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Momentum and Heat Frictions Between Fast Ions and Thermal Plasma Species

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Momentum and heat friction forces between fast ions (isotropic to the lowest order) and thermal bulk plasma species (near Maxwellian) have been derived analytically for the first time using the linearized Fokker-Planck collision operator. Therefore effects of flow-type distortion of test particles and flow-type restoring from field particles are both retained self-consistently. Here, the momentum and heat friction forces are defined as the momentum moments of the Coulomb collision operator weighted by Laguerre polynomials $L_0^{(3/2)}(w^2_\alpha/v^2_{T\alpha})$ and $L_1^{(3/2)}(w^2_\alpha/v^2_{T\alpha})$ respectively, *i*, *e*. momentum friction force $F_0^{\alpha/\beta} \equiv \int m_\alpha w_\alpha L_0^{(3/2)} C_{\alpha\beta}[f_\alpha, f_\beta]\, d^3w_\alpha$ while heat friction force $F_1^{\alpha/\beta} \equiv -\int m_\alpha w_\alpha L_1^{(3/2)} C_{\alpha\beta}[f_\alpha, f_\beta]\, d^3w_\alpha$. Where $w_\alpha \equiv v - V_\alpha$ represents the random velocity measured in moving reference frame of test particles, and the superscript α/β denotes test particles of α species colliding with field particles of β species. The momentum and heat friction forces have been explicitly calculated for electron-fast ion collisions ($F_k^{e/f}$), fast ion-electron collisions ($F_k^{f/e}$), thermal ion-fast ion collisions ($F_k^{i/f}$) and fast ion-

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thermal ion collisions ($F_k^{f/i}$) wherein the subscripts k's correspond to the indices of Laguerre polynomials (k=0, 1). The collisional moment matrix is shown asymmetric in the presence of fast ions because the approximate self-adjointness of the Coulomb collision operator is no longer a valid property.

Keywords : Fast Ions, Friction Forces, Collisions

高速イオンと熱化プラズマ粒子間の粒子流および熱流に対する摩擦力

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（1992年5月25日受理）

線形化されたフォッカー・プランク衝突作用素を用いて，最低次で等方速度分布をもった高速イオンと熱化プラズマ粒子間に働く，粒子流および熱流に対する摩擦力を解析的に評価した。テスト粒子分布関数の流体的変形の効果およびフィールド粒子からの流体的復元力の効果を矛盾なく考慮している。粒子流および熱流に対する摩擦力は0次および1次のレゲーレ多項式と衝突作用素の内積として定義しており，高速イオンとプラズマ電子およびプラズマ・イオン間の各摩擦力の解析的表式を与えた。この解析の結果，高速イオンに対してはクローン衝突作用素の自己共役性が失われ，摩擦係数行列が非対称となることが示された。

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1 INTRODUCTION

Frictional forces specifying the collisional transfer of momentum and energy flux are important to transport theory, parallel neoclassical current, resistive instability studies and neutral beam injection heating *etc.* The friction forces have been well formularized for thermal plasma collisions by many authors.[1-5] However, this problem has not been well answered for fast ion–thermal plasma collisions except the preliminary estimation given in Ref. 3 which considers the parallel momentum friction force for collisions between fast ion test-particles and Maxwellian field-particles only. It emerges necessary to quantify the friction forces for evaluations of bootstrap currents in plasmas with fast ion component, which are of more and more interests for maintaining more stable plasmas in present large tokamak devices.

In this paper, the momentum friction $\mathbf{F}_0$ and heat friction $\mathbf{F}_1$, between fast ions and thermal plasma species, are calculated using the linearized full Fokker-Planck collision operator thereby retained the restoring effects from field-particles. The fast ion distribution is presumably consisted of a lowest order isotropic function[6-9]

$$f_{0f} = \frac{\dot{n}_f \tau_s}{4\pi} \frac{1}{v^3 + v_c^3} H(v_0 - v), \tag{1}$$

and a higher order flow-type distorted function $f_{1f} \propto H(v_0 - v)$.[8,9] Here, $\dot{n}_f$ is the fast ion production rate, τ_s represents the slowing-down time defined by

$$\frac{1}{\tau_s} \equiv \frac{4}{3\sqrt{\pi}} \frac{4\pi n_e Z_f^2 e^4 \ln \Lambda}{m_e m_f v_{T_e}^3}, \tag{2}$$

v_0 denotes the fast ion birth speed assumed much faster than ion thermal speed but much slower than electron thermal speed, *i.e.* $v_{T_i}^2 \ll v_0^2 \ll v_{T_e}^2$, $H(v_0 - v)$ is a unit step function which vanishes abruptly for its negative argument, and v_c represents the critical speed given by

$$v_c^3 \equiv \sum_i \frac{3\sqrt{\pi}}{4} \frac{m_e v_{T_e}^3}{m_i} \frac{n_i Z_i^2}{n_e} = \sum_i v_{ci}^3 \tag{3}$$

beyond which drag by electrons dominant and by ions otherwise. In Sec. 2, the general formulas for calculating friction forces are presented, which can be applied to arbitrary α species test-particles and β species field-particles. Formal expressions for Rosenbluth potentials are also given, assuming that field-particle distribution can be expanded in terms of spherical harmonics for its solid angle dependence, for later reference. In Sec. 3, thermal test-particle species is considered assuming that the distribution function is consisted of a local Maxwellian and a small flow-type distortion (two-polynomials). Then, a formal expression of the frictional moment $\mathbf{M}_k^{\alpha/\beta}$ $(k = 0, 1)$ is presented for thermal test-particle collisions with an arbitrary isotropic, thermal or energetic, field-particles in Sec. 3.3.1. Furthermore, $\mathbf{M}_k^{e/f}$ and $\mathbf{M}_k^{i/f}$ are calculated for electron – fast ion and ion – fast ion collisions accordingly. In parallel, the restoring frictional moment $\mathbf{N}_k^{\alpha/\beta}$ $(k = 0, 1)$ due to collisions with distorted field-particles is discussed and applied to specific calculations for $\mathbf{N}_k^{e/f}$ and $\mathbf{N}_k^{i/f}$ moments in Sec. 3.3.2. Next, Sec. 4 is devoted to the determination of friction moments due to collisions with distorted thermal field-particles. The isotropic test-particle species is kept unspecified at first for more generality in discussing the frictions with maxwellian field-particles and the restoring frictions with the distorted thermal field-particles in subsections 4.4.1 and 4.4.2 respectively. Then, the results are applied to the explicit derivations of $\mathbf{M}_k^{f/e}$, $\mathbf{M}_k^{f/i}$, $\mathbf{N}_k^{f/e}$ and $\mathbf{N}_k^{f/i}$. Finally, in Sec. 5, total friction forces are summarized into matrix forms for electron – fast ion e/f collisions, fast ion – electron f/e collisions, ion – fast ion i/f collisions and fast ion – ion f/i collisions respectively. Brief discussion is given on the asymmetry of these matrix elements caused by the break of the approximate self-adjoint property of the Coulomb collision operator.

2 COLLISIONAL MOMENTS

Define momentum and heat friction forces[10] as the random momentum moments of the Fokker-Planck collision operator weighted by Laguerre polynomials, *i.e.*, momentum friction force

$$\mathbf{F}_0^{\alpha/\beta} \equiv \int m_\alpha \mathbf{w}_\alpha \, L_0^{(3/2)}(x'_\alpha) \, C_{\alpha\beta}[f_\alpha, \, f_\beta] \, d^3 w_\alpha \tag{4}$$

be applied to arbitrary α species test-particles and β species field-particles. Formal expressions for Rosenbluth potentials are also given, assuming that field-particle distribution can be expanded in terms of spherical harmonics for its solid angle dependence, for later reference. In Sec. 3, thermal test-particle species is considered assuming that the distribution function is consisted of a local Maxwellian and a small flow-type distortion (two-polynomials). Then, a formal expression of the frictional moment $\mathbf{M}_k^{\alpha/\beta}$ $(k = 0, 1)$ is presented for thermal test-particle collisions with an arbitrary isotropic, thermal or energetic, field-particles in Sec. 3.3.1. Furthermore, $\mathbf{M}_k^{e/f}$ and $\mathbf{M}_k^{i/f}$ are calculated for electron – fast ion and ion – fast ion collisions accordingly. In parallel, the restoring frictional moment $\mathbf{N}_k^{\alpha/\beta}$ $(k = 0, 1)$ due to collisions with distorted field-particles is discussed and applied to specific calculations for $\mathbf{N}_k^{e/f}$ and $\mathbf{N}_k^{i/f}$ moments in Sec. 3.3.2. Next, Sec. 4 is devoted to the determination of friction moments due to collisions with distorted thermal field-particles. The isotropic test-particle species is kept unspecified at first for more generality in discussing the frictions with maxwellian field-particles and the restoring frictions with the distorted thermal field-particles in subsections 4.4.1 and 4.4.2 respectively. Then, the results are applied to the explicit derivations of $\mathbf{M}_k^{f/e}$, $\mathbf{M}_k^{f/i}$, $\mathbf{N}_k^{f/e}$ and $\mathbf{N}_k^{f/i}$. Finally, in Sec. 5, total friction forces are summarized into matrix forms for electron – fast ion e/f collisions, fast ion – electron f/e collisions, ion – fast ion i/f collisions and fast ion – ion f/i collisions respectively. Brief discussion is given on the asymmetry of these matrix elements caused by the break of the approximate self-adjoint property of the Coulomb collision operator.

