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Masaru TAKAO

日本原子力研究所
Japan Atomic Energy Research Institute

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On FEL Integral Equation and Electron Energy Loss in Intermediate Gain Regime

Masaru TAKAO

Department of Reactor Engineering
Tokai Research Establishment
Japan Atomic Energy Research Institute
Tokai-mura, Naka-gun, Ibaraki-ken

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The FEL pendulum equation in a intermediate gain small signal regime is investigated. By calculating the energy loss of the electron beam in terms of the solution of the pendulum equation, we confirm the consistency of the FEL equation in intermediate gain regime.

Keywords: Free Electron Laser (FEL), Pendulum Equation, Electron Energy Loss, Intermediate Gain Regime, Small Signal Regime.

ＦＥＬの中間利得領域における電子のエネルギー損失について

日本原子力研究所東海研究所原子炉工学部
高雄　　勝

（1994年1月31日受理）

中間利得領域におけるＦＥＬ方程式をレーザー信号の小さい場合について、特に電子の運動方程式に関して詳しく調べ、その方程式の解を用いて中間利得領域における電子のエネルギー損失を求めた。これにより得られたレーザー利得公式はレーザー振幅から計算したものと一致し、中間利得領域におけるＦＥＬ方程式の整合性が確認された。

東海研究所：〒319-11　茨城県那珂郡東海村白方字白根２－４

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Accelerators dedicated to FEL operation are designed to provide an e-beam with large current and good quality, so that the FEL gain values easily exceeding 100 %. Although the FEL gain has been extensively analyzed in the low-gain and high-gain regimes, the feature of FEL operation in the intermediate regime has scarcely explored except [1, 2]. For this range of gain values, the laser dynamics is significantly different from that predicted by the linear theory.

In [1, 2], the dynamics of the laser electric field in the intermediate gain region is extensively investigated, which is reviewed in §2 in the self-contained way. The aim of this paper is to bring insight into the electron dynamics in the intermediate region. In §3 we study the electron pendulum equation in the intermediate gain regime and derive the expression for the energy loss of electron beam. The expression results in the form described by the laser field amplitude, so that the consistency of the FEL equation in the intermediate gain regime is confirmed.

2 FEL Integral Equation and Laser Gain

The FEL equation is the collection of the Maxwell's and the electron pendulum equations [1, 2, 3, 4]. The former describes the evolution of the electromagnetic field, and the latter determines the electron trajectories. To make the equations analytically tractable, we consider the case of neglecting the transverse and longitudinal effects of the electron distribution. Hence the FEL equation can be expressed as follows:

$$\begin{aligned}
&\frac{d}{d\tau}a(\tau) = -j_0\langle e^{-i\zeta}\rangle(\tau)\,, \\
&\frac{d}{d\tau}\zeta(\tau) = \nu(\tau)\,, \\
&\frac{d}{d\tau}\nu(\tau) = \frac{1}{2}a(\tau)e^{i\zeta(\tau)} + c.c.\,.
\end{aligned} \tag{1}$$

Here, for the sake of simplicity, we have introduced the dimensionless variables and parameters explained in Table 1. In a planer undulator case, the complex field amplitude

complex field amplitude	$a(\tau)$
electron phase	$\zeta(\tau)$
electron phase velocity	$\nu(\tau)$
dimensionless current density	j_0
characteristic time	τ

Table 1: Dimensionless variables and parameters in FEL equation.

$a(\tau)$ is related to the electric field amplitude $E(\tau)$ as follows:

$$a(\tau) = \frac{2\pi e[JJ]K\lambda_U N^2}{m_0 c^2\gamma_0^2}E(\tau)\,, \tag{2}$$

1 Introduction

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\tag{2}
$$

where λ_U is a period of an undulator, N the number of the period, K the undulator parameter, $[JJ]$ the famous Bessel factor for a planer undulator. On the other hand, the dimensionless current density is expressed as

$$j_0 = (2\pi[JJ]K)^2 \left(\frac{N}{\gamma_0}\right)^3 \frac{I\lambda_U^2}{I_A \Sigma_e}, \tag{3}$$

where I is an electron beam current, I_A Alfvén current, Σ_e the cross section of the electron beam.

