JAERI-M
9716

SELF-SIMILARITY IN APPLIED SUPERCONDUCTIVITY

September 1981

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(Received September 10, 1981)

Self-similarity is a descriptive term applying to a family of curves. It means that the family is invariant to a one-parameter group of affine (stretching) transformations. The property of self-similarity has been exploited in a wide variety of problems in applied superconductivity, namely, (i) transient distribution of the current among the filaments of a superconductor during charge-up, (ii) steady distribution of current among the filaments of a superconductor near the current leads, (iii) transient heat transfer in superfluid helium, (iv) transient diffusion in cylindrical geometry (important in studying the growth rate of the reacted layer in A15 materials), (v) thermal expulsion of helium from quenching cable-in-conduit conductors, (vi) eddy current heating of irregular plates by slow, ramped fields, and (vii) the specific heat of type-II superconductors. Most, but not all, of the applications involve differential equations, both ordinary and partial. The novel methods explained in this report should prove of great value in other fields, just as they already have done in applied superconductivity.

Keywords; Superconductivity, Differential Equations, Lie Groups, Similarity Solutions

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*On assignment by the Oak Ridge National Laboratory of the U. S. Department of Energy under the terms of the Japan-U. S. Personnel Exchange Program AL-4 in Fusion Energy.

応用超電導工学における自己相似性

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(1981年9月10日受理)

自己相似性とは，曲線群を表現する際に用いられる用語である。その意味する所は，単一パラメータによる拡大変換に対して，不変な性質である。

自己相似性は，応用超電導工学における種々の問題に適用されてきた。その事例としては，(1)励磁中の超電導線におけるフィラメント間の過渡電流分布，(2)電流リード近傍の超電導線におけるフィラメント間の定常電流分布，(3)超流動ヘリウムへの過渡熱伝達，(4)軸対称の過渡的拡散現象（これは，A15型超電導材料の生成反応層の成長率の研究に重要である），(5)常電導転移したチューブ型導体よりの冷媒熱吐出，(6)時間に比例して，ゆっくり増大する変動磁界による任意形状平板のうず電流損失，(7)第2種超電導体の比熱，が挙げられる。

一部の例外を除き，ほとんどの応用例は，通常微分又は，偏微分方程式を含んでいる。この論文で紹介する新しい手法は，他の分解においても，その有用性を発揮するものである。

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Contents

"Holmes laughed. 'Watson insists that I am the dramatist in real life,' said he. 'Some touch of the artist wells up within me, and calls insistently for a well-staged performance. Surely our profession...would be a drab...one if we did not sometimes set the scene so as to glorify our results....The quick inference...the clever forecast of coming events, the triumphant vindication of bold theories--are these not the pride and justification of our life's work?'"

---------A. Conan Doyle, *The Valley of Fear.*

SELF-SIMILARITY IN APPLIED SUPERCONDUCTIVITY

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I. Introduction

Self-similarity is a property of a one-parameter family of curves: it means any two curves of the family may be brought into congruence by stretching transformations of the ordinate and abscissa of one of them. An example is the family of temperature profiles arising from a pulsed point source in an infinite medium, $T = \exp(-r^2/4Dt)/(4\pi Dt)^{3/2}$. From the temperature profile at time t, we can find the temperature profile at time $t'=\lambda^2 t$ with the stretching transformations $T' = \lambda^{-3}T$, $r' = \lambda r$.

Self-similarity is a rather descriptive term. In more mathematical language, it means the family of curves is invariant to a one-parameter group of affine (stretching) transformations. The curves of the family are caried into one another by the transformations of the group so that the family as a whole is carried into itself. As in many problems involving group invariance, we can draw useful conclusions from the group invariance alone without making detailed calculations.

Dimensional analysis bears a strong relation to self-similarity: dimensional homogeneity requires invariance to certain groups of affine transformations. But many physical problems are invariant to affine groups not derivable from dimensional arguments. So self-similarity is a broader notion than dimensional homogeneity.

Self-similarity has been applied to three kinds of problems in applied superconductivity. First there are applications that involve solving a partial differential equation. Here the group invariance of the partial differential equation allows us to reduce the partial differential equation to an ordinary differential equation, a great step forward. Furthermore, the group invariance enables us, using methods developed a century ago by Sophus Lie,[1] to simplify treatment of the ordinary differential equation. Problems that can be dealt with this way include (i) transient distribution

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of the current among the filaments of a superconductor during charge-up, (ii) steady distribution of current among the filaments of a superconductor near the current leads, (iii) transient heat transfer in superfluid helium, (iv) transient diffusion in cylindrical geometry (important in studying the growth rate of the reacted layer in A15 materials), and (v) thermal expulsion of helium from quenching cable-in-conduit superconductors. A second kind of application, also involving a partial differential equation, which is not, however, reduced to an ordinary differential equation, is to eddy current heating of irregular plates by slow, ramped fields. A third kind of application, not involving partial differential equations at all, is the specific heat of type-II superconductors.

