\beta}{\alpha}\right)^2-\frac{175}{128}\left(\frac{\beta}{\alpha}\right)^3\left(1-\frac{1}{5}\frac{\beta}{\alpha}\right)\right]\left[(\gamma+\ln x)x^4+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}\right]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+8)!}\left\{\frac{1}{378}\left[1-\frac{325}{128}\frac{\beta}{\alpha}+\frac{345}{128}\left(\frac{\beta}{\alpha}\right)^2-\frac{175}{128}\left(\frac{\beta}{\alpha}\right)^3\left(1-\frac{1}{5}\frac{\beta}{\alpha}\right)\right](n^8+37n^7+583n^6\right.$$

$$+5119n^5+27568n^4)+\left[\frac{93907}{378}-\frac{7706725}{12096}\frac{\beta}{\alpha}+\frac{2726995}{4032}\left(\frac{\beta}{\alpha}\right)^2\left(1-\frac{35}{69}\frac{\beta}{\alpha}+\frac{7}{69}\left(\frac{\beta}{\alpha}\right)^2\right)\right]n^3$$

$$+\left[\frac{7293}{14}-\frac{157425}{112}\frac{\beta}{\alpha}+\frac{170085}{112}\left(\frac{\beta}{\alpha}\right)^2\left(1-\frac{35}{69}\frac{\beta}{\alpha}+\frac{7}{69}\left(\frac{\beta}{\alpha}\right)^2\right)\right]n^2+\left[\frac{3575}{6}-\frac{5525}{3}\frac{\beta}{\alpha}+\frac{4375}{2}\left(\frac{\beta}{\alpha}\right)^2\right.$$

$$\left.\left.-\frac{3500}{3}\left(\frac{\beta}{\alpha}\right)^3\left(1-\frac{1}{5}\frac{\beta}{\alpha}\right)\right]+\frac{525}{2}\left(1-\frac{\beta}{\alpha}\right)^4\right\}(-x)^{n+3},$$

$$C_{53}=\frac{5}{128}\left(1-\frac{\beta}{\alpha}\right)\left[1-14\left(\frac{\beta}{\alpha}\right)^2+21\left(\frac{\beta}{\alpha}\right)^4-\frac{4}{3}\left(1-\frac{\beta}{\alpha}\right)\frac{\beta}{\alpha}\left(1-3\left(\frac{\beta}{\alpha}\right)^2\right)x-\frac{1}{12}\left(1-\frac{\beta}{\alpha}\right)^2\left(1-9\left(\frac{\beta}{\alpha}\right)^2\right)x^2\right.$$

$$\left.+\frac{1}{6}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^3x^3\right]-\frac{1}{1536}\left(1-\frac{\beta}{\alpha}\right)^5\left[(\gamma+\ln x)x^4+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}\right]$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+9)!}\left\{\frac{1}{1536}\left(1-\frac{\beta}{\alpha}\right)^5(n^9+46n^8+916n^7+10366n^6+73639n^5)+\left[\frac{84781}{384}\right.\right.$$

$$\left.-\frac{428705}{384}\frac{\beta}{\alpha}\left(1-2\frac{\beta}{\alpha}+2\left(\frac{\beta}{\alpha}\right)^2-\left(\frac{\beta}{\alpha}\right)^3+\frac{1}{5}\left(\frac{\beta}{\alpha}\right)^4\right)\right]n^4+\left[\frac{82167}{128}-\frac{439635}{128}\frac{\beta}{\alpha}+\frac{444435}{64}\left(\frac{\beta}{\alpha}\right)^2\right.$$

$$\left.\times\left(1-\frac{\beta}{\alpha}+\frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2-\frac{1}{10}\left(\frac{\beta}{\alpha}\right)^3\right)\right]n^3+\left[\frac{33773}{32}-\frac{212865}{32}\frac{\beta}{\alpha}+\frac{229065}{16}\left(\frac{\beta}{\alpha}\right)^2-\frac{233265}{16}\left(\frac{\beta}{\alpha}\right)^3\right.$$

$$\left.\times\left(1-\frac{1}{2}\frac{\beta}{\alpha}+\frac{1}{10}\left(\frac{\beta}{\alpha}\right)^2\right)\right]n^2+\left[\frac{2895}{4}-\frac{27525}{4}\frac{\beta}{\alpha}+17850\left(\frac{\beta}{\alpha}\right)^2-\frac{40425}{2}\left(\frac{\beta}{\alpha}\right)^3\right.$$

$$\left.\left.+\frac{42525}{4}\left(\frac{\beta}{\alpha}\right)^4\left(1-\frac{1}{5}\frac{\beta}{\alpha}\right)\right]n-\frac{4725}{2}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^4\right\}(-x)^{n+3},$$

$$C_{63}=-\frac{25}{128}\left(1-\frac{\beta}{\alpha}\right)\left[\frac{\beta}{\alpha}\left(1-6\left(\frac{\beta}{\alpha}\right)^2+\frac{33}{5}\left(\frac{\beta}{\alpha}\right)^4\right)+\frac{1}{30}\left(1-\frac{\beta}{\alpha}\right)\left(1-18\left(\frac{\beta}{\alpha}\right)^2+33\left(\frac{\beta}{\alpha}\right)^4\right)x-\frac{1}{20}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^2\right.$$

$$\left.\times\left(1-\frac{11}{3}\left(\frac{\beta}{\alpha}\right)^2\right)x^2+\frac{1}{300}\left(1-\frac{\beta}{\alpha}\right)^3\left(1-11\left(\frac{\beta}{\alpha}\right)^2\right)x^3\right]+\frac{1}{1536}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^5[(\gamma+\ln x)x^4$$