If we eliminate the electron phase ζ and the phase velocity ν in the FEL equation (1), we obtain the single mode FEL integral equation [1, 2]

$$\frac{d}{d\tau}a(\tau) = i\pi g_0 e^{-i\nu_0\tau} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 e^{i\nu_0\tau_2} a(\tau_2), \tag{4}$$

with setting $2\pi g_0 = j_0$. This integral equation can be easily converted into the third order differential equation:

$$\left(\frac{d^3}{d\tau^3} + 2i\nu_0\frac{d^2}{d\tau^2} - \nu_0^2\frac{d}{d\tau} - i\pi g_0\right)a(\tau) = 0, \tag{5}$$

which is easier to be handled. With the initial condition

$$a(0) = a_0, \quad \dot{a}(0) = 0, \quad \ddot{a}(0) = 0, \tag{6}$$

the complex amplitude $a(\tau)$ is determined by the differential equation (5) as follows [2]

$$\begin{aligned} a(\tau) = \frac{a_0 e^{-i\frac{2}{3}\nu_0\tau}}{3(\mathcal{R}_+ + \nu_0)} \Bigg[(\mathcal{R}_+ - \nu_0) e^{-\frac{i}{3}\mathcal{R}_+\tau} + 2(\mathcal{R}_+ + 2\nu_0) e^{\frac{i}{6}\mathcal{R}_+\tau} \Bigg\{ \cosh\left[\frac{\sqrt{3}}{6}(\mathcal{R}_- + i\mathcal{I}_-)\tau\right] \\ + i\frac{\sqrt{3}\nu_0}{\mathcal{R}_- + i\mathcal{I}_-} \sinh\left[\frac{\sqrt{3}}{6}(\mathcal{R}_- + i\mathcal{I}_-)\tau\right] \Bigg\} \Bigg], \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mathcal{R}_\pm &= \Re(p \pm q), \\ \mathcal{I}_\pm &= \Im(p \pm q), \end{aligned} \tag{8}$$

with

$$\begin{aligned} p &= \left[\frac{1}{2}\left(r + \sqrt{d}\right)\right]^{1/3}, \\ q &= \left[\frac{1}{2}\left(r - \sqrt{d}\right)\right]^{1/3}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} r &= 27\pi g_0 - 2\nu_0^3, \\ d &= 27\pi g_0(27\pi g_0 - 4\nu_0^3). \end{aligned} \tag{10}$$

After getting the amplitude $a(\tau)$ from the FEL integral equation (4), we can easily obtain the gain $G(\nu_0, \tau)$ by the usual definition:

$$G(\nu_0, \tau) = \frac{|a(\tau)|^2 - |a_0|^2}{|a_0|^2}. \tag{11}$$

Inserting the solution (7) into eq.(11), we obtain the exact form of the intermediate gain case [2]:

$$
\begin{aligned}
&G(\nu_0,\tau) \\
&= \frac{1}{9(\mathcal{R}_+ + \nu_0)^2}\Bigg[\!\Bigg[(\mathcal{R}_+ - \nu_0)^2 + \frac{2(\mathcal{R}_+ + 2\nu_0)^2}{\mathcal{R}_-^2 + \mathcal{I}_-^2}\Bigg\{(\mathcal{R}_-^2 + \mathcal{I}_-^2 + 3\nu_0^2)\cosh\left(\frac{\mathcal{R}_-\tau}{\sqrt{3}}\right) \\
&+(\mathcal{R}_-^2 + \mathcal{I}_-^2 - 3\nu_0^2)\cos\left(\frac{\mathcal{I}_-\tau}{\sqrt{3}}\right) - 2\sqrt{3}\nu_0\left[\mathcal{R}_-\sin\left(\frac{\mathcal{I}_-\tau}{\sqrt{3}}\right) - \mathcal{I}_-\sinh\left(\frac{\mathcal{R}_-\tau}{\sqrt{3}}\right)\right]\Bigg\} \\
&+4(\mathcal{R}_+ - \nu_0)(\mathcal{R}_+ + 2\nu_0) \\
&\times\Bigg\{\cos\left(\frac{\mathcal{R}_+\tau}{2}\right)\cosh\left(\frac{\mathcal{R}_-\tau}{2\sqrt{3}}\right)\cos\left(\frac{\mathcal{I}_-\tau}{2\sqrt{3}}\right) - \sin\left(\frac{\mathcal{R}_+\tau}{2}\right)\sinh\left(\frac{\mathcal{R}_-\tau}{2\sqrt{3}}\right)\sin\left(\frac{\mathcal{I}_-\tau}{2\sqrt{3}}\right) \\
&-\frac{\sqrt{3}\nu_0}{\mathcal{R}_-^2 + \mathcal{I}_-^2}\Bigg[\Bigg\{\mathcal{R}_-\sin\left(\frac{\mathcal{R}_+\tau}{2}\right) - \mathcal{I}_-\cos\left(\frac{\mathcal{R}_+\tau}{2}\right)\Bigg\}\sinh\left(\frac{\mathcal{R}_-\tau}{2\sqrt{3}}\right)\cos\left(\frac{\mathcal{I}_-\tau}{2\sqrt{3}}\right) \\
&\qquad+\Bigg\{\mathcal{R}_-\cos\left(\frac{\mathcal{R}_+\tau}{2}\right) + \mathcal{I}_-\sin\left(\frac{\mathcal{R}_+\tau}{2}\right)\Bigg\}\cosh\left(\frac{\mathcal{R}_-\tau}{2\sqrt{3}}\right)\sin\left(\frac{\mathcal{I}_-\tau}{2\sqrt{3}}\right)\Bigg]\Bigg\}\Bigg]\!\Bigg] - 1\,. \quad (12)
\end{aligned}
$$

For the low gain case ($27\pi g_0/4\nu^3 \ll 1$), this exact form is converted into the well-known formula

$$G(\nu_0,\tau) \cong \frac{2\pi g_0}{\nu_0^3}[2(1-\cos\nu_0\tau) - \nu_0\tau\sin\nu_0\tau]\,. \tag{13}$$

As shown in this section, we can calculate the exact form of the intermediate gain case by means of solving the complex amplitude $a(\tau)$. In the next section, we confirm the correspondence of the energy loss of the electron beam to the electromagnetic energy gain.

3 FEL Pendulum Equation and Electron Energy Loss

For the purpose of confirming the consistency of the FEL equations, we calculate the energy loss of the electron beam in intermediate gain regime [5, 6]. If we scale out the initial amplitude a_0 from the complex amplitude $a(\tau)$ as

$$a(\tau) = a_0 g(\tau)\,. \tag{14}$$

we have the following electron pendulum equation

$$\frac{d^2}{d\tau^2}\zeta(\tau) = \frac{1}{2}|a_0|g(\tau)e^{i[\zeta(\tau)+\phi_0]} + c.c.\,. \tag{15}$$

Here ϕ_0 is the phase of the complex amplitude a_0. The constant field amplitude a_0 (*cf.* eq.(2)) is explicitly given by

$$a_0 = 2\pi e K\lambda_U[JJ]\frac{N^2}{m_0\gamma_0^2c^2}E_0\,, \tag{16}$$

with E_0 an initial electric field amplitude.