The foregoing applications all break new ground. Their number and diversity show the broad applicability of the idea of self-similarity. Only a few will be dealt with in detail in this review. However, we begin with an already solved problem, simpler than any of the foregoing, in order first to show the reader how the notion of self-similarity is used and how Lie applied it to the treatment of ordinary differential equations.

II. Partial Flux Penetration in a Hard Superconductor

(1) When a magnetic field is created parallel to the face of a superconducting slab, shielding currents are induced in the slab that oppose entry of the magnetic field. If the critical current J_c is treated as independent of the local magnetic field, the profiles of magnetic induction B are parallel straight lines of slope $-\mu_0 J_c$ that form a self-similar family (at least until the field penetrates to the middle of the slab). Is there any other dependence of J_c on B for which the B-profiles are self similar, as illustrated in Fig.1b?

Self-similarity of the B-profile means that the image $B'(x')$ of a flux profile $B(x)$ under the transformation

$$B' = \lambda^a B \tag{1a}$$

$$x' = \lambda x \tag{1b}$$

is also a flux profile. All the flux profiles must satisfy the same first-order differential equation

$$\frac{dB}{dx} = -\mu_0 J_c(B) \tag{2}$$

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so this differential equation itself must be invariant to the group of transformations (1). What is the most general first-order differential equation invariant to the group (1)?

Any differential equation of first-order in B takes the form $F(x,B,\dot{B})=0$ where F is an as yet undetermined function and $\dot{B}$ is an abbreviation for dB/dx. Since the primed variables are supposed to satisfy this differential equation as well, we must have

$$F(\lambda x,\ \lambda^{a}B,\ \lambda^{a-1}\dot{B}) = 0 \tag{3}$$

for all λ. Differentiate (3) with respect to λ and set $\lambda = 1$:

$$xF_{x} + aBF_{B} + (a-1)\dot{B}F_{\dot{B}} = 0 \tag{4}$$

Here the subscripts indicate partial differentiation of $F(x,B,\dot{B})$ with respect to its various arguments.

The standard method of finding the general solution of a first-order, linear partial differential equation like (4) is to set F equal to an arbitrary function of two independent integrals of the so-called characteristic equations:

$$\frac{dx}{x} = \frac{dB}{aB} = \frac{d\dot{B}}{(a-1)\dot{B}} \tag{5}$$

Two such integrals are B/x^{a} and $B/\dot{B}^{(a-1)/a}$; thus

$$F(x,B,\dot{B}) = G\left(\frac{\dot{B}}{B^{(a-1)/a}},\ \frac{B}{x^{a}}\right) \tag{6}$$

where F is as yet entirely undetermined. Setting $G = 0$ is the same as writing

$$\dot{B} = B^{(a-1)/a}\ H\left(\frac{B}{x^{a}}\right) \tag{7}$$

where H is another as yet undetermined function.

If (7) must be the same as (2), H can only be a constant in order that the right-hand side of (7) be independent of x. Thus

$$J_c \sim B^{(a-1)/a} \tag{8}$$

is a necessary (and as it turns out also sufficient) condition that the penetrating field profiles be self-similar.

The total magnetic flux ϕ that has penetrated the sample is $\phi = \int_0^\infty B dx$ per unit width. The penetrating flux ϕ' corresponding to the field profile $B'(x')$ is

$$\phi' = \int_0^\infty B' dx' = \lambda^{a+1} \int_0^\infty B dx = \lambda^{a+1} \phi \tag{9}$$

The parameter λ is related to the ratio of the external magnetic fields B'_o and B_o : $\lambda = (B'_o/B_o)^{1/a}$. Now, (9) can be written

$$\frac{\phi'}{B'^{(a+1)/a}_o} = \frac{\phi}{B_o^{(a+1)/a}} \tag{10}$$

Thus the pentrating flux varies as the $(a+1)/a$ power of the applied field when the critical current varies as the $(a-1)/a$ power of the field. This conclusion is entirely a consequence of the group invariance of the problem.

(2) Brechna[2] has given a direct solution of the flux penetration problem. when $J_c = J_* B_*/(B+B_*)$. If we take the limit of his formulas when $B_* \to 0$, with $J_* B_*$ approaching a finite limit, we should obtain the case $J_c \sim B^{-1}$, i.e., the case $a=1/2$. Then $\phi \sim B^3$, and this in fact is what Brechna's formulas reduce to.

(3) The same result is obtainable by dimensional analysis. For if $J_c = CB^{(a-1)/a}$, the proportionality constant C must have the dimensions $A\ m^{-2}\ T^{(1-a)/a}$. The penetrating flux has the dimensions T m, and the only other relevant parameters are B_o (dimensions T) and μ_0 (dimensions henry/m). There is no quantity having the dimensions of length because we are confining our interest to the case of partial penetration, in which the field has not reached the centerline of the slab. Then dimensional homogeneity requires that $\phi \sim B_o^{(a+1)/a}/\mu_0 C$.