$$+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}]-\sum_{n=1}^{\infty}\frac{1}{(n+10)!}\{\frac{1}{1536}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^5(n^{10}+56n^9+1376n^8+19526n^7+177299n^6)$$

$$+[\frac{5}{2}+\frac{539677}{768}\frac{\beta}{\alpha}-\frac{2698385}{768}(\frac{\beta}{\alpha})^2(1-2\frac{\beta}{\alpha}+2(\frac{\beta}{\alpha})^2-(\frac{\beta}{\alpha})^3+\frac{1}{5}(\frac{\beta}{\alpha})^4)]n^5+[\frac{125}{2}+\frac{1103991}{384}\frac{\beta}{\alpha}$$

$$-\frac{5620355}{384}(\frac{\beta}{\alpha})^2(1-2\frac{\beta}{\alpha}+2(\frac{\beta}{\alpha})^2-(\frac{\beta}{\alpha})^3+\frac{1}{5}(\frac{\beta}{\alpha})^4)]n^4+\{575+\frac{467181}{64}\frac{\beta}{\alpha}-\frac{2655105}{64}(\frac{\beta}{\alpha})^2$$

$$+\frac{2688705}{32}(\frac{\beta}{\alpha})^3(1-\frac{\beta}{\alpha}+\frac{1}{2}(\frac{\beta}{\alpha})^2-\frac{1}{10}(\frac{\beta}{\alpha})^3)]n^3+[2375+\frac{136245}{16}\frac{\beta}{\alpha}-\frac{1193625}{16}(\frac{\beta}{\alpha})^2$$

$$+\frac{1311225}{8}(\frac{\beta}{\alpha})^3-\frac{1336425}{8}(\frac{\beta}{\alpha})^4(1-\frac{1}{2}\frac{\beta}{\alpha}+\frac{1}{10}(\frac{\beta}{\alpha})^2)]n^2+[\frac{8445}{2}-2415\frac{\beta}{\alpha}-67200(\frac{\beta}{\alpha})^2$$

$$+195300(\frac{\beta}{\alpha})^3-\frac{448875}{2}(\frac{\beta}{\alpha})^4+118125(\frac{\beta}{\alpha})^5(1-\frac{1}{5}\frac{\beta}{\alpha})]+\frac{4725}{2}(1-\frac{\beta}{\alpha})^4[1-11(\frac{\beta}{\alpha})^2]\}(-x)^{n+3},$$

$$C_{73}=-\frac{25}{1024}(1-\frac{\beta}{\alpha})[1-27(\frac{\beta}{\alpha})^2+99(\frac{\beta}{\alpha})^4-\frac{429}{5}(\frac{\beta}{\alpha})^6-\frac{4}{3}(1-\frac{\beta}{\alpha})\frac{\beta}{\alpha}(1-\frac{22}{3}(\frac{\beta}{\alpha})^2+\frac{143}{15}(\frac{\beta}{\alpha})^4)x$$

$$-\frac{1}{25}(1-\frac{\beta}{\alpha})^2(1-22(\frac{\beta}{\alpha})^2+\frac{143}{4}(\frac{\beta}{\alpha})^4)x^2+\frac{2}{25}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^3(1-\frac{13}{3}(\frac{\beta}{\alpha})^2)x^3]$$

$$+\frac{1}{18432}(1-\frac{\beta}{\alpha})^5[1-13(\frac{\beta}{\alpha})^2][(\gamma+\ln x)x^4+\sum_{n=2}^{\infty}\frac{(-x)^{n+3}}{(n-1)(n-1)!}]-\sum_{n=1}^{\infty}\frac{1}{(n+11)!}$$

$$\times\{\frac{1}{18432}(1-\frac{\beta}{\alpha})^5[1-13(\frac{\beta}{\alpha})^2](n^{11}+67n^{10}+1992n^9+34662n^8+392085n^7)$$

$$+[\frac{341747}{2048}-\frac{1683135}{2048}\frac{\beta}{\alpha}-\frac{1009881}{2048}(\frac{\beta}{\alpha})^2(1-\frac{55}{3}\frac{\beta}{\alpha}+\frac{125}{3}(\frac{\beta}{\alpha})^2-43(\frac{\beta}{\alpha})^3+\frac{65}{3}(\frac{\beta}{\alpha})^4-\frac{13}{3}(\frac{\beta}{\alpha})^5)]n^6$$

$$+[\frac{8852749}{9216}-\frac{41568065}{9216}\frac{\beta}{\alpha}-\frac{24553767}{9216}(\frac{\beta}{\alpha})^2(1-\frac{55}{3}\frac{\beta}{\alpha}+\frac{125}{3}(\frac{\beta}{\alpha})^2-43(\frac{\beta}{\alpha})^3+\frac{65}{3}(\frac{\beta}{\alpha})^4$$

$$-\frac{13}{3}(\frac{\beta}{\alpha})^5)]n^5+[\frac{19150907}{4608}-\frac{86020135}{4608}\frac{\beta}{\alpha}-\frac{14139707}{1536}(\frac{\beta}{\alpha})^2+\frac{857517485}{4608}(\frac{\beta}{\alpha})^3$$

$$\times(1-\frac{25}{11}\frac{\beta}{\alpha}+\frac{129}{55}(\frac{\beta}{\alpha})^2-\frac{13}{11}(\frac{\beta}{\alpha})^3+\frac{13}{55}(\frac{\beta}{\alpha})^4)]n^4+[\frac{3242657}{256}-\frac{15627685}{256}\frac{\beta}{\alpha}$$