Now, we solve the above equation perturbatively with respect to the field amplitude $|a_0|$, so we expand the electron phase as follows:

$$\zeta(\tau) = \zeta_0 + \nu_0\tau + \frac{|a_0|}{\nu_0^2}\zeta_1(\tau) + \left(\frac{|a_0|}{\nu_0^2}\right)^2\zeta_2(\tau) + \cdots\,. \tag{17}$$

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This expansion has the advantage indicating that the phase $\zeta(\tau)$ is manifestly real. Then, since the pendulum equation can be expanded to the order of $|a_0|^2$ as

$$\frac{d^2}{d\tau^2}\zeta(\tau) \cong \frac{1}{2}|a_0|g(\tau)e^{i(\nu_0\tau+\zeta_0+\phi_0)}\left[1 + i\frac{|a_0|}{\nu_0^2}\zeta_1(\tau)\right] + c.c.\,, \tag{18}$$

we have the following iterative equation

$$\frac{d^2}{d\tau^2}\zeta_1(\tau) = \frac{\nu_0^2}{2}g(\tau)e^{i(\nu_0\tau+\zeta_0+\phi_0)} + c.c.\,, \tag{19}$$

$$\frac{d^2}{d\tau^2}\zeta_2(\tau) = i\frac{\nu_0^2}{2}\left[g(\tau)e^{i(\nu_0\tau+\zeta_0+\phi_0)} - c.c.\right]\zeta_1(\tau)\,. \tag{20}$$

The energy change in electrons is expressed in terms of the change of the phase velocity as follows:

$$\begin{aligned}\frac{\Delta\gamma}{\gamma_0} &\equiv \frac{1}{4\pi N}\langle\Delta\nu(\zeta_0)\rangle_{\zeta_0} \\ &= \frac{1}{4\pi N}\left[\frac{|a_0|}{\nu_0^2}\left\langle\frac{d}{d\tau}\zeta_1(\tau)\right\rangle_{\zeta_0} + \frac{|a_0|^2}{\nu_0^4}\left\langle\frac{d}{d\tau}\zeta_2(\tau)\right\rangle_{\zeta_0}\right],\end{aligned} \tag{21}$$

where $\langle\cdot\rangle_{\zeta_0}$ denoting the average over the initial phase ζ_0:

$$\langle f\rangle_{\zeta_0} = \frac{1}{2\pi}\int_0^{2\pi} d\zeta_0 f(\zeta_0)\,. \tag{22}$$

Since the linear term $\zeta_1(\tau)$ is periodic in ζ_0, the contribution of the linear deviation to the energy change automatically vanishes:

$$\frac{\Delta\gamma}{\gamma_0} = \frac{|a_0|^2}{4\pi N\nu_0^4}\left\langle\frac{d}{d\tau}\zeta_2(\tau)\right\rangle_{\zeta_0}\,. \tag{23}$$

The change in energy of the electromagnetic field is minus $\Delta\gamma/\gamma_0$ multiplied by the initial energy $m_0\gamma_0 c^2$, and by the electron density n_e. Hence, the FEL gain is given by

$$G(\nu_0,\tau) = \frac{\dfrac{\Delta\gamma}{\gamma_0}m_0\gamma_0 c^2 n_e}{\dfrac{1}{2}\epsilon_0|E_0|^2}\,. \tag{24}$$

Comparing eq.(16) with the definition of the dimensionless current (3), we have

$$|a_0|^2 = \frac{4\pi N}{\mu_0 m_0\gamma_0 c^4 n_e}j_0|E_0|^2\,. \tag{25}$$

After all, the gathering of the above expressions results in the following gain formula:

$$G(\nu_0,\tau) = -\frac{4\pi g_0}{\nu_0^4}\left\langle\frac{d}{d\tau}\zeta_2(\tau)\right\rangle_{\zeta_0}\,. \tag{26}$$