2 COLLISIONAL MOMENTS

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and heat friction force

$$\mathbf{F}_1^{\alpha/\beta} \equiv -\int m_\alpha \mathbf{w}_\alpha \, L_1^{(3/2)}(x'_\alpha)\, C_{\alpha\beta}[f_\alpha,\, f_\beta]\, d^3 w_\alpha \tag{5}$$

where $\mathbf{w}_\alpha \equiv \mathbf{v} - \mathbf{V}_\alpha$ represents the random velocity measured in moving reference frame of test particles (α species), and $x'_\alpha \equiv w_\alpha^2/v_{T_\alpha}^2$ is the normalized random kinetic energy. Combining Eqs. (4) and (5), we can express the frictional forces in the laboratory reference frame in a more compact form

$$\widehat{\mathbf{F}}_k^{\alpha/\beta} \equiv (-1)^k \int m_\alpha \mathbf{v}\, L_k^{(3/2)}(x_\alpha)\, C_{\alpha\beta}[f_\alpha,\, f_\beta]\, d^3 v \tag{6}$$

for convenience of derivations, where $x_\alpha \equiv v^2/v_{T_\alpha}^2$ is the normalized kinetic energy. Obviously, the momentum friction forces defined in two reference frame are equivalent to each other, *i.e.*,

$$\mathbf{F}_0^{\alpha/\beta} = \widehat{\mathbf{F}}_0^{\alpha/\beta}, \tag{7}$$

which indicates the Galilean invariance physically. However, the heat friction forces in two reference frames are related by

$$\mathbf{F}_1^{\alpha/\beta} \simeq \widehat{\mathbf{F}}_1^{\alpha/\beta} - \frac{10}{3}\frac{Q_{\alpha\beta}}{v_{T_\alpha}^2}\mathbf{V}_\alpha, \tag{8}$$

with $Q_{\alpha\beta} = \int d^3v\, \frac{1}{2} m_\alpha v^2 C_{\alpha\beta}$ denoting the collisional energy exchange rate. The second term in Eq. (8) is usually second order and hence can be neglected if α and β are both thermal species. Then, using Fokker-Planck form of the Coulomb collision operator[11,12] (also given in Appendix A) to integrate by parts and using the tensor relation given in Eq. (D1) we can obtain

$$\begin{aligned}
&\int m_\alpha \mathbf{v}\, L_k^{(3/2)}(x_\alpha)\, C_{\alpha\beta}[f_\alpha,\, f_\beta]\, d^3 v \\
&= m_\alpha \int d^3 v\, \mathbf{J}_v^{\alpha/\beta} \cdot \frac{\partial}{\partial \mathbf{v}} \left[\mathbf{v} L_k^{(3/2)}(x_\alpha)\right] \\
&= m_\alpha \gamma_{\alpha\beta} \int d^3 v\, f_\alpha \left\{ \left(1 + \frac{m_\alpha}{m_\beta}\right) \left[L_k^{(3/2)}(x_\alpha)\mathbf{I} + \frac{2}{v_{T_\alpha}^2} \dot{L}_k^{(3/2)} \mathbf{v}\mathbf{v} \right] \cdot \frac{\partial h_\beta}{\partial \mathbf{v}} \right. \\
&\quad \left. + \frac{2}{v_{T_\alpha}^2} \dot{L}_k^{(3/2)} \left(\mathbf{v} h_\beta + \mathbf{v} \cdot \frac{\partial^2 g_\beta}{\partial \mathbf{v} \partial \mathbf{v}} \right) \right\}, \qquad \text{for } k = 0, 1.
\end{aligned} \tag{9}$$

Where, $\mathbf{I}$ is a unit tensor, the dot on top of Laguerre polynomials denotes the derivative with respect to its argument x_α, and Rosenbluth potentials, $h_\beta = h[f_\beta]$ and $g_\beta = g[f_\beta]$, are functional of the field particle distribution. In deriving Eq. (9), all of the surface integrals in velocity space have been dropped off considering that energy and momentum moments *etc.* are finite thereby the distribution function has to vanish fast enough on the infinite boundary. The relation $\nabla_v^2 g_\beta = 2h_\beta$ has also been used.

From Eq. (9) it is not difficult to see that collisions of isotropic test particles with isotropic field particles do not contribute to frictional forces which are odd moments of velocity $\mathbf{v}$. Therefore, linearizing the Coulomb collision operator, the frictional forces can be reduced to the following form

$$(-1)^k \hat{\mathbf{F}}_k^{\alpha/\beta} \simeq \frac{n_\alpha}{\tau_{\alpha\beta}} \left(\mathbf{M}_k^{\alpha/\beta} + \mathbf{N}_k^{\alpha/\beta} \right). \tag{10}$$

Here, the relaxation time $\tau_{\alpha\beta}$ for $\alpha - \beta$ collisions is defined as

$$\frac{1}{\tau_{\alpha\beta}} \equiv \frac{4}{3\sqrt{\pi}} \frac{4\pi n_\beta e_\alpha^2 e_\beta^2 \ln \Lambda_{\alpha\beta}}{m_\alpha^2 v_{T_\alpha}^3} \tag{11}$$

wherein $\ln \Lambda_{\alpha\beta} \simeq \ln \Lambda$ will be used consistently throughout this paper, and

$$\frac{n_\alpha}{\tau_{\alpha\beta}} \mathbf{M}_k^{\alpha/\beta} \equiv \int d^3v \, m_\alpha \mathbf{v} \, L_k^{(3/2)}(x_\alpha) \, C_{\alpha\beta}[f_{1\alpha}, f_{0\beta}] \tag{12}$$

represents frictions arose from collisions of test particles (described by the first order flow-distorted distribution $f_{1\alpha}$) with field particles (describe by the the lowest order isotropic distribution $f_{0\beta}$), whereas

$$\frac{n_\alpha}{\tau_{\alpha\beta}} \mathbf{N}_k^{\alpha/\beta} \equiv \int d^3v \, m_\alpha \mathbf{v} \, L_k^{(3/2)}(x_\alpha) \, C_{\alpha\beta}[f_{0\alpha}, f_{1\beta}] \tag{13}$$

represents restoring frictions due to collisions of test particles (described by the lowest order isotropic distribution $f_{0\alpha}$) with field particles (described by the first order flow-distorted distribution $f_{1\beta}$). Having obtained the above three equations, the frictional forces can be calculated by substituting in appropriate distribution functions for test and field particles.

To determine the Rosenbluth potentials needed, we assume that the distribution function for field particles can be expressed in terms of spherical harmonics $Y_{\ell m}(\mathbf{\Omega})$ for its solid angle variation

$$f_\beta = \sum_{\ell,m} G_{\ell m}^{(\beta)}(v) Y_{\ell m}(\mathbf{\Omega}). \tag{14}$$

Where the energy dependent coefficient $G_{\ell m}^{(\beta)}(v)$ can be further expanded in terms of Laguerre polynomials for thermal plasmas. Substituting this distribution function into Eq. (A5), we can obtain

$$\begin{aligned}
h_\beta &= \int d^3v' \frac{f_\beta(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} \\
&= \sum_{\ell,m}\sum_{\ell',m'} \frac{4\pi}{2\ell+1} Y_{\ell m}(\mathbf{\Omega}) \int v'^2 dv' \frac{v_<^\ell}{v_>^{\ell+1}} G_{\ell'm'}^{(\beta)}(v') \\
&\quad \times \int d\mathbf{\Omega}' Y_{\ell'm'}(\mathbf{\Omega}') Y_{\ell m}^*(\mathbf{\Omega}') \\
&= \sum_{\ell,m} \frac{4\pi}{2\ell+1} Y_{\ell m}(\mathbf{\Omega}) \int v'^2 dv' \frac{v_<^\ell}{v_>^{\ell+1}} G_{\ell m}^{(\beta)}(v'). \qquad (15)
\end{aligned}$$