(4) In this simple problem, the same conclusion can be reached in several ways. But the first way, emphasizing the property of self-similarity, helped us to introduce some of the basic ideas we shall be using in the rest of this paper. Moreover, as we shall see next, it paves way for the explanation of an

old and powerful method of treating ordinary differential equations due to Sophus Lie, which, sorprisingly, is not used very much much these days.

III. Application of Self-Similarity to First-Order Differential Equations

(1) The most general first-order differential equation involving the dependent variable y and the independent variable x can be written $f(x,y,\dot{y})=0$, where f can be any function. The solution of this differential equation is a family of curves $\phi(x,y)=C$ labelled by the single parameter c, the so-called constant of integration. If this family is self-similar, i.e., if a transformation like (1), namely, $x'=\lambda x$, $y'=\lambda^{\alpha}y$, carries one curve of the family into another, then it must transform the differential equation into itself. Then, repeating the steps of (3) - (6), we see that

$$f(x,y,\dot{y}) = g\left(\frac{y}{x^{\alpha}}, \frac{\dot{y}}{x^{\alpha-1}}\right) \tag{11}$$

where g is an arbitrary function. The seemingly mild restriction of $f(x,y,\dot{y})$ to the form g(u,v), where $u = y/x^{\alpha}$, $v = \dot{y}/x^{\alpha-1}$, and g is arbitrary, is the only effect of self-similarity.

But a moment's work more will show that if we take u to be the new dependent variable, the differential equation $f(x,y,\dot{y})=0$ will go over into one in which the variables u and x are separable. For

$$x\frac{du}{dx} = \frac{\dot{y}}{x^{\alpha-1}} - \frac{\alpha y}{x^{\alpha}} = v - \alpha u = F(u) - \alpha u \tag{12}$$

since g(u,v) = 0 can be solved in the form v = F(u). It is important to note here that the differential equation has not been assumed to be linear and does not have to be.

(2) Often first-order differential equations are written in the form

$$M(x,y)\ dx + N(x,y)\ dy = 0 \tag{13}$$

Now since

$$\phi_x\ dx + \phi_y\ dy = d\phi = dc = 0 \tag{14}$$

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we must have

$$\frac{\phi_x}{M} = \frac{\phi_y}{N} = \mu(x,y) \tag{15}$$

where $\mu(x,y)$ is an as yet undetermined function. The function $\mu(x,y)$ is an integrating factor because if we multiply (13) by it, the left-hand side of (13) becomes the perfect differential left-hand side of (14).

The requirement of self-similarity means that $\phi(x,y) = C$ transforms into $\phi(x',y') = C'$, where $C' = C'(\lambda,C)$:

$$\phi(\lambda x, \lambda^{\alpha} y) = C'(\lambda,C) \tag{16}$$

Again we can use the trick of differentiating with respect to λ and setting $\lambda = 1$:

$$x\phi_x + \alpha y\phi_y = \left(\frac{\partial C'}{\partial \lambda}\right)_{\lambda=1} = \begin{array}{l}\text{a function of } C\\ \text{independent of } x,y\end{array} \tag{17}$$

Substituting for ϕ_x and ϕ_y from (15), we find

$$\mu(xM + \alpha yN) = \left(\frac{\partial C'}{\partial \lambda}\right)_{\lambda=1} \tag{18}$$

Ignoring the constant, as we may, we see from (18) that $(xM + \alpha yN)^{-1}$ is an integrating factor for (13).

(3) This last conclusion is a special case of a much more general theorem of Lie's. Lie did not restrict himself to stretching groups but studied the completely general group

$$x' = X(x,y;\lambda) \tag{19a}$$

$$y' = Y(x,y;\lambda) \tag{19b}$$

Now when we differentiate $\phi(x',y') = C'$ with respect to λ and set $\lambda = 1$, we get

$$\xi\phi_x + \eta\phi_y = \left(\frac{\partial C'}{\partial \lambda}\right)_{\lambda=1} \tag{20}$$

where $\xi(x,y) = (\partial X/\partial\lambda)_{\lambda=1}$ and $\eta(x,y) = (\partial Y/\partial\lambda)_{\lambda=1}$. Following the rest of the previous derivation without change, we find the celebrated formula of Lie for the integrating factor, μ:

$$\mu = (\xi M + \eta N)^{-1} \tag{21}$$

IV. Application of Self-Similarity to Second-Order Differential Equations

The most general second-order differential equation involving the dependent variable y can be written $f(x,y,\dot{y},\ddot{y}) = 0$, where f again can be any function. As before, the invariance of the differential equation to the transformation $x' = \lambda x$, $y' = \lambda^{\alpha} y$ requires that f have the form

$$f(x,y,\dot{y},\ddot{y}) = g\left(\frac{y}{x^{\alpha}}, \frac{\dot{y}}{x^{\alpha-1}}, \frac{\ddot{y}}{x^{\alpha-2}}\right) \tag{22}$$

when g can be any function. In view of (22), the stipulation f = 0 is equivalent to

$$\frac{\ddot{y}}{x^{\alpha-2}} = h(u,v) \tag{23}$$

where $u = y/x^{\alpha}$, $v = \dot{y}/x^{\alpha-1}$, and h can be any function.