Before calculating the gain based on the electron energy loss in the intermediate gain regime, we derive the low gain formula. In the low gain regime the energy change of the electron beam is calculated with setting $g(\tau) \equiv 1$. Then, the iterative pendulum equations (19) and (20) becomes

$$\frac{d^2}{d\tau^2}\zeta_1(\tau) = \nu_0^2 \cos(\zeta_0 + \nu_0\tau + \phi_0)\,, \tag{27}$$

$$\frac{d^2}{d\tau^2}\zeta_2(\tau) = -\nu_0^2 \sin(\zeta_0 + \nu_0\tau + \phi_0)\zeta_1(\tau)\,. \tag{28}$$

Under the initial condition

$$\zeta_1(0) = \frac{d}{d\tau}\zeta_1(0) = 0\,, \tag{29}$$

the linear deviation $\zeta_1(\tau)$ is

$$\zeta_1(\tau) = \nu_0^2 \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cos(\nu_0\tau_2 + \zeta_0 + \phi_0)\,. \tag{30}$$

Note that it is easier to calculate the gain after averaging over the electron initial phase ζ_0 than before. Similarly, with the initial condition

$$\zeta_2(0) = \frac{d}{d\tau}\zeta_2(0) = 0\,, \tag{31}$$

we obtain the first integral of $d^2\zeta_2/d\tau^2$:

$$\frac{d}{d\tau}\zeta_2(\tau) = -\nu_0^4 \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \sin(\nu_0\tau_1 + \zeta_0 + \phi_0)\cos(\nu_0\tau_3 + \zeta_0 + \phi_0)\,. \tag{32}$$

Averaging the above expression over the electron initial phase ζ_0, we obtain

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = -\frac{\nu_0^4}{2} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \sin \nu_0(\tau_1 - \tau_3)\,. \tag{33}$$

Then, the integration of eq.(33) is easily calculated and results in

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = \frac{\nu_0}{2}[2(\cos \nu_0\tau - 1) + \nu_0\tau \sin \nu_0\tau]\,. \tag{34}$$

Inserting the above equation into the expression of the gain (26), we obtain the low gain formula:

$$G(\nu_0, \tau) = \frac{2\pi g_0}{\nu_0^3}[2(1 - \cos \nu_0\tau) - \nu_0\tau \sin \nu_0\tau]\,, \tag{35}$$

which agrees with (13).

Let us calculate the energy loss of the electron in intermediate gain regime. Before averaging the expressions over the electron phase ζ_0, the linear and the square deviation $\zeta_1(\tau)$ and $\zeta_2(\tau)$ are given by

$$\zeta_1(\tau) = \frac{\nu_0^2}{2} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \Big[g(\tau_2) e^{i(\nu_0\tau_2 + \zeta_0 + \phi_0)} + c.c. \Big]\,, \tag{36}$$

and

$$\frac{d}{d\tau}\zeta_2(\tau) = i\frac{\nu_0^2}{2} \int_0^\tau d\tau_1 \Big[g(\tau_1) e^{i(\nu_0\tau_1 + \zeta_0 + \phi_0)} - c.c. \Big] \zeta_1(\tau_1)\,. \tag{37}$$

After averaging the above expression over the electron initial phase ζ_0, we obtain

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = i\frac{\nu_0^4}{4}\Big[\int_0^\tau d\tau_1 g(\tau_1)e^{i\nu_0\tau_1}\int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 g^*(\tau_3)e^{-i\nu_0\tau_3} - c.c.\Big]. \tag{38}$$

Here, from eq.(4), we know

$$\frac{d}{d\tau}g(\tau) = i\pi g_0 e^{-i\nu_0\tau}\int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 e^{i\nu_0\tau_2} g(\tau_2), \tag{39}$$

so that we have

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = -\frac{\nu_0^4}{4\pi g_0}\Big[\int_0^\tau d\tau_1 g(\tau_1)\frac{d}{d\tau_1}g^*(\tau_1) + c.c.\Big]. \tag{40}$$

With the initial condition $g(0) = 1$, we obtain

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = -\frac{\nu_0^4}{4\pi g_0}[|g(\tau)|^2 - 1]. \tag{41}$$

Taking the expression of the gain (26) into account, we can confirm the correspondence between the energy change in the electron beam and the evolution of the electromagnetic field amplitude.