Where we have used the expansion given in Eq. (A7) and the orthogonality of spherical harmonics $\int d\mathbf{\Omega}' Y_{\ell m}^*(\mathbf{\Omega}') Y_{\ell m}(\mathbf{\Omega}) = \delta_{\ell\ell'}\delta_{mm'}$. Similarly, we can obtain

$$\begin{aligned}
g_\beta &= \int d^3v' f_\beta(\mathbf{v}')|\mathbf{v}-\mathbf{v}'| \\
&= \sum_{\ell,m}\sum_{\ell',m'} \frac{4\pi}{2\ell+1} Y_{\ell m}(\mathbf{\Omega}) \int d\mathbf{\Omega}' Y_{\ell'm'}(\mathbf{\Omega}') Y_{\ell m}^*(\mathbf{\Omega}') \\
&\quad \times \int v'^2 dv' \frac{v_<^\ell}{v_>^{\ell-1}} \left[\frac{1}{2\ell+3}\left(\frac{v_<}{v_>}\right)^2 - \frac{1}{2\ell-1}\right] G_{\ell'm'}^{(\beta)}(v') \\
&= \sum_{\ell,m} \frac{4\pi}{2\ell+1} Y_{\ell m}(\mathbf{\Omega}) \int v'^2 dv' \frac{v_<^\ell}{v_>^{\ell-1}} \left[\frac{1}{2\ell+3}\frac{v_<^2}{v_>^2} - \frac{1}{2\ell-1}\right] G_{\ell m}^{(\beta)}(v'), \qquad (16)
\end{aligned}$$

using Eq. (A8). Here, $v_< \equiv \min(v, v')$ whereas $v_> \equiv \max(v, v')$. If the field-particle distribution is isotropic, then the above results in Eqs. (15) and (16) will be reduced to the following form:

$$\begin{aligned}
h_\beta^{(0)} &= \int d^3v' \frac{f_{0\beta}(v')}{|\mathbf{v}-\mathbf{v}'|} \\
&= \int v'^2\, dv' \int d\Omega' \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{v_<^\ell}{v_>^{\ell+1}} Y_{\ell m}^*(\mathbf{\Omega}') Y_{\ell m}(\mathbf{\Omega}) f_{0\beta}(v') \\
&= 4\pi \int v'^2\, dv' \frac{1}{v_>} f_{0\beta}(v') \qquad (17)
\end{aligned}$$

$$\begin{aligned}
g_\beta^{(0)} &= \int d^3v' f_{0\beta}(\mathbf{v}')|\mathbf{v}-\mathbf{v}'| \\
&= 4\pi \int v'^2\, dv'\, v_> \left[\frac{1}{3}\left(\frac{v_<}{v_>}\right)^2 + 1\right] f_{0\beta}(v'), \qquad (18)
\end{aligned}$$

since all $G_{\ell m}^{(\beta)}(v')$ but $G_{00}^{(\beta)}(v')$ are zero.

3. THERMAL TEST PARTICLES

The distribution function for thermal plasma species can be approximated by a lowest order Maxwellian plus a flow-type fluid distortion of the form[4]

$$f_{1\alpha} = \frac{2}{v_{T_\alpha}^2}\mathbf{v}\cdot\left[\mathbf{V}_\alpha L_0^{(3/2)}(x_\alpha) - \frac{2}{5p_\alpha}\mathbf{q}_\alpha L_1^{(3/2)}(x_\alpha)\right] f_M^{(\alpha)}, \tag{19}$$

where

$$f_M^{(\alpha)} \equiv \frac{n_\alpha}{\pi^{3/2}v_{T_\alpha}^3}\exp\{-x_\alpha\} \tag{20}$$

is the Maxwellian distribution, n_α the density, T_α the temperature, $v_{T_\alpha} = \sqrt{2T_\alpha/m_\alpha}$ the thermal speed, p_α the pressure, $\mathbf{V}_\alpha$ the mass flow and $\mathbf{q}_\alpha$ the heat flux for α species.

3.1 $\mathbf{M}_k^{\alpha/\beta}$ for thermal test particles

Substituting in the distribution function given in Eq. (19) and using Eqs. (9) and (12), frictions of distorted thermal test particles with isotropic field particles can be written as

$$\begin{aligned}\frac{n_\alpha}{\tau_{\alpha\beta}}\mathbf{M}_k^{\alpha/\beta} &= m_\alpha\gamma_{\alpha\beta}\frac{n_\alpha}{\pi^{3/2}v_{T_\alpha}^3}\int d^3v\frac{2}{v_{T_\alpha}^2}\mathbf{v}\cdot\left(\mathbf{V}_\alpha L_0^{(3/2)} - \frac{2}{5p_\alpha}\mathbf{q}_\alpha L_1^{(3/2)}\right)e^{-x_\alpha}\\ &\times\left\{\left(1+\frac{m_\alpha}{m_\beta}\right)\left[L_k^{(3/2)}(x_\alpha) + 2x_\alpha\dot{L}_k^{(3/2)}(x_\alpha)\right]\frac{\mathbf{v}}{v}\frac{\partial h_\beta^{(0)}}{\partial v}\right.\\ &\left.+\frac{2}{v_{T_\alpha}^2}\dot{L}_k^{(3/2)}(x_\alpha)\left[\mathbf{v}h_\beta^{(0)} + v\frac{\partial}{\partial v}\left(\frac{\mathbf{v}}{v}\frac{\partial g_\beta^{(0)}}{\partial v}\right)\right]\right\}, \quad \text{for } k = 0, 1.\end{aligned}$$

Where $h_\beta^{(0)} \equiv h[f_{0\beta}]$ and $g_\beta^{(0)} \equiv g[f_{0\beta}]$ are functional of the isotropic distribution for field particles of β species. Considering that

$$\nabla_v^2 g_\beta^{(0)} = \frac{\partial^2 g_\beta^{(0)}}{\partial v^2} + \frac{2}{v}\frac{\partial g_\beta^{(0)}}{\partial v} = 2h_\beta^{(0)},$$

the above equation can be reduced to a more compact form

$$\mathbf{M}_k^{\alpha/\beta} = M_{k0}^{\alpha/\beta}\mathbf{u}_{0\alpha} + M_{k1}^{\alpha/\beta}\mathbf{u}_{1\alpha}. \tag{21}$$

Where we have introduced the notations

$$\begin{aligned}\mathbf{u}_{0\alpha} &\equiv \mathbf{V}_\alpha\\ \mathbf{u}_{1\alpha} &\equiv -\frac{2}{5p_\alpha}\mathbf{q}_\alpha\end{aligned}$$

as mass flow and normalized heat flux respectively, and the matrix element

$$
\begin{aligned}
M_{kj}^{\alpha/\beta} \equiv\ & 2\frac{m_\alpha}{n_\beta}\int_0^\infty v^2\,dv\; x_\alpha L_j^{(3/2)}(x_\alpha)e^{-x_\alpha} \\
& \times\left\{\left(1+\frac{m_\alpha}{m_\beta}\right)\left[L_k^{(3/2)}(x_\alpha)+2x_\alpha \dot{L}_k^{(3/2)}(x_\alpha)\right]\frac{1}{v}\frac{\partial h_\beta^{(0)}}{\partial v}\right. \\
& \left.+\frac{2}{v_{T_\alpha}^2}\dot{L}_k^{(3/2)}(x_\alpha)\left(3h_\beta^{(0)}-\frac{2}{v}\frac{\partial g_\beta^{(0)}}{\partial v}\right)\right\}, \quad \text{for } k,j=0,1. \qquad (22)
\end{aligned}
$$

To proceed further, we will consider e/f and i/f collisions separately below.