$$-\frac{1903971}{256}(\frac{\beta}{\alpha})^2+\frac{126029335}{256}(\frac{\beta}{\alpha})^3-\frac{291012125}{256}(\frac{\beta}{\alpha})^4(1-\frac{129}{125}\frac{\beta}{\alpha}+\frac{13}{25}(\frac{\beta}{\alpha})^2-\frac{13}{125}(\frac{\beta}{\alpha})^3)]n^3$$

$$+[\frac{1460125}{64}-\frac{9215025}{64}\frac{\beta}{\alpha}+\frac{5814345}{64}(\frac{\beta}{\alpha})^2+\frac{52264475}{64}(\frac{\beta}{\alpha})^3-\frac{135483625}{64}(\frac{\beta}{\alpha})^4$$

$$+\frac{142679805}{64}(\frac{\beta}{\alpha})^5(1-\frac{65}{129}\frac{\beta}{\alpha}+\frac{13}{129}(\frac{\beta}{\alpha})^2)]n^2+[\frac{136275}{8}-\frac{1491315}{8}\frac{\beta}{\alpha}+\frac{2686635}{8}(\frac{\beta}{\alpha})^2$$

$$+\frac{4559625}{8}(\frac{\beta}{\alpha})^3-\frac{19317375}{8}(\frac{\beta}{\alpha})^4+\frac{23336775}{8}(\frac{\beta}{\alpha})^5-\frac{12387375}{8}(\frac{\beta}{\alpha})^6(1-\frac{1}{5}\frac{\beta}{\alpha})]n$$

$$-\frac{155925}{2}\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})^4[1-\frac{13}{3}(\frac{\beta}{\alpha})^2]\}(-x)^{n+3}. \qquad (A.77)$$

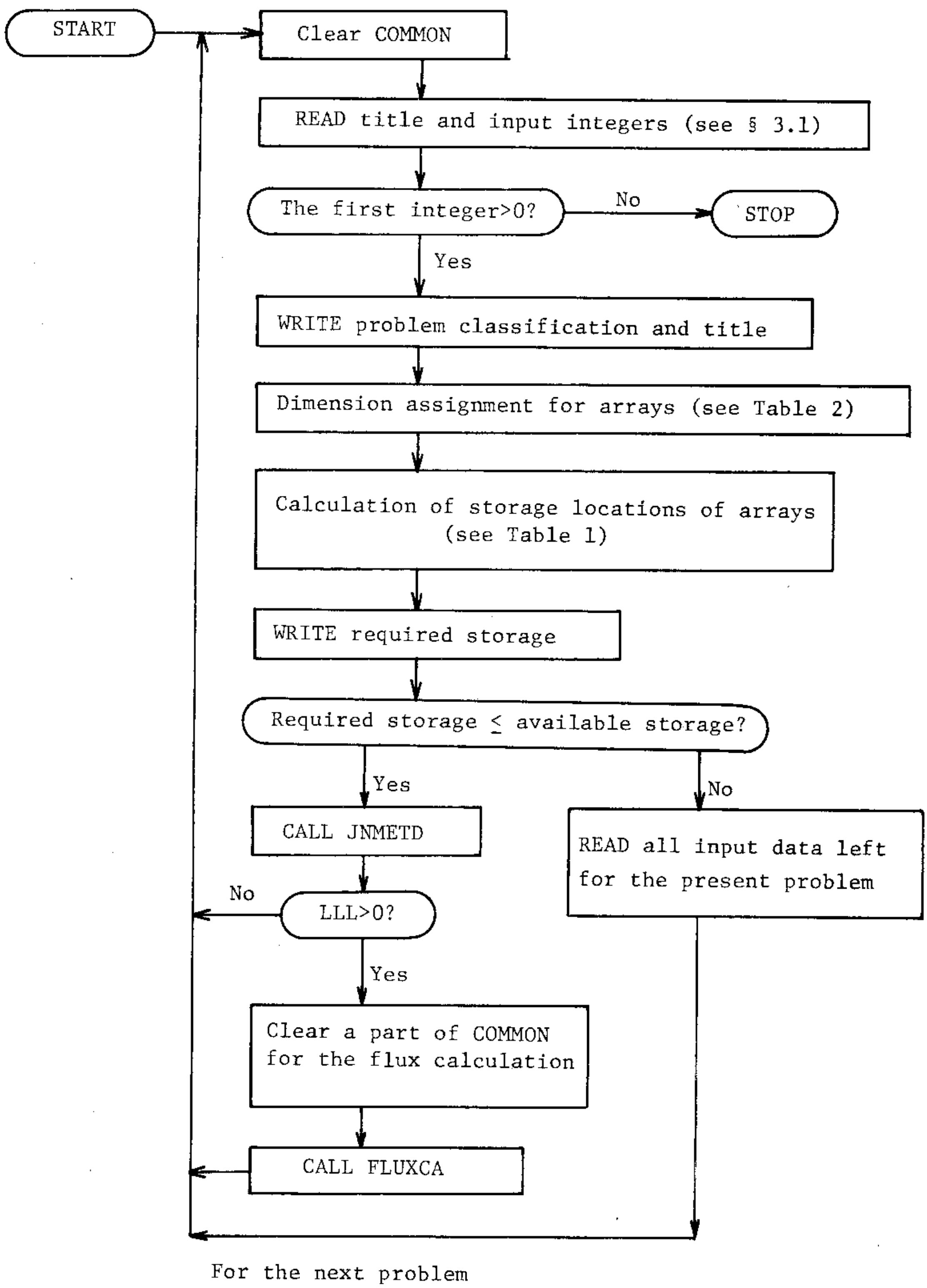
The series expansion for the expression (A.67) can be obtained by regarding the formulae (A.77) as the series expansion for $2(i)^{-q}W_{q\ell}(\alpha,\beta)$ $\div(\alpha+\beta)$.