4 Conclusion

As shown in this paper, the FEL integral equation (4) plays an important role in electron dynamics in the intermediate gain regime as well as laser field evolution. In other words, when we study the electron dynamics in the intermediate regime, we should take the time evolution of the laser field into account.

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After averaging the above expression over the electron initial phase ζ_0, we obtain

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = i\frac{\nu_0^4}{4}\left[\int_0^\tau d\tau_1 g(\tau_1)e^{i\nu_0\tau_1}\int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 g^*(\tau_3)e^{-i\nu_0\tau_3} - c.c.\right]. \qquad (38)$$

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$$\frac{d}{d\tau}g(\tau) = i\pi g_0 e^{-i\nu_0\tau}\int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 e^{i\nu_0\tau_2} g(\tau_2), \qquad (39)$$

so that we have

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = -\frac{\nu_0^4}{4\pi g_0}\left[\int_0^\tau d\tau_1 g(\tau_1)\frac{d}{d\tau_1}g^*(\tau_1) + c.c.\right]. \qquad (40)$$

With the initial condition $g(0) = 1$, we obtain

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = -\frac{\nu_0^4}{4\pi g_0}[|g(\tau)|^2 - 1]. \qquad (41)$$

Taking the expression of the gain (26) into account, we can confirm the correspondence between the energy change in the electron beam and the evolution of the electromagnetic field amplitude.

4 Conclusion

As shown in this paper, the FEL integral equation (4) plays an important role in electron dynamics in the intermediate gain regime as well as laser field evolution. In other words, when we study the electron dynamics in the intermediate regime, we should take the time evolution of the laser field into account.

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After averaging the above expression over the electron initial phase ζ_0, we obtain

$$\left\langle \frac{d}{d\tau}\zeta_2(\tau) \right\rangle_{\zeta_0} = i\frac{\nu_0^4}{4}\left[\int_0^\tau d\tau_1 g(\tau_1)e^{i\nu_0\tau_1}\int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 g^*(\tau_3)e^{-i\nu_0\tau_3} - c.c.\right]. \tag{38}$$

Here, from eq.(4), we know

$$\frac{d}{d\tau}g(\tau) = i\pi g_0 e^{-i\nu_0\tau}\int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 e^{i\nu_0\tau_2} g(\tau_2), \tag{39}$$

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References

[1] G. Dattoli, A. Torre, C. Centioli and M. Richetta, *Free-Electron Laser Operation in the Intermediate Gain Regime*, IEEE J. Quantum Electron. **QE-25** (1989) 2327.

[2] G. Dattoli, A. Renieri and A. Torre, *Lectures on the Free Electron Laser Theory and Related Topics*, World Scientific Publishing Co. Pre. Ltd., Singapore, 1993.

[3] W.B. Colson and S.K. Ride, *The Nonlinear Wave Equation for Free Electron Lasers Driven by Single-Particle Currents*, Phys. Lett. **76A** (1980) 379.

[4] W.B. Colson and S.K. Ride, *The Free Electron Laser: Maxwell's Equations Driven by Single-Particle Currents*, in Physics of Quantum Electronics **7** (1980) 377, *ed. S.F. Jacobs, et al*, Addison Wesley Publishing Co. Inc., Massachusetts.

[5] W.B. Colson, *Theory of a Free Electron Laser*, Phys. Lett. **59A** (1976) 187; *One Body Electron Dynamics in a Free Electron Laser*, Phys. Lett. **64A** (1977) 190.

[6] W.B. Colson, *One-Body Analysis of Free Electron Lasers*, in Physics of Quantum Electronics **5** (1978) 157, *ed. S.F. Jacobs, et al*, Addison Wesley Publishing Co. Inc., Massachusetts.