A. $M_{kj}^{e/f}$ friction coefficients

For electro–fast ion collisions, Eq. (22) becomes

$$
\begin{aligned}
M_{kj}^{e/f} \simeq\ & 2\frac{m_e}{n_f}\int_{v_0}^\infty v^2\,dv\; x_e L_j^{(3/2)}(x_e)e^{-x_e} \\
& \times\left\{\left[L_k^{(3/2)}(x_e)+2x_e \dot{L}_k^{(3/2)}(x_e)\right]\frac{1}{v}\frac{\partial h_f^{(0)}}{\partial v}\right. \\
& \left.+\frac{2}{v_{T_e}^2}\dot{L}_k^{(3/2)}(x_e)\left(3h_f^{(0)}-\frac{2}{v}\frac{\partial g_f^{(0)}}{\partial v}\right)\right\}, \quad \text{for } k,j=0,1. \qquad (23)
\end{aligned}
$$

neglecting the integrals over $v \in [0, v_0]$, which are $O(v_0^2/v_{T_e}^2) \ll 1$, under the standard fast ion approximation ($v_{T_i}^2 \ll v_0^2 \ll v_{T_e}^2$). Straightforward calculations of integrating by parts will yield the following matrix elements:

$$
\begin{pmatrix} M_{00}^{e/f} & M_{01}^{e/f} \\ M_{10}^{e/f} & M_{11}^{e/f} \end{pmatrix} \simeq -m_e \begin{pmatrix} 1 & 3/2 \\ 3/2 & 13/4 \end{pmatrix} \qquad (24)
$$

Which, in turn, yields the friction forces caused by Coulomb collisions of electrons with isotropic fast ions

$$
\frac{n_e}{\tau_{ef}}\begin{pmatrix} \mathbf{M}_0^{e/f} \\ \mathbf{M}_1^{e/f} \end{pmatrix} = -\frac{n_e m_e}{\tau_{ee}}\frac{n_f Z_f^2}{n_e}\begin{pmatrix} 1 & 3/2 \\ 3/2 & 13/4 \end{pmatrix}\begin{pmatrix} \mathbf{u}_{0e} \\ \mathbf{u}_{1e} \end{pmatrix}. \qquad (25)
$$

Here, we have utilized results for $h_f^{(0)}$ and $g_f^{(0)}$ given in Appendix B, the electron–fast ion collisional relaxation time

$$
\frac{1}{\tau_{ef}} = \frac{1}{\tau_{ee}}\frac{n_f Z_f^2}{n_e}, \qquad (26)
$$

and the fast ion density $n_f \equiv n_f(v_0)$.

B. $M_{kj}^{i/f}$ friction coefficients

Considering thermal ion–fast ion collisions, Eq. (22) becomes

$$\begin{aligned} M_{kj}^{i/f} &\equiv 2\frac{m_i}{n_f}\int_0^\infty v^2\,dv\; x_i L_j^{(3/2)}(x_i)e^{-x_i} \\ &\times\left\{\left(1+\frac{m_i}{m_f}\right)\left[L_k^{(3/2)}(x_i)+2x_i\dot{L}_k^{(3/2)}\right]\frac{1}{v}\frac{\partial h_f^{(0)}}{\partial v}\right. \\ &\left.+\frac{2}{v_{T_i}^2}\dot{L}_k^{(3/2)}\left(3h_f^{(0)}-\frac{2}{v}\frac{\partial g_f^{(0)}}{\partial v}\right)\right\}, \quad \text{for } k,j=0,1. \end{aligned} \tag{27}$$

For $k=0$, the above result is further reduced to

$$M_{0j}^{i/f}=2\frac{m_i}{n_f}\left(1+\frac{m_i}{m_f}\right)\int_0^\infty v^2dvx_iL_j^{(3/2)}(x_i)e^{-x_i}\frac{1}{v}\frac{dh_f^{(0)}}{dv} \tag{28}$$

because all the other terms proportional to the derivative of Laguerre polynomials vanish. The integral above can be carried out by using Eq. (B5) and changing the order of the double integrations over v and v', *e.g.*,

$$\begin{aligned} M_{0j}^{i/f} &\propto \int_0^\infty v^2dv\,x_iL_j^{(3/2)}(x_i)e^{-x_i}\frac{1}{v}\frac{dh_f^{(0)}}{dv} \\ &= -\int_0^\infty v^2dv\,x_iL_j^{(3/2)}(x_i)e^{-x_i}\frac{1}{v^3}\int_0^v 4\pi v'^2dv' f_{0f}(v') \\ &= -\int_0^\infty 4\pi v'^2dv' f_{0f}(v')\int_{v'}^\infty dv\,\frac{x_i}{v}L_j^{(3/2)}(x_i)e^{-x_i} \\ &\simeq 0. \end{aligned} \tag{29}$$

Where, those exponentially vanishing terms have been neglected considering the fast ion approximation $(v_0/v_{T_i})^2 \gg 1$. Similarly, for $k=1$, we can obtain (referring Appendix D for more details)

$$M_{1j}^{i/f} \simeq -\frac{15\sqrt{\pi}}{4}\frac{m_i}{n_f}\int d^3v\,\frac{v_{T_i}}{v}f_{0f}(v)\delta_{j0}, \tag{30}$$

with δ_{ij} being the Kronecker notation. It is interesting to notice that the integral above can be cast into a form like

$$\int d^3v\,\frac{v_{T_i}}{v}f_{0f}(v)=\frac{1}{2}\dot{n}_f\tau_s\frac{v_0^3}{v_{ci}^3}\frac{v_{T_i}}{v_0}G_i, \tag{31}$$

which correlates to the fraction of energy transfered from fast ions to thermal ions in the slowing down process:

$$G_i \equiv \frac{2}{v_0^2}\int_0^{v_0} v\,dv\,\frac{v_{ci}^3}{v^3+v_c^3}. \tag{32}$$

Therefore, the frictional forces for thermal ion and isotropic fast ion collisions can be written in a matrix expression

$$\frac{n_i}{\tau_{if}}\begin{pmatrix}\mathbf{M}_0^{i/f}\\ \mathbf{M}_1^{i/f}\end{pmatrix} = -\frac{n_e m_e}{\tau_{ee}}\frac{n_f Z_f^2}{n_e}\begin{pmatrix}0 & 0\\ \bar{M}_{10}^{i/f} & 0\end{pmatrix}\begin{pmatrix}\mathbf{u}_{0i}\\ \mathbf{u}_{1i}\end{pmatrix}. \tag{33}$$

$$\bar{M}_{10}^{i/f} \equiv \frac{5}{2}\frac{v_0^2}{v_{T_i}^2}\frac{\dot{n}_f\tau_s}{n_f}G_i \tag{34}$$

3.2 $\mathbf{N}_k^{\alpha/\beta}$ for thermal test particles

Frictions caused by collisions of Maxwellian test particles with flow-distorted field particles can expressed as

$$\begin{aligned}\mathbf{N}_k^{\alpha/\beta} &= \frac{3}{4\pi}\frac{m_\alpha}{n_\beta}\int d^3v\left\{\left(1+\frac{m_\alpha}{m_\beta}\right)\left[L_k^{(3/2)}(x_\alpha)\mathbf{I}+\frac{2}{v_{T_\alpha}^2}\dot{L}_k^{(3/2)}(x_\alpha)\mathbf{vv}\right]\cdot\frac{\partial h_\beta^{(1)}}{\partial\mathbf{v}}\right.\\ &\left.+\frac{2}{v_{T_\alpha}^2}\dot{L}_k^{(3/2)}(x_\alpha)\left[\mathbf{v}h_\beta^{(1)}+\mathbf{v}\cdot\frac{\partial^2 g_\beta^{(1)}}{\partial\mathbf{v}\partial\mathbf{v}}\right]\right\}e^{-x_\alpha}, \quad \text{for } k=0,1.\end{aligned}$$

using Eqs. (9) and (13). Where, $h_\beta^{(1)} = h[f_{1\beta}]$ and $g_\beta^{(1)} = g[f_{1\beta}]$ are functional given in Section II. Integrating by parts and dropping out the contributionless surface integrals, the above results can be further cast into the following form:

$$\begin{aligned}\mathbf{N}_k^{\alpha/\beta} &= \frac{3}{4\pi}\frac{m_\alpha}{n_\beta}\left\{\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{2}{v_{T_\alpha}^2}\int d^3v\;\mathbf{v}h_\beta^{(1)}e^{-x_\alpha}\right.\\ &\times\left[L_k^{(3/2)}(x_\alpha)-2\dot{L}_k^{(3/2)}L_1^{(3/2)}(x_\alpha)+\frac{\dot{L}_k^{(3/2)}}{1+m_\alpha/m_\beta}\right]\\ &\left.-\frac{8}{v_{T_\alpha}^4}\int d^3v\;\mathbf{v}g_\beta^{(1)}e^{-x_\alpha}\dot{L}_k^{(3/2)}L_1^{(3/2)}(x_\alpha)\right\}, \quad \text{for } k=0,1\end{aligned} \tag{35}$$