Appendix 2. Input Data for Three Sample Problems

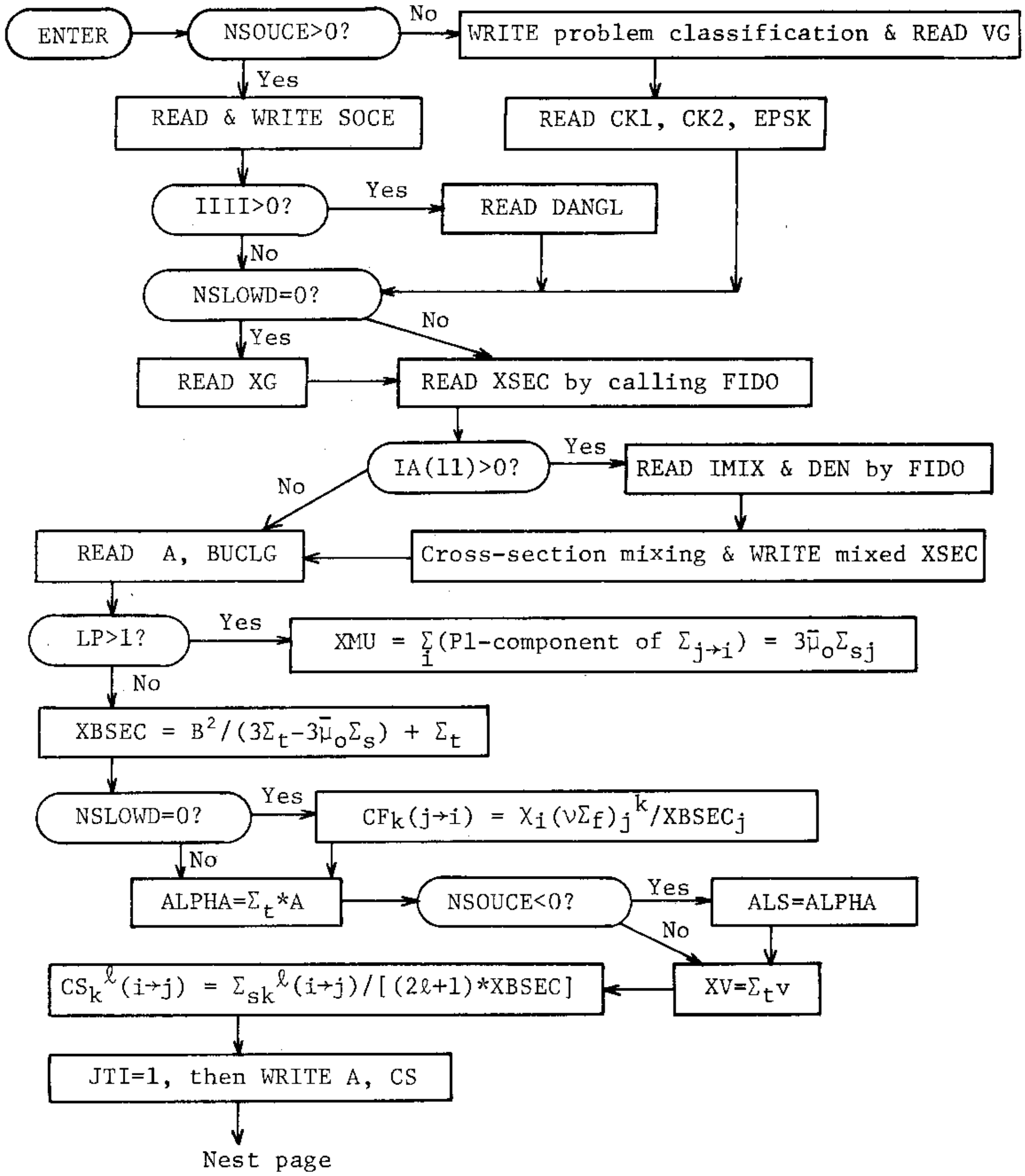
```
 TEST CASE 1      1-GROUP      (NSOUCE=0,NSLOWD=0,LP=2)
  5                   2  1  3  4  4  2  2  1 16  1  1  6  3
1.          .95         .00001
1.
14*
   .1    +00    .8    +00    1.1    +00    .9    +00    0.0          0.0
   0.0          .3    +00
   .1    +00    .8    +00    1.1    +00    .9    +00    0.0          0.0
   0.0          .3    +00
  T
.5

.5


  1.
    6
 TEST CASE 2   POLYETHYLENE   (7-GROUP,NSOUCE=-1,NSLOWD=1,LP=2)
  7    -1  1  2  1  7  3  4  7  2  1     27
285.       171.2       82.24       18.45       2.118       .2402       .024484
.000042    .000041     .000001
14*
 2R0.0           .2076601    .0680886   3R0.0
 2R0.0           .3793791    .1825128    .104429     2R0.0
 2R0.0           .7587582    .5095126    .177199     .0319487     0.0
 2R0.0           1.5774185   1.2088507   .241389     .0193013     .0031948
   .00031222     0.0         1.8210198   1.5562732   .3670025     .0078566
   .000366
   .0024435      0.0         1.839773    1.5268467   .2626097     .0015653
   0.0
   .024043      0.0          4.813576    4.789533    .31048285    .0018247
   0.0
 3R0.0           .331857    3R0.0
 3R0.0           .6746253   3R0.0
 3R0.0           1.3360566  3R0.0
 3R0.0           2.7552478  3R0.0
 3R0.0           2.8671703  3R0.0
 3R0.0           2.76267    3R0.0
 3R0.0           4.8409093  3R0.0
  T
11*                   0               1  T
12*              0.0          1.0        T
7.
.03       .048        .048        .026        .015        .007        .007
 TEST CASE 3    WATER   (7-GROUP,NSOUCE=1,NSLOWD=1,LP=2)
  5    1  1  4  1  7  3  4  7  2  2  1 30  2  1  6  3  1
.10757     .36278      .50403      .02559      .00003
14*
   .0013372     0.0          .19492189   .07978801  3R0.0
 2R0.0           .29709322   .13158687   .08437803  2R0.0
 2R0.0           .64341711   .4364553    .14904233   .02674421    0.0
 2R0.0           1.2676869   .96144278   .20038489   .0161576     .00267442
   .00026137     0.0         1.7969246   1.5766734   .30493378    .00657692
   .00030642
   .00204551     0.0         1.9091561   1.6487843   .21846222    .00131032
   0.0
   .01947        0.0         3.57954     3.56007     .25832622    .00152754
   0.0
 3R0.0           .3310298   3R0.0
 3R0.0           .5080294   3R0.0
 3R0.0           1.1002432  3R0.0
 3R0.0           2.2438058  3R0.0
 3R0.0           3.2878937  3R0.0
 3R0.0           3.4900124  3R0.0
 3R0.0           4.39065    3R0.0
  T
11*                   0               1           0           1  T
12*              0.0          1.0         0.0         1.0        T
 1.
.009      .012        .016        .018        .019        .019        .021
 3.
.009      .012        .016        .018        .019        .019        .021
  1  7
  3  3
```