Lie noticed that if we form the derivative dv/du, it will only involve functions of u and v. This means that the second-order differential equation f = 0, when written in terms of the variables u and v, becomes a first-order differential equation. The calculation is a short one:

$$\frac{du}{dx} = \frac{\dot{y}}{x^{\alpha}} - \frac{\alpha y}{x^{\alpha+1}} = \frac{1}{x}(u-\alpha v) \tag{24a}$$

$$\frac{dv}{dx} = \frac{\ddot{y}}{x^{\alpha-1}} - \frac{(\alpha-1)\dot{y}}{x^{\alpha}} = \frac{1}{x}(h(u,v) - (\alpha-1)v) \tag{24b}$$

so that, dividing, we obtain

$$\frac{dv}{du} = \frac{h(u,v) - (\alpha-1)v}{u - \alpha v} \tag{24c}$$

a first-order equation for v in terms of u.

This reduction does not alter the necessity for two integrations to find y in terms of x. If we succeed in integating (24c) to get v in where $\xi(x,y) = (\partial X/\partial\lambda)_{\lambda=1}$ and $\eta(x,y) = (\partial Y/\partial\lambda)_{\lambda=1}$. Following the rest of the previous derivation without change, we find the celebrated formula of Lie for the integrating factor, μ:

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This reduction does not alter the necessity for two integrations to find y in terms of x. If we succeed in integating (24c) to get v in

terms of u, we have really gotten a first-order equation for y in terms of x. This second equation must also be integrated.
However, it, too, is invariant to the group $x' = \lambda x$, $y' = \lambda^{\alpha} y$ and hence can be treated by the methods of section III.

What if we cannot integrate (24c), by far the commonest situation? All is far from lost: because (24c) is of first order, we can study its direction field and determine much useful information about the solutions of the second-order equation $f = 0$. How this is done will be made clear next by working an example. The second-order differential equation will be obtained in the course of solving a partial differential equation exactly as announced in the Introduction.

V. Partial Differential Equations: Example-Unsteady Heat Conduction in He-II

(1) Heat conduction in He-II, an unusual low-temperature liquid phase of helium, is characterized not by Fourier's linear law but the non-linear Görter-Mellink law $q = -k(\partial T/\partial z)^{1/3}$. Here q is the heat flux, $\partial T/\partial z$ the temperature gradient, and k a thermal conductance parameter. The heat balance equation $\rho c_p(\partial T/\partial t) + (\partial q/\partial z) = 0$ combined with the Görter-Mellink law leads, when suitably dimensionalized*, to the non-linear partial differential equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)^{1/3} \tag{25}$$

Eq.(25) is invariant to the stretching transformations

$$\left.\begin{aligned} T' &= \lambda^{\alpha} T \\ t' &= \lambda^{\beta} t \\ z' &= \lambda z \end{aligned}\right\} \qquad 0 < \lambda < \infty \tag{26a}$$

where

$$2\alpha - 3\beta + 4 = 0 \tag{26b}$$

What is the most general solution, $T = f(z,t)$, of (25) invariant to (26)? Following our usual procedure, we note that invariance means $T' = f(z',t')$ or

* We shall play fast and loose with physical constants in this section because our purpose here is to explore the treatment of partial differential equations rather than derive specific practical results.

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$$\lambda^{\alpha} f(z,t) = \lambda^{\alpha} T = f(\lambda z, \lambda^{\beta} t) \tag{27}$$

Differentiating with respect to λ and setting $\lambda=1$, we get the first-order linear partial differential equalion

$$\alpha f = z\frac{\partial f}{\partial z} + \beta t \frac{\partial f}{\partial t} \tag{28}$$

whose characteristic equations are

$$\frac{dz}{z} = \frac{dt}{\beta t} = \frac{df}{\alpha f} \tag{29}$$

Two independent integrals of (29) are $z/t^{1/\beta}$ and $f/t^{\alpha/\beta}$ so the most general solution of (28) has the form

$$T = f(z,t) = t^{\alpha/\beta}\, y\left(\frac{z}{t^{1/\beta}}\right) \tag{30}$$

where y is an arbitrary function.

Eq.(30) is the most general form an invariant solution of (25) may have. If we substitute (30) into (25), the partial derivatives of T will all be expressed in terms of the ordinary derivative $\dot{y}$ of the function y:

$$\frac{\partial T}{\partial t} = t^{(\alpha/\beta)-1}\left(\frac{\alpha}{\beta}y - \frac{1}{\beta}\, x\dot{y}\right) \quad \text{where } x = \frac{z}{t^{1/\beta}} \tag{31a}$$

$$\frac{\partial T}{\partial z} = t^{(\alpha-1)/\beta}\dot{y} \tag{31b}$$

Substituting (31a) and (31b) into (25) and also using (26b) we find

$$\beta \frac{d}{dx}\left(\frac{dy}{dx}\right)^{1/3} + x\frac{dy}{dx} - \alpha y = 0 \tag{32}$$

Any solution of the ordinary differential equation (32) will provide a solution of the partial differential equation (25) invariant to the group (26a).