A. $\mathbf{N}_k^{e/f}$ restoring frictions

For electron–fast ion collisions, neglecting the electron – fast ion mass ratio m_e/m_f and small integrals $O(v_0^2/v_{T_e}^2)$, Eq. (35) gives

$$\mathbf{N}_0^{e/f} \simeq \frac{3}{4\pi}\frac{m_e}{n_f}\int_{v_0}^{\infty} x_e e^{-x_e}\,dv^2\int_{v>v_0} d\mathbf{\Omega}\hat{\mathbf{v}}h_f^{(1)},$$

with $\hat{\mathbf{v}}$ representing the unit velocity vector. Substituting in Eq. (15) for $h_f^{(1)}$ and assuming $G_{\ell m}^f(v) \propto H(v_0 - v)$,[9] we have

$$\int_{v>v_0} d\mathbf{\Omega}\hat{\mathbf{v}}h_f^{(1)} = \frac{4\pi}{3}\frac{1}{v^2}\sum_{\ell,m}\int d\mathbf{\Omega}'\,\hat{\mathbf{v}}'Y_{\ell m}(\mathbf{\Omega}')\int_0^{v_0} v'^2\,dv'\,v'G_{\ell m}^{(f)}(v')$$

$$= \frac{4\pi}{3}\frac{1}{v^2}n_f\mathbf{V}_f. \tag{36}$$

Then, the restoring momentum friction force for electron–fast ion collisions is obtained

$$\mathbf{N}_0^{e/f} \simeq m_e\mathbf{V}_f \tag{37}$$

presuming the fast ion approximation $v_0^2/v_{T_e}^2 \ll 1$. For $k = 1$, similar calculations lead us to an integral expression as follows

$$\begin{aligned} \mathbf{N}_1^{e/f} \;&\simeq\; \frac{3}{4\pi}\frac{m_e}{n_f}\left[\int_{v_0}^{\infty} dv^2\, x_e e^{-x_e}\left(\frac{13}{2} - 3x_e\right)\int_{v>v_0} d\Omega\,\hat{\mathbf{v}}h_f^{(1)} \right. \\ &\left. +\frac{4}{v_{T_e}^2}\int_{v_0}^{\infty} dv^2\, x_e e^{-x_e} L_1^{(3/2)}(x_e)\int_{v>v_0} d\Omega\,\hat{\mathbf{v}}g_f^{(1)}\right]. \end{aligned}$$

Substituting in Eq. (16) for $g_f^{(1)}$ and following the same procedure as has been used in deriving Eq. (36), we find

$$\begin{aligned} \int_{v>v_0} d\Omega\,\hat{\mathbf{v}}g_f^{(1)} &= \frac{4\pi}{3}\int d^3v'\mathbf{v}'\left[\frac{1}{5}\left(\frac{v'}{v}\right)^2 - 1\right] f_{1f}(\mathbf{v}') \\ &= \frac{4\pi}{3}\left(\frac{2}{5}\frac{\mathbf{Q}_f}{m_f v^2} - n_f\mathbf{V}_f\right), \end{aligned} \tag{38}$$

where $\mathbf{Q}_f \equiv \mathbf{q}_f + \frac{5}{2}p_f\mathbf{V}_f$ is the total fast ion energy flux. Therefore, the restoring heat friction force acted on electrons by distorted field fast ions can be approximated as

$$\mathbf{N}_1^{e/f} \simeq \frac{3}{2}m_e\mathbf{V}_f, \tag{39}$$

by completing energy dependent integrals. Where, the energy flux term proportional to $\mathbf{Q}_f$ is of higher order, [*e.g.* $O(v_0^2/v_{T_e}^2) \ll 1$], compared with the mass flow term and henceforth neglected in deriving Eq. (39). Rewriting Eqs. (37) and (39) in a matrix form, we thus obtain

$$\frac{n_e}{\tau_{ef}}\begin{pmatrix} \mathbf{N}_0^{e/f} \\ \mathbf{N}_1^{e/f} \end{pmatrix} = \frac{n_e m_e}{\tau_{ee}}\frac{n_f Z_f^2}{n_e}\begin{pmatrix} 1 & 0 \\ 3/2 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{u}_{0f} \\ \mathbf{u}_f \end{pmatrix}. \tag{40}$$

Here, the distribution f_{1f} has been replaced by f_f considering that the isotropic part does not contribute to odd moments.

B. $\mathbf{N}_k^{i/f}$ restoring frictions

Recalling Eq. (35), for Maxwellian ions being test particles while distorted fast ions being field particles, we find

$$\mathbf{N}_0^{i/f} = \frac{3}{4\pi}\frac{m_i}{n_f}\left(1+\frac{m_i}{m_f}\right)\int_0^\infty dv^2\, x_i e^{-x_i}\int d\Omega\,\hat{\mathbf{v}}h_f^{(1)}.$$

Because the solid angle integral above can be written as

$$\int d\Omega\,\hat{\mathbf{v}}h_f^{(1)} = \frac{4\pi}{3}\int d\Omega'\,\hat{\mathbf{v}}'\int_0^\infty v'^2\,dv'\,\frac{v_<}{v_>^2}f_{1f}(\mathbf{v}'), \tag{41}$$

the restoring momentum friction force can be approximately expressed as

$$\mathbf{N}_0^{i/f} \simeq \frac{3\sqrt{\pi}}{4}\frac{m_i}{n_f}\left(1+\frac{m_i}{m_f}\right)\int d^3v\,\mathbf{v}\frac{v_{T_i}^3}{v^3}f_{1f}(\mathbf{v}) \tag{42}$$

reverting the order of the double integrations over v and v' and utilizing the fast ion approximations. Following the same procedure and noticing that the solid angle integration of $\hat{\mathbf{v}}g_f^{(1)}$ can be cast into the form

$$\int d\Omega\,\hat{\mathbf{v}}g_f^{(1)} = \frac{4\pi}{3}\int d\Omega'\,\hat{\mathbf{v}}'\int_0^\infty dv'\,v'^2 v_<\left(\frac{1}{5}\frac{v_<^2}{v_>^2}-1\right)f_{1f}(\mathbf{v}'), \tag{43}$$

we can obtain the restoring heat friction force arose from Coulomb collisions of Maxwellian ions with field fast ions

$$\mathbf{N}_1^{i/f} \simeq \frac{3\pi}{4}\frac{m_i}{n_f}\int d^3v\,\mathbf{v}\frac{v_{T_i}^3}{v^3}f_{1f}(\mathbf{v}). \tag{44}$$

Utilizing Eq. (11), restoring forces given in Eqs. (42) and (44) can be summarized in the following matrix form:

$$\frac{n_i}{\tau_{if}}\begin{pmatrix}\mathbf{N}_0^{i/f}\\ \mathbf{N}_1^{i/f}\end{pmatrix} = \frac{n_e m_e}{\tau_{ee}}\frac{n_f Z_f^2}{n_e}\frac{1}{n_f}\int d^3v\,\mathbf{v}\frac{v_c^3}{v^3}f_f(\mathbf{v})\begin{pmatrix}\bar{N}_{00}^{i/f}\\ \bar{N}_{10}^{i/f}\end{pmatrix}. \tag{45}$$

$$\begin{pmatrix}\bar{N}_{00}^{i/f}\\ \bar{N}_{10}^{i/f}\end{pmatrix} \equiv \frac{v_{ci}^3}{v_c^3}\begin{pmatrix}1+(m_i/m_f)\\ 1\end{pmatrix} \tag{46}$$

Again, the distorted fast ion distribution function has been replaced by its total distribution since the isotropic part does not contribute.

4. THERMAL FIELD PARTICLES

4.1 $\mathbf{M}_k^{f/\beta}$ for thermal field particles

For collisions of fast ions with Maxwellian field particles, Coulomb collision operator can be simplified to:

$$C_{fe}[f_f, f_M^{(e)}] \simeq \frac{1}{\tau_s}\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3 f_f\right) \tag{47}$$

$$C_{fi}[f_f, f_M^{(i)}] \simeq \frac{1}{\tau_s}\left(\frac{m_i}{m_f}\frac{v_{ci}^3}{v^3}\pounds f_f + \frac{v_{ci}^3}{v^2}\frac{\partial}{\partial v}f_f\right). \tag{48}$$

Wherein $\pounds$ represents the pitch-angle scattering operator, v_{ci} is the critical speed for a single ion species [see Eq. (3)].

A. $\mathbf{M}_k^{f/e}$ friction moments

Then, taking flow-type moments, $m_f\mathbf{v}L_k^{(3/2)}$, of the collision operator given in Eq. (47), we find

$$\frac{n_f}{\tau_{fe}}\mathbf{M}_k^{f/e} \simeq -\frac{1}{\tau_s}\int d^3v\, m_f\mathbf{v}\left[L_k^{(3/2)}(x_f) + 2x_f\dot{L}_k^{(3/2)}\right]f_f. \tag{49}$$

Which in turn yields momentum and heat friction moments

$$\frac{n_f}{\tau_{fe}}\begin{pmatrix}\mathbf{M}_0^{f/e}\\ \mathbf{M}_1^{f/e}\end{pmatrix} = -\frac{n_e m_e}{\tau_{ee}}\frac{n_f Z_f^2}{n_e}\begin{pmatrix}1 & 0\\ -5 & 15/2\end{pmatrix}\begin{pmatrix}\mathbf{u}_{0f}\\ \mathbf{u}_{1f}\end{pmatrix}, \tag{50}$$

caused by collisions with Maxwellian electrons.