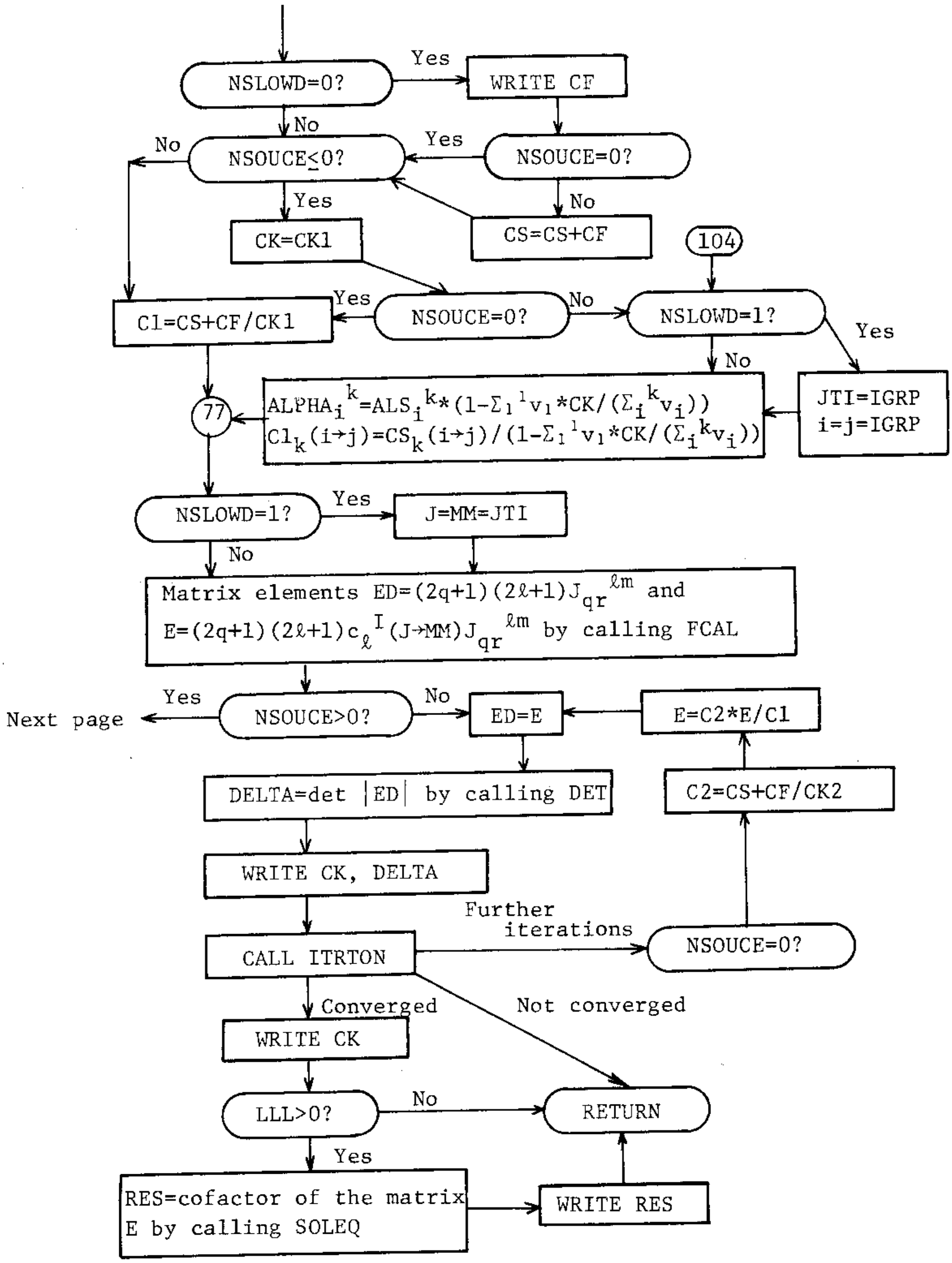
Appendix 3. Flow Diagrams of Computer Programs