What about the values of α and β? These are determined by the boundary conditions as we shall see next.

(2) Let us consider the problem of a of half-space, $z > 0$, the temperature of whose front surface $z = 0$ is suddenly raised at $t = 0$ and thereafter held constant. The boundary conditions corresponding to this problem are

$$T(0,t) = 1 \qquad t > 0 \tag{33a}$$

$$T(\infty,t) = 0 \qquad t > 0 \tag{33b}$$

$$T(z,0) = 0 \qquad z > 0 \tag{33c}$$

Eqs(33b) and (33c) are invariant to (26a) whatever the values of α and β, but (33a) is invariant only if $\alpha = 0$ and $\beta = 4/3$. Then (33a-c) become, when written in terms of y,

$$(33a): \qquad y(0) = 1 \tag{34a}$$

$$(33b,c): \qquad y(\infty) = 0 \tag{34b}$$

These two boundary conditions are just sufficient to select a unique solution of second order deq (32), taken for $\alpha = 0$ and $\beta = 4/3$. It is fortunate that boundary conditions (33b) and (33c) for T collapse to the same boundary condition (34b) for y, so that three conditions become two. This will not always happen, and the fact that it does not means that the partial differential equation (25) has many more solutions than the ordinary differential equation (32). Or said in other words, the solutions of (25) invariant to the transformations (26) are only a small subclass of all its solutions.

When $\alpha = 0$ and $\beta = 4/3$, (32) can easily be solved by introducing $\dot{y}^3$ as a new independent variable. Mark well, this will not always happen either. The solution , which the reader can verify by differentiation is

$$y = 1 - \frac{x}{\left(\frac{8}{3\sqrt{3}} + x^2\right)^{1/2}} \tag{35}$$

But whether or not we bothered to solve for y, we could already have seen at the time we had determined that $\alpha = 0$ and $\beta = 4/3$ that a point on front of rising temperature marked by a constant value of x would advance in a time t a distance z proportional to $t^{3/4}$.

(3) Let us now consider the problem of a instantaneous heat source in the plane $z = 0$ pulsed at $t = 0$.
The boundary conditions for this problem are

$$T(z,0) = 0 \qquad |z| > 0 \tag{36a}$$

$$T(\infty,t) = 0 \qquad t > 0 \tag{36b}$$

$$\int_0^\infty T\, dz = 1 \tag{36c}$$

plus the obvious symmetry condition T(z) = T(-z). Eq.(36c) requires that $\alpha = -1$ to be invariant to (26). Then $\beta = 2/3$. Again (36a) and (36b) collapse to the same condition

$$y(\infty) = 0 \tag{37a}$$

while (36c) becomes

$$\int_0^\infty y\, dx = 1 \tag{37b}$$

[Actually, for (37a) to satisfy (36a), y must vanish at infinity faster then $\frac{1}{x}$ because $T = t^{-3/2}\, y(z/t^{3/2}) = \frac{1}{z} \cdot xy(x)$. We shall have to verify this for the solution we actually obtain.]

When $\alpha = -1$ and $\beta = 2/3$, (32) is once again easily soluble, for when $\alpha = -1$, the last two terms become just d(xy)/dx. The solution, which the reader again can verify by differentiation, is

$$y = \frac{4}{3\sqrt{3}} \;(x^4 + a^4)^{-1/2} \tag{38a}$$

and the value of a determined from the integral in (36c) is

$$a = \frac{[\Gamma(\frac{1}{4})]^2}{3\sqrt{3\pi}} = 1.42727 \tag{38b}$$

It is already clear, even without solving (32), that the central temperature falls as $t^{-3/2}$.

(4) In the previous two examples, the ordinary differential equation that arose from the search for invariant solutions could be solved and an explicit formula for the temperature distribution achieved. In the problem we discuss now, this is not the case and we shall have to proceed somewhat differently. We consider the clamped-flux problem in which the plane z = 0 is imagined to be a heater that is turned on suddenly and delivers a constant flux of heat to the half-space.

The boundary conditions for this problem are

$$\frac{\partial T}{\partial z}(0,t) = -1 \qquad t > 0 \tag{39a}$$

$$T(\infty,t) = 0 \qquad t>0 \tag{39b}$$

$$T(z,0) = 0 \qquad z>0 \tag{39c}$$

Eq.(39a) requires that $\alpha = 1$, so that $\beta = 2$. Written in terms of y now, (39a) becomes

$$\dot{y}(0) = -1 \tag{40a}$$

and (39b) and (39c) will both be satisfied if

$$y\ (\infty) = 0 \tag{40b}$$

[Actually this condition is stronger than necessary to satisfy (39c) but is sufficient.] Eq.(32) now becomes

$$2\frac{d}{dx}\left(\frac{dy}{dx}\right)^{\frac{1}{3}} + x\frac{dy}{dx} - y = 0 \tag{41}$$

which is not simply integrable in terms of elementary functions. How shall we now proceed?