B. $\mathbf{M}_k^{f/i}$ friction moments

Similarly, taking flow-type moments of the collision operator given by Eq. (48), we find

$$\frac{n_f}{\tau_{fi}}\mathbf{M}_k^{f/i} \simeq -\frac{1}{\tau_s}\int d^3v\, m_f\mathbf{v}\frac{v_{ci}^3}{v^3}\left[\left(1+\frac{m_i}{m_f}\right)L_k^{(3/2)}(x_f) + 2x_f\dot{L}_k^{(3/2)}\right]f_f. \tag{51}$$

Thus, the matrix elements of the above expression (for $k = 0, 1$) can be written as

$$\frac{n_f}{\tau_{fi}}\begin{pmatrix}\mathbf{M}_0^{f/i}\\ \mathbf{M}_1^{f/i}\end{pmatrix} \simeq -\frac{n_e m_e}{\tau_{ee}}\frac{n_f Z_f^2}{n_e}\begin{pmatrix}\bar{M}_{00}^{f/i} & 0\\ \bar{M}_{10}^{f/i} & \bar{M}_{11}^{f/i}\end{pmatrix}\begin{pmatrix}\mathbf{U}_{0f}\\ \mathbf{U}_{1f}\end{pmatrix}, \tag{52}$$

where the fast ion "flows"

$$\mathbf{U}_{0f} \equiv \frac{1}{n_f}\int d^3v\,\mathbf{v}\frac{v_c^3}{v^3}f_f, \tag{53}$$

$$\mathbf{U}_{1f} \equiv \frac{1}{n_f}\int d^3v\,\mathbf{v}\frac{v_c}{v}f_f, \tag{54}$$

characterize fast ion flows weighted by the ion-drag factor v_c/v, whereas

$$\begin{aligned}\bar{M}_{00}^{f/i} &\equiv \left(1+\frac{m_i}{m_f}\right)\frac{v_{ci}^3}{v_c^3}\\ \bar{M}_{10}^{f/i} &\equiv \frac{5}{2}\left(1+\frac{m_i}{m_f}\right)\frac{v_{ci}^3}{v_c^3}\\ \bar{M}_{11}^{f/i} &\equiv -\frac{v_c^2}{v_{T_f}^2}\left(3+\frac{m_i}{m_f}\right)\frac{v_{ci}^3}{v_c^3}\end{aligned} \tag{55}$$

are dimensionless coefficients. The order of unity factors v_{ci}^3/v_c^3 an d v_{ci}/v_c will be simplified to unity for single ion species plasmas.

4.2 $\mathbf{N}_k^{f/\beta}$ for thermal field particles

Given $h_\beta^{(1)}$ and $g_\beta^{(1)}$ for distorted thermal field particles (see Appendix C), the restoring friction forces can be reduced to

$$\frac{n_\alpha}{\tau_{\alpha\beta}}\mathbf{N}_k^{\alpha/\beta} = \frac{n_\alpha m_\alpha}{\tau_{\alpha\beta}}\frac{v_{T_\alpha}^3}{v_{T_\beta}^3}\left[N_{k0}^{\alpha/\beta}\mathbf{u}_{0\beta} + N_{k1}^{\alpha/\beta}\mathbf{u}_{1\beta}\right]. \tag{56}$$

Wherein the coefficients (for $k = 0, 1$ only) are defined as

$$\begin{aligned}N_{k0}^{\alpha/\beta} \equiv\ & \frac{\sqrt{\pi}}{2}\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{1}{n_f}\int d^3v\left\{L_k^{(3/2)}(x_\alpha)\frac{1}{x^{1/2}}\psi'(x)\right.\\ &\left.+2\frac{v_{T_\beta}^2}{v_{T_\alpha}^2}\dot{L}_k^{(3/2)}\left[x^{1/2}\psi'(x) - \frac{m_\alpha/m_\beta}{1+m_\alpha/m_\beta}\frac{1}{x^{1/2}}\psi(x)\right]\right\}f_{0\alpha}\end{aligned} \tag{57}$$

$$\begin{aligned}N_{k1}^{\alpha/\beta} \equiv\ & \frac{\sqrt{\pi}}{2}\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{1}{n_f}\int d^3v\left[L_k^{(3/2)}(x_\alpha)\frac{1}{x^{1/2}}\left(\frac{3}{2}-x\right)\right.\\ &\left.+\frac{v_{T_\beta}^2}{v_{T_\alpha}^2}\dot{L}_k^{(3/2)}x^{1/2}\left(\frac{4+m_\alpha/m_\beta}{1+m_\alpha/m_\beta} - \frac{2}{3}x - \frac{2}{3}x^2\right)\right]\psi'(x)f_{0\alpha}.\end{aligned} \tag{58}$$

It is important to keep remembering that $x_\alpha = v^2/v_{T_\alpha}^2$, whereas x is a simple notation for $x^{\alpha/\beta} = v^2/v_{T_\beta}^2$ in the above equations.

A. $\mathbf{N}_k^{f/e}$ restoring frictions

Straightforward calculations using Eqs. (56) – (58) and $x^{f/e} \ll 1$ approximation give us the following result:

$$\frac{n_f}{\tau_{fe}} \begin{pmatrix} \mathbf{N}_0^{f/e} \\ \mathbf{N}_1^{f/e} \end{pmatrix} \simeq \frac{n_e m_e}{\tau_{ee}} \frac{n_f Z_f^2}{n_e} \begin{pmatrix} 1 & 3/2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0e} \\ \mathbf{u}_{1e} \end{pmatrix}. \tag{59}$$

B. $\mathbf{N}_k^{f/i}$ restoring frictions

Using Eqs. (56) – (58) and the fast ion approximation, $x^{f/i} \gg 1$, we find

$$\frac{n_f}{\tau_{fi}} \begin{pmatrix} \mathbf{N}_0^{f/i} \\ \mathbf{N}_1^{f/i} \end{pmatrix} \simeq \frac{n_e m_e}{\tau_{ee}} \frac{n_f Z_f^2}{n_e} \begin{pmatrix} 0 & 0 \\ \bar{N}_{10}^{f/i} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0i} \\ \mathbf{u}_{1i} \end{pmatrix}, \tag{60}$$

$$\bar{N}_{10}^{f/i} \equiv \frac{2}{3} \frac{v_0}{v_{T_f}} \frac{v_0}{v_{T_i}} \frac{\dot{n}_f \tau_s}{n_f} G_i. \tag{61}$$

Here, those terms proportional to $\psi'(x^{f/i})$ have been neglected because they are exponentially small.

5. CONCLUSION

Recalling Eqs. (10), (25) and (40), total frictional forces for e/f collisions can be expressed as

$$\begin{pmatrix} \widehat{\mathbf{F}}_0^{e/f} \\ -\widehat{\mathbf{F}}_1^{e/f} \end{pmatrix} = -\frac{n_e m_e}{\tau_{ee}} \frac{n_f Z_f^2}{n_e} \left[\begin{pmatrix} 1 & 3/2 \\ 3/2 & 13/4 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0e} \\ \mathbf{u}_{1e} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3/2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0f} \\ \mathbf{u}_{1f} \end{pmatrix} \right] \tag{62}$$