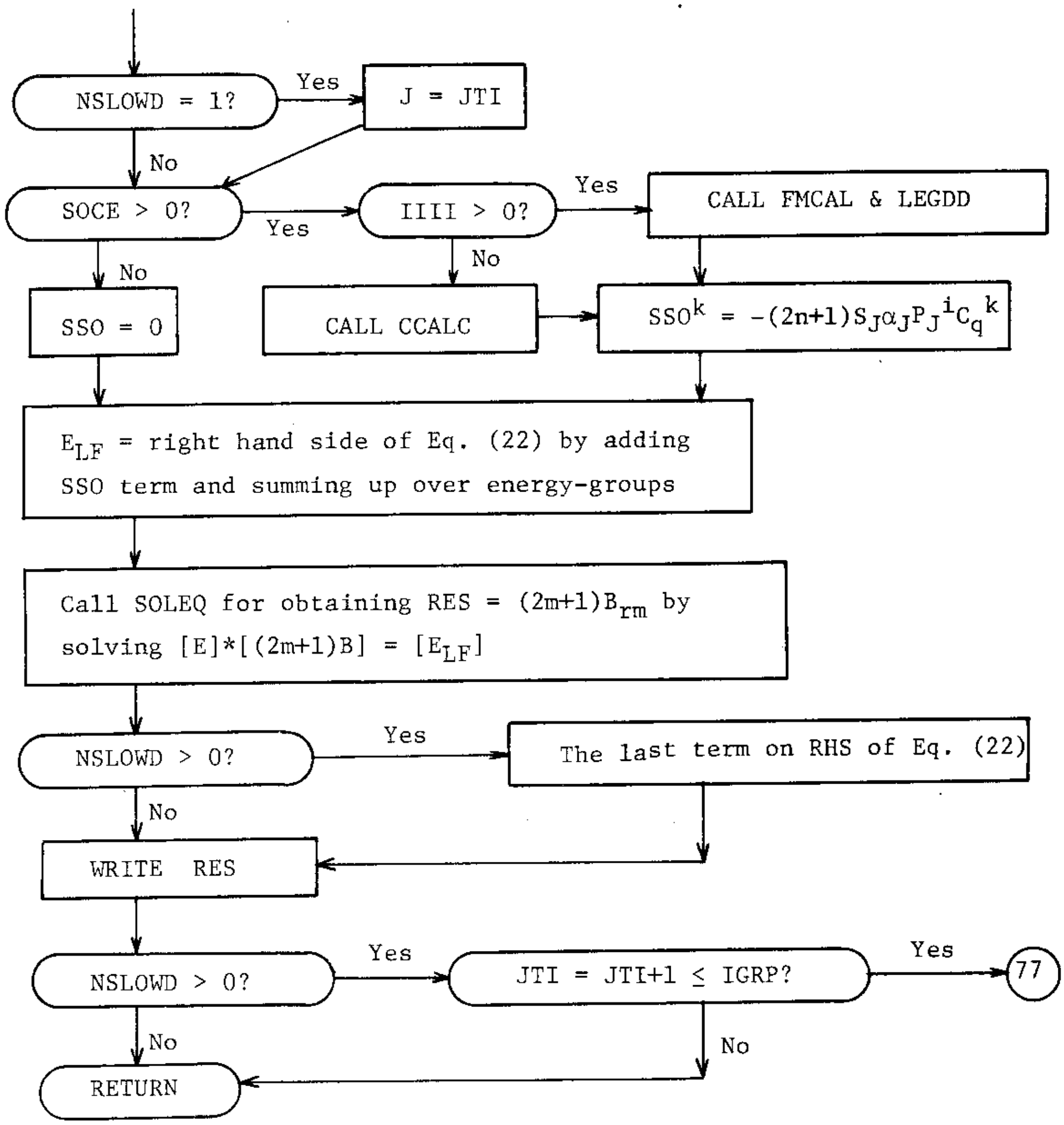
1. MAIN



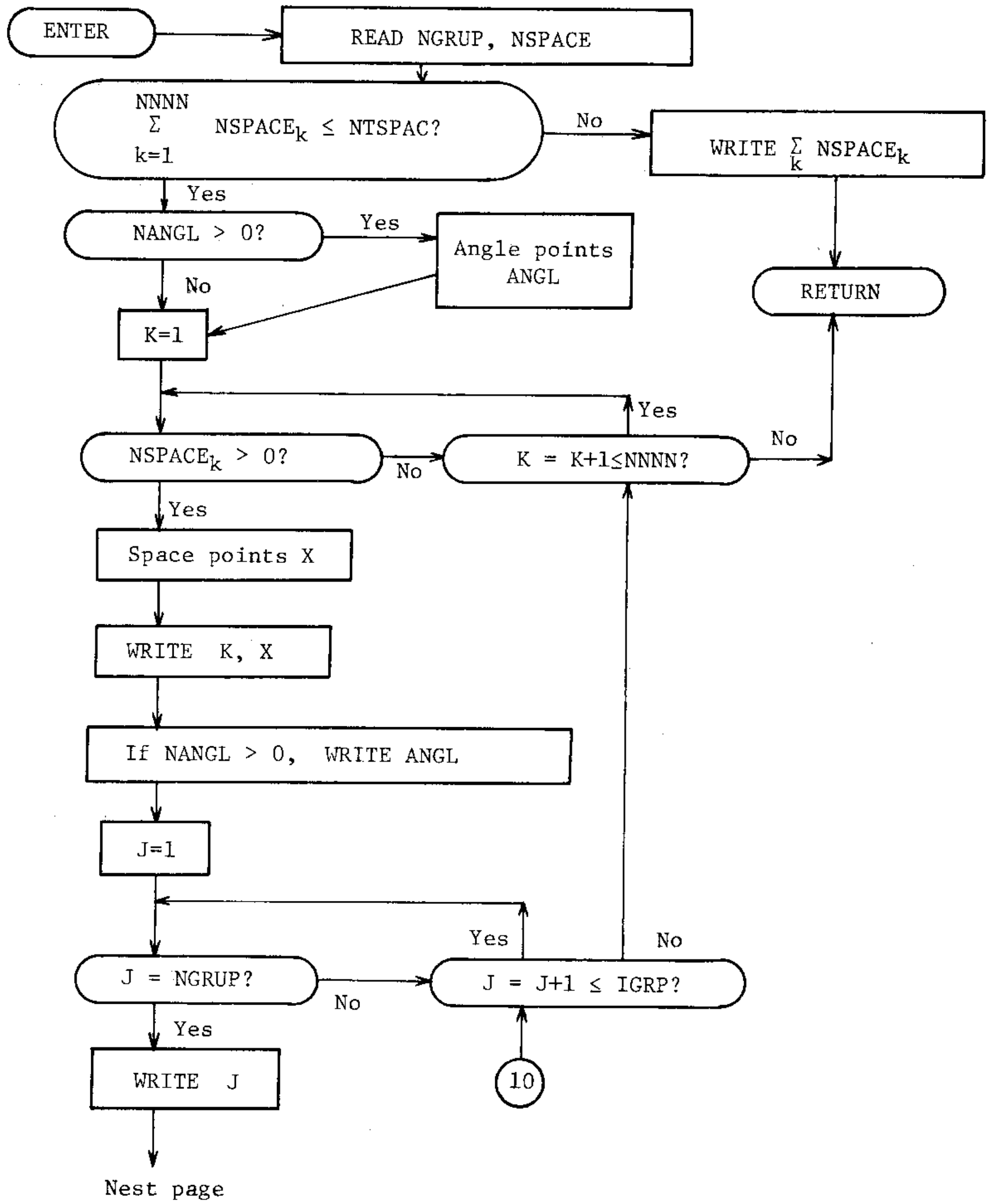
2. JNMETD

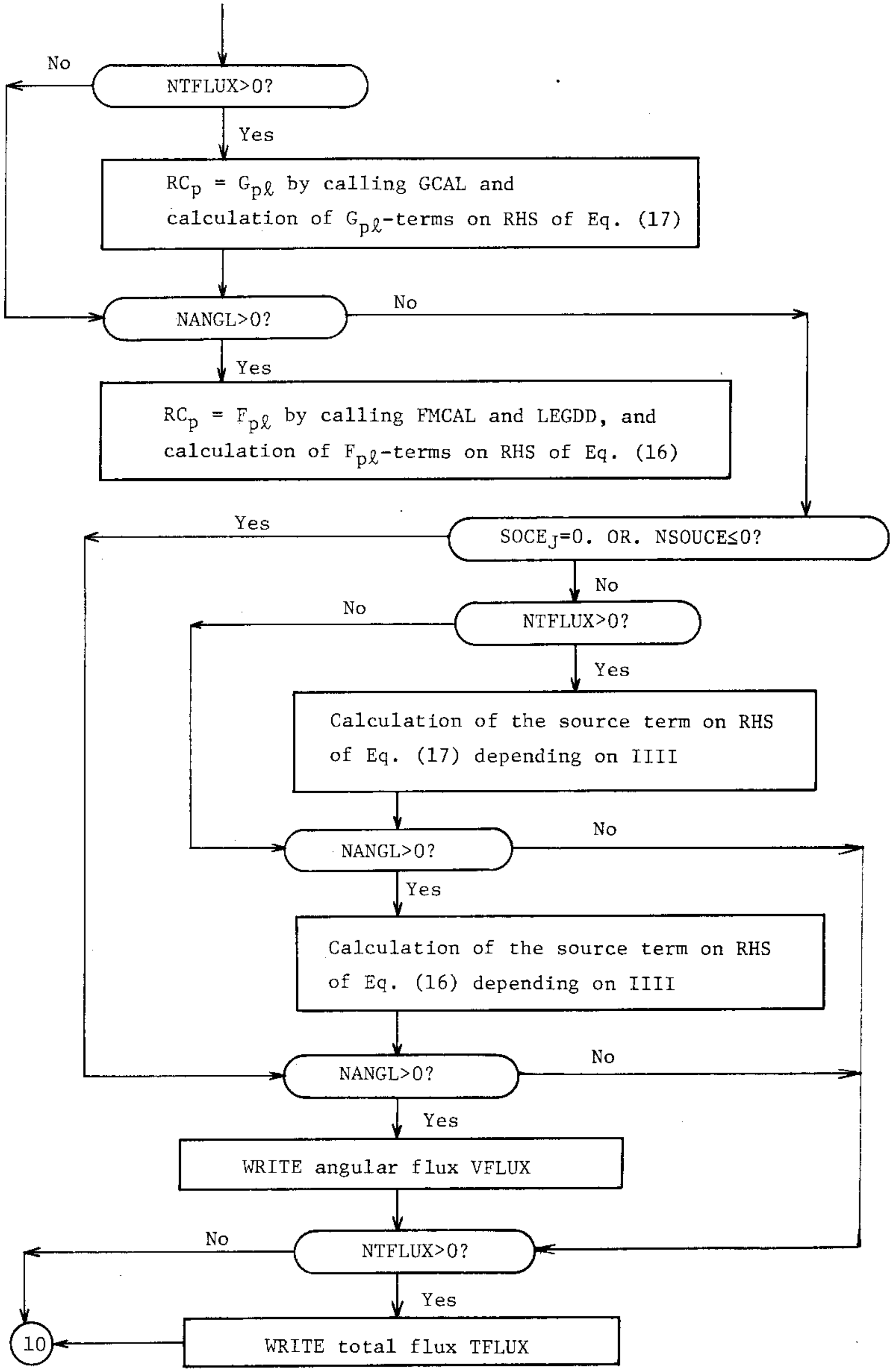

ENTER
NSOUCE>0?
No
WRITE problem classification & READ VG
Yes
READ & WRITE SOCE
READ CK1, CK2, EPSK
III>0?
Yes
READ DANGL
No
NSLOWD=0?
No
Yes
READ XG
READ XSEC by calling FIDO
IA(11)>0?
Yes
READ IMIX & DEN by FIDO
No
READ A, BUCLG
Cross-section mixing & WRITE mixed XSEC
LP>1?
Yes
XMU = $\sum_i$(P1-component of $\Sigma_{j\to i}$) = $3\bar{\mu}_o\Sigma_{sj}$
No
XBSEC = $B^2/(3\Sigma_t-3\bar{\mu}_o\Sigma_s) + \Sigma_t$
NSLOWD=0?
Yes
$CF_k(j\to i) = \chi_i(\nu\Sigma_f)_j{}^k/XBSEC_j$
No
ALPHA=Σ_t*A
NSOUCE<0?
Yes
ALS=ALPHA
No
XV=$\Sigma_t v$
$CS_k{}^\ell(i\to j) = \Sigma_{sk}{}^\ell(i\to j)/[(2\ell+1)*XBSEC]$
JTI=1, then WRITE A, CS
Nest page


NSLOWD=0?
Yes
WRITE CF
No
NSOUCE≤0?
Yes
NSOUCE=0?
No
Yes
CK=CK1
No
CS=CS+CF
104
C1=CS+CF/CK1