(5) Eq.(32) is invariant to the stretching group

$$\left.\begin{aligned} y' &= \mu^{-2}y \\ x' &= \mu x \end{aligned}\right\} \qquad 0 < \mu < \infty \tag{42}$$

This is an easy thing to discover after only the briefest calculation, but it is no lucky accident like the easy integrability of the ordinary differential equation (32) in the first two problems we did. In fact, the group (42) could have been written down from the information contained in Eq.(26) even before Eq.(32) was worked out. The proof of these assertions with be given later.

The group invariance of (32) will allow us to reduce (32) to a first-order differential equation using Lie's theorem described in Section IV. We take $v = x\dot{y}^{1/3}$ as the first differential invariant and $u = xy^{1/2}$ as the invariant. Then

$$x\frac{dv}{dx} = x\dot{y}^{1/3} + x^2\frac{d}{dx}(\dot{y}^{\ 1/3}) = v + \frac{\alpha}{\beta}u^2 - \frac{v^3}{\beta} \tag{43a}$$

$$x\,\frac{du}{dx} = xy^{1/2} + \frac{1}{2}x^2y^{-1/2}\dot{y} = u + \frac{v^3}{2u} \tag{43b}$$

so that

$$\frac{dv}{du} = \frac{u(2\beta v - 2v^3 + 2\alpha u^2)}{2\beta u^2 + \beta v^3} \tag{44}$$

or, when $\alpha = 1$ and $\beta = 2$,

$$\frac{dv}{du} = \frac{u(2v - v^3 + u^2)}{2u^2 + v^3} \tag{45}$$

We now proceed by analyzing the direction field of the first-order equation (45). This stage of the analysis is closely reasoned and some might consider it tedious. I personally find it interesting. In any case the reader is strongly advised to study the next sections carefully because it exemplifies a pattern found in many problems.

(5) Because the temperature rise is positive and falls as we move away from the heated plate, $u > 0$ and $v < 0$. Hence we shall only be interested in the fourth quadrant of the (u,v)-plane (see Fig.2). First we find the curves on which dv/du equals either zero or infinity. Only on these curves does dv/du change sign, and only where curves of $dv/du = 0$ and $dv/du = \infty$ intersect are there singular points. Now $dv/du = 0$ on the v-axis: $u = 0$, and on the curve C_1: $u^2 = v^3 - 2v$. Furthermore, $dv/du = \infty$ on the curve C_2: $2u^2 + v^3 = 0$. The origin O and the point P are singular points.

Since $y(0)$ and $\dot{y}(0)$ are finite, when $z = 0$, u and v also equal 0. Therefore, we are interested in integral curves that pass through the origin O. The integral curves in the fourth quadrant passing through the origin are of two kinds, those that eventually intersect the curve C_1 and those that eventually intersect the curve C_2. These two kinds are separated by a single, exceptional integral curve, S, called a separatrix, that joins the singularities O and P. Curves intersecting curve C_1 eventually intersect the u-axis. There $v = 0$ and $u > 0$, i.e., $\dot{y} = 0$ and $y > 0$. Such curves have a minimum and so do not conform to the temperature profiles we are seeking. Curves intersecting C_2 eventually intersect the v-axis: $u = 0$, $v < 0$, i.e., $y = 0$, $\dot{y} < 0$. Such curves reach zero temperature with a negative slope, a possibility we also reject. All that is left is the separatrix. Later we will see that it is possible to predict that points of the separatrix near the singularity P correspond to the asymptotic behavior $y \sim 4\sqrt{3}/9x^2$. Such behavior is what we are looking for in the solution we seek. To find the separatrix, we must perform a numerical integration. Owing to the divergence of the integral curves as we approach P and their convergence as we approach 0, we expect integration in the direction $P \to 0$ to be stable and integration in the direction $0 \to P$ to be unstable. Furthermore integration in the direction $P \to 0$ involves no trial and error, as integration in the direction $0 \to P$ would. Application of L'Hospital's rule shows that $(dv/du)_P = -(3+\sqrt{17})/2 \cdot 3^{3/4} = -1.5624$. We use this slope to advance a short distance from P and then we continue by numerical integration towards 0. (All numerical computations were carried out on a programmable desk calculator.) The resulting curve for the separatrix approaches the origin with a slope of -1.095.