While the total friction forces for f/e collisions can be summarized into a matrix form

$$\begin{pmatrix} \widehat{\mathbf{F}}_0^{f/e} \\ -\widehat{\mathbf{F}}_1^{f/e} \end{pmatrix} = -\frac{n_e m_e}{\tau_{ee}} \frac{n_f Z_f^2}{n_e} \left[\begin{pmatrix} 1 & 0 \\ -5 & 15/2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0f} \\ \mathbf{u}_{1f} \end{pmatrix} - \begin{pmatrix} 1 & 3/2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0e} \\ \mathbf{u}_{1e} \end{pmatrix} \right]. \tag{63}$$

recalling Eqs. (50) and (59). Also, using Eqs. (33), (45), (52) – (55), we find the friction forces

$$\begin{pmatrix} \widehat{\mathbf{F}}_0^{i/f} \\ -\widehat{\mathbf{F}}_1^{i/f} \end{pmatrix} = -\frac{n_e m_e}{\tau_{ee}} \frac{n_f Z_f^2}{n_e} \left[\begin{pmatrix} 0 & 0 \\ \bar{M}_{10}^{i/f} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0i} \\ \mathbf{u}_{1i} \end{pmatrix} - \begin{pmatrix} \bar{N}_{00}^{i/f} & 0 \\ \bar{N}_{10}^{i/f} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_{0f} \\ \mathbf{U}_{1f} \end{pmatrix} \right] \quad (64)$$

for i/f collisions, and

$$\begin{pmatrix} \widehat{\mathbf{F}}_0^{f/i} \\ -\widehat{\mathbf{F}}_1^{f/i} \end{pmatrix} = -\frac{n_e m_e}{\tau_{ee}} \frac{n_f Z_f^2}{n_e} \left[\begin{pmatrix} \bar{M}_{00}^{f/i} & 0 \\ \bar{M}_{10}^{f/i} & \bar{M}_{11}^{f/i} \end{pmatrix} \begin{pmatrix} \mathbf{U}_{0f} \\ \mathbf{U}_{1f} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \bar{N}_{10}^{f/i} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0i} \\ \mathbf{u}_{1i} \end{pmatrix} \right] \quad (65)$$

for f/i collisions. The asymmetry of these friction force matrixes may seem to be surprising since it is contrary to the well known symmetry of the friction matrix elements[4] which are derived utilizing the self-adjoint property[13] of the linearized Coulomb collision operator. However, the self-adjointness is an approximate property presuming that the temperature difference between two species is negligible,[5] *e.g.*,

$$\sqrt{\frac{m_<}{m_>}} \frac{|\Delta T|}{T} \ll 1,$$

wherein $m_< \equiv \min(m_\alpha, m_\beta)$, $m_> \equiv \max(m_\alpha, m_\beta)$ and $\Delta T \equiv T_\alpha - T_\beta$. The linearized Fokker-Planck operator for collisions between fast ion and thermal plasma species is no longer self-adjoint thereby breaks the symmetry of friction matrix elements. Another interesting feature is the appearance of the flows

$$\begin{aligned} \mathbf{U}_{0f} &= \frac{1}{n_f} \int d^3v \, \mathbf{v} \frac{v_c^3}{v^3} f_f \\ \mathbf{U}_{1f} &= \frac{1}{n_f} \int d^3v \, \mathbf{v} \frac{v_c}{v} f_f \end{aligned}$$

instead of the normal mass flows $\mathbf{V}_f$ and heat flux $\mathbf{q}_f$ in i/f and f/i friction forces. The flow moments $\mathbf{U}_{0f}$ and $\mathbf{U}_{1f}$ are at most the same order of magnitude compared with the normal mass flow and heat flux moments since the fast ion speed is typically $O(1)$ compared with the critical speed v_c for many practical applications.

The parallel component of momentum friction force $\widehat{\mathbf{F}}_0^{i/f}$ given in Eq. (64) agrees with Hirshman and Sigmar's result [*c.f.* Eq. (8.23) in Ref. 4], whereas $\widehat{\mathbf{F}}_0^{e/f}$ given in Eq. (62) has additional restoring terms [*c.f.* Eq. (8.22) in Ref. 4]. In this work, the test-particle distortion and field-particle restoring effects are both considered self-consistently using small flow distortion assumptions in linearizing Fokker-Planck collision operator. It is explicitly calculated that the frictional forces $\widehat{\mathbf{F}}_k^{e/f}$, $\widehat{\mathbf{F}}_k^{f/e}$, $\widehat{\mathbf{F}}_k^{i/f}$ and $\widehat{\mathbf{F}}_k^{f/i}$ for $k = 0, 1$. The momentum conservation properties of momentum friction forces are explicitly demonstrated here observing that $\mathbf{F}_0^{e/f} = -\mathbf{F}_0^{f/e}$ and $\mathbf{F}_0^{i/f} = -\mathbf{F}_0^{f/i}$. While in Ref. 4, only collisions between beam induced fast ions and Maxwellian particles are considered for the frictional moments $\widehat{\mathbf{F}}_0^{i/f}$, $\widehat{\mathbf{F}}_0^{e/f}$ and $\widehat{\mathbf{F}}_1^{e/f}$.

The friction forces quantify the momentum and energy flux transferring between fast ions and thermal bulk plasma species and hence the collisional coupling. In combination with viscous forces, the coupled parallel momentum and heat flux balance equations for thermal electrons, ions and fast ions can be inverted to evaluate the bootstrap current for plasmas with fast ion component.[14,15]

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The friction forces quantify the momentum and energy flux transferring between fast ions and thermal bulk plasma species and hence the collisional coupling. In combination with viscous forces, the coupled parallel momentum and heat flux balance equations for thermal electrons, ions and fast ions can be inverted to evaluate the bootstrap current for plasmas with fast ion component.[14,15]

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Appendix A COULOMB COLLISION

The Coulomb collision operator can be written in terms of the Fokker-Planck coefficients $\mathbf{A}_{\alpha\beta}$ and $\mathbf{D}_{\alpha\beta}$ as follows[11,12]

$$C_\alpha[f_\alpha] = \sum_\beta C_{\alpha\beta}[f_\alpha, f_\beta], \tag{A1}$$

$$C_{\alpha\beta}[f_\alpha, f_\beta] = -\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{J}_v^{\alpha/\beta} = -\frac{\partial}{\partial \mathbf{v}}\cdot\left[\mathbf{A}_{\alpha\beta} f_\alpha - \frac{1}{2}\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{D}_{\alpha\beta} f_\alpha)\right] \tag{A2}$$

where $C_{\alpha\beta}$ represents the collisions of test particles α on field particles β. Here, the dynamical friction vector is given by

$$\mathbf{A}_{\alpha\beta} \equiv \gamma_{\alpha\beta}\left(1 + \frac{m_\alpha}{m_\beta}\right)\frac{\partial h_\beta}{\partial \mathbf{v}} \tag{A3}$$

wherein $\gamma_{\alpha\beta} \equiv 4\pi e_\alpha^2 e_\beta^2 \ln \Lambda_{\alpha\beta}/m_\alpha^2$, velocity diffusion tensor

$$\mathbf{D}_{\alpha\beta} \equiv \gamma_{\alpha\beta}\frac{\partial^2 g_\beta}{\partial \mathbf{v}\partial \mathbf{v}}, \tag{A4}$$

and Rosenbluth potentials

$$h_\beta \equiv \int d^3v' \frac{f_\beta(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|}, \tag{A5}$$

$$g_\beta \equiv \int d^3v' f_\beta(\mathbf{v}')|\mathbf{v}-\mathbf{v}'|. \tag{A6}$$

The following expansions in terms of spherical harmonic function, $Y_{\ell m}(\mathbf{\Omega})$, are useful in calculating Rosenbluth potentials:

$$\frac{1}{|\mathbf{v}-\mathbf{v}'|} = \sum_{\ell,m=0}^{\infty} \frac{4\pi}{2\ell+1}\frac{v_<^\ell}{v_>^{\ell+1}} Y_{\ell m}^*(\mathbf{\Omega}')Y_{\ell m}(\mathbf{\Omega}) \tag{A7}$$

$$|\mathbf{v}-\mathbf{v}'| = \sum_{\ell,m=0}^{\infty} \frac{4\pi}{2\ell+1}\frac{v_<^\ell}{v_>^{\ell+1}}\left(\frac{1}{2\ell+3}\frac{v_<^2}{v_>^2} - \frac{1}{2\ell-1}\right) Y_{\ell m}^*(\mathbf{\Omega}')Y_{\ell m}(\mathbf{\Omega}) \tag{A8}$$

where $v_< \equiv \min(v, v')$, $v_> \equiv \max(v, v')$ and $\mathbf{\Omega}$ is the solid angle of $\mathbf{v}$.