Yes
NSOUCE=0?
No
NSLOWD=1?
Yes
No
77
ALPHA_i^k=ALS_i^k*(1-Σ_1^l v_l*CK/(Σ_i^k v_i))
C1_k(i→j)=CS_k(i→j)/(1-Σ_1^l v_l*CK/(Σ_i^k v_i))
JTI=IGRP
i=j=IGRP
NSLOWD=1?
Yes
J=MM=JTI
No
Matrix elements ED=(2q+1)(2ℓ+1)J_qr^ℓm and
E=(2q+1)(2ℓ+1)c_ℓ^I(J→MM)J_qr^ℓm by calling FCAL
Next page
Yes
NSOUCE>0?
No
ED=E
E=C2*E/C1
DELTA=det |ED| by calling DET
C2=CS+CF/CK2
WRITE CK, DELTA
Further iterations
CALL ITRTON
NSOUCE=0?
Converged
Not converged
WRITE CK
LLL>0?
No
RETURN
Yes
RES=cofactor of the matrix E by calling SOLEQ
WRITE RES


NSLOWD = 1?
Yes
J = JTI
No
SOCE > 0?
Yes
IIII > 0?
Yes
CALL FMCAL & LEGDD
No
No
SSO = 0
CALL CCALC
$SSO^k = -(2n+1)S_J\alpha_J P_J{}^i C_q{}^k$
E_{LF} = right hand side of Eq. (22) by adding
SSO term and summing up over energy-groups
Call SOLEQ for obtaining RES = $(2m+1)B_{rm}$ by
solving $[E]*[(2m+1)B] = [E_{LF}]$
NSLOWD > 0?
Yes
The last term on RHS of Eq. (22)