To find y(x), one procedure is the following. Near the origin

$$\frac{\dot{y}^{1/3}}{y^{1/2}} = \frac{v}{u} = -1.095 \tag{46}$$

so that if we arbitrarily normalize $y(0) = 1$, then $\dot{y}(0) = -1.313$. With this initial condition, we can integrate forwards numerically to obtain y(x). The results of such an integration are shown as small circles in Fig.3. Because we are integrating in the direction $0 \to P$, the integration eventually becomes unstable. But we can integrate close enough to the asymptotic limit $y = 4\sqrt{3}/9x^2$ that there is no difficulty in graphically continuing the numerical solution valid for small x.

VI. Partial Differential Equations: Recapitulation

(1) Our starting point was the group invariance of the partial differential equation. We saw that solutions of the pde invariant to the group involved only a function of one variable. So for these solutions the partial differential equation must reduce to an ordinary differential equation. Then, we have avoided all the difficulties ordinarily associated with pdes, especially non-linear pdes (where we lose the property of superposition).

(2) The initial and boundary conditions that determine an invariant solution must themselves be invariant to the group. The initial and boundary conditions needed to specify a unique solution of a pde are more extensive than those needed to specify a unique solution of an ode. Some of the initial and boundary conditions for the pde must "collapse" to the same condition the solution of the ode. Otherwise, the initial and boundary conditions overdetermine the solution the ode and generally cannot be fulfilled simultaneously.

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(3) The ordinary differential equation can sometimes be solved in terms of the elementary functions, but usually it cannot. Very often, however, the ode is invariant to a group of stretching transformations--this is no coincidence, as we shall see in sec. VII--and the group of the ode is related to the group of the pde. If the ode is of second order, which is the usual case, its group invariance can be used to reduce it to a first-order ode.

(4) The first-order ode can be dealt with by studying its direction field. A large part of this study involves the behavior of the direction field near singular points. The differential equation can often be simplified near the singular points because some terms may become small compared with others. Occasionally, when some terms are dropped, the simplified first-order ode is invariant to a group of stretching transformations and can be readily integrated.

(5) Quite often, the integral curve of the first-order ode being sought is a separatrix joining two singular points. The behavior of the separatrix near its two ends can often be represented analytically, and these analytical representations usually can be converted into simple descriptions of the asymptotic behavior of the solution of the pde.

(6) Often a numerical integration is necessary to calculate the entire separatrix. Here especially, knowledge of the direction field is valuable in choosing the direction of stable integration.

This procedure, in the hands of a practitioner who has had the patience to become skilled in its use, is very powerful, as the reader perhaps will now appreciate from the example just worked.

VII. Group Invariance of the Ode.

(1) The group invariance of the ode is connected with Eq.(26a)'s representing not just a group of stretching transformations but a one-parameter family of groups of stretching transformations. The boundary conditions determine the particular values of α and β, say α_0 and β_0, that rule the solution we seek

$$f(z,t) = t^{\alpha_0/\beta_0}\, y\left(\frac{z}{t^{1/\beta_0}}\right) \tag{47}$$

where α_0 and β_0 satisfy

$$M\alpha_0 + N\beta_0 = L \tag{48}$$

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this being a generalization of linear relation (26b) with coefficients M, N, and L. If we transform (47) with respect to transformations of (26a) belonging to the values of α_0 and β_0 we recover (47) again. If we transform (47) with respect to transformations of Eq.(26a) belonging to other values of α and β, we must still get a solution of the partial differential equation. Transforming and rearranging, we find

$$f(z,t) = \lambda^{(\beta\alpha_0 - \alpha\beta_0)/\beta_0} \, t^{\alpha_0/\beta_0} \, y\left(\lambda^{1-\beta/\beta_0} \frac{z}{t^{1/\beta_0}}\right) \tag{49}$$

is also a solution of the pde. If we set $\lambda^{1-\beta/\beta_0} = \mu$, (49) can be written

$$f(z,t) = \mu^{\frac{\beta\alpha_0 - \alpha\beta_0}{\beta_0 - \beta}} \, t^{\alpha_0/\beta_0} \, y\left(\mu \frac{z}{t^{1/\beta_0}}\right) \tag{50a}$$

$$= t^{\alpha_0/\beta_0} \quad \mu^{-\frac{L}{M}} \quad y(\mu\, x) \tag{50b}$$

where $x = z/t^{1/\beta_0}$ and (48) has been used to show that

$$\frac{\beta\alpha_0 - \alpha\beta_0}{\beta_0 - \beta} = -\frac{L}{M} \tag{51}$$

Note that the function f(z,t) in (50b) is invariant to (26a) when $\alpha = \alpha_0$ and $\beta = \beta_0$. Therefore the function $\mu^{-L/M} y(\mu\, x)$ will provide an invariant solution if the function y(x) does. The function $\mu^{-\frac{L}{M}} y(\mu\, x)$ is an image of y(x) under the group

$$y' = \lambda^{\frac{L}{M}} y \qquad 0 < \lambda < \infty \qquad x' = \lambda x \tag{52}$$