Appendix B CALCULATION OF $h_f^{(0)}$ AND $g_f^{(0)}$

Substituting Eq. (1) into Eqs. (17) and (18) for the lowest order distribution functiuon, we can then obtain

$$\begin{aligned} h_f^{(0)} &= \frac{1}{v}\int_0^v 4\pi v'^2 dv' f_{0f}(v') + \int_v^\infty 4\pi v' dv' f_{0f}(v') \\ &= \frac{1}{v} n_f(\bar{v}_<) + \dot{n}_f \tau_s \int_v^{v_0} \frac{v\,dv}{v^3+v_c^3} H(v-v_0) \end{aligned} \tag{B1}$$

$$\begin{aligned} g_f^{(0)} &= \int_0^v 4\pi v'^2 dv' v \left[\frac{1}{3}\left(\frac{v'}{v}\right)^2 + 1\right] f_{0f}(v') \\ &\quad + \int_v^\infty 4\pi v'^2 dv' v' \left[\frac{1}{3}\left(\frac{v}{v'}\right)^2 + 1\right] f_{0f}(v') \\ &= \frac{1}{m_f v} p_f(\bar{v}_<) + v n_f(\bar{v}_<) \\ &\quad + \dot{n}_f \tau_s \left[\frac{v^2}{3}\int_v^{v_0} \frac{v\,dv}{v^3+v_c^3} + \int_v^{v_0} \frac{v^3\,dv}{v^3+v_c^3}\right] H(v-v_0) \end{aligned} \tag{B2}$$

where $\bar{v}_< \equiv \min(v, v_0)$, the fast ion "density"

$$n_f(\bar{v}_<) \equiv \int_0^v 4\pi v^2\, dv\, f_{0f}(v) = \frac{\dot{n}_f \tau_s}{3} \ln\left(1 + \frac{\bar{v}_<^3}{v_c^3}\right). \tag{B3}$$

and the fast ion "pressure"

$$p_f(\bar{v}_<) \equiv \int_0^v 4\pi v^2\, dv\, \frac{1}{3} m_f v^2 f_{0f}(v) = \frac{1}{3} m_f \dot{n}_f \tau_s \int_0^{\bar{v}_<} \frac{v^4\, dv}{v^3+v_c^3}. \tag{B4}$$

From the above results, it is trivial to verify that

$$\frac{dh_f^{(0)}}{dv} = -\frac{1}{v^2} n_f(\bar{v}_<) \tag{B5}$$

$$3h_f^{(0)} - \frac{2}{v}\frac{dg_f^{(0)}}{dv} = \frac{1}{v} n_f(\bar{v}_<) + \frac{2}{m_f v^3} p_f(\bar{v}_<) + \frac{5}{3}\int_v^\infty 4\pi v\, dv\, f_{0f}(v) \tag{B6}$$

Appendix C. THERMAL SPECIES $h_\beta^{(1)}$ AND $g_\beta^{(1)}$

To determine $h_\beta^{(1)}$ and $g_\beta^{(1)}$, we need to utilize Eqs. (14) – (20) and orthogonality of spherical harmonics (only $\ell = 1$ component here). Thus, we can obtain:

$$h_\beta^{(1)} = \frac{n_\beta}{v_{T_\beta}^3} \mathbf{v} \cdot \left(H_{0\beta}\mathbf{u}_{0\beta} + H_{1\beta}\mathbf{u}_{1\beta}\right), \tag{C1}$$

$$g_\beta^{(1)} = \frac{n_\beta}{v_{T_\beta}} \mathbf{v} \cdot (G_{0\beta}\mathbf{u}_{0\beta} + G_{1\beta}\mathbf{u}_{1\beta}) \,. \tag{C2}$$

Where the coefficients are defined and evaluated, replacing $x^{\alpha/\beta}$ and x'_β $(= v'^2/v_{T_\beta}^2)$ with x and t respectively for simplicity, as follows

$$\begin{aligned} H_{0\beta} &\equiv \frac{8}{3\sqrt{\pi}} \frac{1}{x^{1/2}} \int_0^\infty dv' \, t^{3/2} \frac{v_<}{v_>^2} e^{-t} L_0^{(3/2)} \\ &= \frac{1}{x^{3/2}} \psi(x), \end{aligned} \tag{C3}$$

$$\begin{aligned} H_{1\beta} &\equiv \frac{4}{3\sqrt{\pi}} \frac{1}{x^{1/2}} \int_0^\infty dv' \, t^{3/2} \frac{v_<}{v_>^2} e^{-t} L_1^{(3/2)}(t) \\ &= \frac{1}{x^{1/2}} \psi'(x), \end{aligned} \tag{C4}$$

$$\begin{aligned} G_{0\beta} &\equiv \frac{4}{3\sqrt{\pi}} \int_0^\infty dt \, te^{-t} \frac{v_<}{v} \left(\frac{1}{5} \frac{v_<^2}{v_>^2} - 1 \right) L_0^{(3/2)} \\ &= \frac{1}{x^{1/2}} \left[\left(\frac{1}{2x} - 1 \right) \psi(x) - \psi'(x) \right], \end{aligned} \tag{C5}$$

$$\begin{aligned} G_{1\beta} &\equiv \frac{4}{3\sqrt{\pi}} \int_0^\infty dt \, te^{-t} \frac{v_<}{v} \left(\frac{1}{5} \frac{v_<^2}{v_>^2} - 1 \right) L_1^{(3/2)}(t) \\ &= -\frac{1}{2x^{3/2}} \psi(x). \end{aligned} \tag{C6}$$

Here, the function $\psi(x^{\alpha/\beta})$ is defined as

$$\psi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x t^{1/2} e^{-t} \, dt \tag{C7}$$

which preserves the following property

$$\psi''(x) = \left(\frac{1}{2x} - 1 \right) \psi'(x) \tag{C8}$$

and asymptotic expansions for large and small arguments

$$\left.\begin{aligned} \psi(x) &\simeq \tfrac{4}{3\sqrt{\pi}} x^{3/2} \left(1 - \tfrac{3}{5}x + \cdots\right) \\ \psi'(x) &\simeq \tfrac{2}{\sqrt{\pi}} x^{1/2} (1 - x + \cdots) \end{aligned}\right\} \quad \text{for } x \ll 1,$$
$$\left.\begin{aligned} \psi(x) &\simeq 1 - \tfrac{2}{\sqrt{\pi}} x^{1/2} e^{-x} \left(1 + \tfrac{1}{2x} + \cdots\right) \\ \psi'(x) &= \tfrac{2}{\sqrt{\pi}} x^{1/2} e^{-x} \end{aligned}\right\} \quad \text{for } x \gg 1. \tag{C9}$$

Appendix D MISCELLANEOUS

For an arbitrary symmetric tensor **T**, vectors **A** and **B** we can obtain

$$\begin{aligned}\left(\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{T}\right)\cdot(\mathbf{AB}) &= \sum_{ij}\frac{\partial T_{ij}}{\partial v_i}A_j\mathbf{B} \\ &= \sum_{ij}\left[\frac{\partial}{\partial v_i}(T_{ij}A_j\mathbf{B}) - T_{ij}\frac{\partial}{\partial v_i}(A_j\mathbf{B})\right] \\ &= \frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{T}\cdot\mathbf{AB}) - \mathbf{T}:\left(\frac{\partial \mathbf{A}}{\partial \mathbf{v}}\mathbf{B} + \mathbf{A}\frac{\partial \mathbf{B}}{\partial \mathbf{v}}\right).\end{aligned} \tag{D1}$$

For $M_{1j}^{i/f}$, in Section 2, we have

$$\begin{aligned}M_{1j}^{i/f} &= 2\frac{m_i}{n_f}\int_0^\infty v^2\,dv\,x_i L_j^{(3/2)}(x_i)e^{-x_i}\left[\left(1+\frac{m_i}{m_f}\right)\left(\frac{5}{2}-3x_i\right)\frac{1}{v}\frac{dh_f^{(0)}}{dv}\right. \\ &\left. -\frac{2}{v_{T_i}^2}\left(3h_f^{(0)} - \frac{2}{v}\frac{dg_f^{(0)}}{dv}\right)\right],\end{aligned} \tag{D2}$$

using Eq. (27). Where the first integral can be shown negligible, *i.e.*,

$$\begin{aligned}&\int_0^\infty v^2\,dv\,x_i L_j^{(3/2)}(x_i)e^{-x_i}\left(\frac{5}{2}-3x_i\right)\frac{1}{v}\frac{dh_f^{(0)}}{dv} \\ &= -\int_0^\infty 2\pi v'^2\,dv' f_{0f}(v')\int_{v'}^\infty dx_i\,L_j^{(3/2)}(x_i)e^{-x_i}\left(\frac{5}{2}-3x_i\right) \\ &\simeq 0\end{aligned}$$

by reversing the order of integrations over v and v' and neglecting exponentially small terms. The remaining integral in Eq. (D2) yields, substituting in Eq. (B6),

$$\begin{aligned}&\int_0^\infty v^2\,dv\,x_i L_j^{(3/2)}(x_i)e^{-x_i}\left(3h_f^{(0)} - \frac{2}{v}\frac{dg_f^{(0)}}{dv}\right) \\ &\simeq \frac{5}{4}v_{T_i}^3\int_0^\infty 4\pi v'\,dv'\,f_{0f}(v')\int_0^{x_i'} dx_i\,x_i^{3/2}L_j^{(3/2)}(x_i)e^{-x_i} \\ &\simeq \frac{5}{4}\frac{3\sqrt{\pi}}{4}v_{T_i}^3\int_0^\infty 4\pi v\,dv\,f_{0f}(v)\delta_{j0}\end{aligned}$$

adopting the same methodology and the orthogonality of Laguerre Polynomials. Thus, Eq. (30) is proved.