No
WRITE RES
NSLOWD > 0?
Yes
JTI = JTI+1 ≤ IGRP?
Yes
77
No
No
RETURN

3. FLUXCA


ENTER
READ NGRUP, NSPACE
$\sum_{k=1}^{NNNN}$ NSPACE$_k$ ≤ NTSPAC?
No
WRITE $\sum_k$ NSPACE$_k$
Yes
NANGL > 0?
Yes
Angle points ANGL
No
RETURN
K=1
Yes
No
NSPACE$_k$ > 0?
No
K = K+1≤NNNN?
Yes
Space points X
WRITE K, X
If NANGL > 0, WRITE ANGL
J=1
Yes
No
J = NGRUP?
No
J = J+1 ≤ IGRP?
Yes
WRITE J
10
Nest page


No
NTFLUX>0?
Yes
RCp = Gpℓ by calling GCAL and
calculation of Gpℓ-terms on RHS of Eq. (17)
NANGL>0?
No
Yes
RCp = Fpℓ by calling FMCAL and LEGDD, and
calculation of Fpℓ-terms on RHS of Eq. (16)
Yes
SOCEJ=0. OR. NSOUCE≤0?
No
No
NTFLUX>0?
Yes
Calculation of the source term on RHS
of Eq. (17) depending on IIII
NANGL>0?
No
Yes
Calculation of the source term on RHS
of Eq. (16) depending on IIII
NANGL>0?
No
Yes
WRITE angular flux VFLUX
No
NTFLUX>0?
Yes
10
WRITE total flux TFLUX