[Proof: Take $\lambda = \mu^{-1}$. Then $y'(x') = \mu^{-\frac{L}{M}} y(x) = \mu^{-\frac{L}{M}} y(\mu\, x')$.]
So each function y(x) that provides and invariant solution to the pde when $\alpha = \alpha_0$ and $\beta = \beta_0$ gives rise to an entire one-parameter family of functions that do the same thing. These families are invariant to (52) because they consist of an entirety of curves that map into each other. The families all taken together represent the solution of the ordinary differential equation for the function y.* Since they are invariant to (52), so, too, must be the

* For the more mathematically inclined reader, I mention that imaging under (52) is an equivalence relation that therefore defines a partion of the integral curves.

ode. Since (52) does not depend on the particular values of α and β, this conclusion is true for any α and β obeying (48). Therefore, the ode is invariant to the group (52).

In the example of section V, L/M = -2. It is easy to verify directly that (32) is invariant to (52) with this value of L/M.

(2) Knowing only the structure of the groups (26) and (52), we can go one step further. An invariant and a first differential invariant of (52) are $u = y/x^a$ and $v = \dot{y}/x^{a-1}$, where a is an abbreviation for L/M. Then

$$du = (v - au)\ \frac{dx}{x} \tag{53a}$$

and

$$dv = [\ \frac{\ddot{y}}{x^{a-2}} - (a-1)v]\ \frac{dx}{x} = (F(u,v)-(a-1)v)\ \frac{dx}{x} \tag{53b}$$

because Lie's theorem tells us that $\ddot{y}/x^{a-2}$ must be a function of u and v. Thus

$$\frac{dv}{du} = \frac{F(u,v)-(a-1)v}{v-au} \tag{53c}$$

We know nothing about the function F(u,v) because it depends on the differential equation and not just on the group alone. In general, however, the singular points of (53c) are the roots of the simultaneous equations

$$F(u,v) = (a-1)v \tag{54a}$$

$$v = au \tag{54b}$$

A pair of values of u_0 and v_0 connected by (54b) correspond to an exceptional solution of the ode for y(x) of the form $y = u_0 x^a$. Integral curves in the (u,v)-plane which enter this singularity will have $u_0 x^a$ as a limiting behavior. This is how we found the asymptotic behavior $4\sqrt{3}/9x^2$ for the separatrix in section V(5).*

(3) Remark on Separatrices. In my experience with problems of diffusion, heat and mass transfer, and applied superconductivity, the integral curve of the first-order equation (53c) we seek is often a separatrix. If we do not know a group to which the first-order ode is invariant, as was the case in

* A word of caution: because the choice of an invariant and first-differential invariant in section V(5) is slightly different than that used here, there $y = u_0^2 x^a$. From Fig.3, we see that $u_0 = 2\sqrt[4]{3}/3$.

the example of sections V:(4)-(5), we must usually resort to some numerical method of claculation. On the other hand, if we know a group to which the first-order ode (53c) is invariant, its solution can be expressed in terms of an indefinite integral. There is no guarantee. however, that the integral can be carried out. We can find separatices without any integration at all by noting that they are invariant curves of the group. [If this is not already clear to the reader, let him consider Fig.2, for example. Invariance of the ode to a group means the image of Fig.2 under transformations of the group is again Fig.2. Since the deq has the same form in the primed variables as in the unprimed, singular points like O and P have the same coordinates in the primed variables as in the unprimed, i.e., they are their own images. But so then must the separatrix S, which passes through O and P, be its own image.]

Suppose the group of (53c) is the general group

$$\left.\begin{aligned} u' &= U(u,v\ ;\lambda) \\ v' &= V(u,v\ ;\lambda) \end{aligned}\right\} \qquad 0 < \lambda < \infty \tag{55}$$

The most general curve $f(u,v) = 0$ invariant to (55) obeys the condition

$$f(u',v') = 0$$

too. Differentiating wrt λ and then setting $\lambda=1$ as before, we find

$$\xi\frac{\partial f}{\partial u} + \eta\frac{\partial f}{\partial v} = 0 \tag{57}$$

where $\xi = (dU/d\lambda)_{\lambda=1}$ and $\eta = (dV/d\lambda)_{\lambda=1}$. The level lines of the functions f that solve (57)(i.e., the curves in the (u,v)-plane on which such functions f are constant) are the curves

$$\frac{\partial f}{\partial u}\ du + \frac{\partial f}{\partial v}\ du = 0 \tag{58}$$

Eqs.(57) and (58) imply then that on level lines of f

$$\frac{dv}{du} = \frac{\eta}{\xi} \tag{59}$$

If we replace the left hand-side of (53c) by the ratio η/ξ we obtain an algebraic equation for the invariant curves of (53c). These include but are not limited to, the separatrices (e.g., envelopes, i.e., singular solutions, are also obtained by the same calculation).

VIII. Approximate Solutions: Diffusion in Cylindrical Geometry

(1) The procedure outlined so far gives exact solutions to the pde in some, but not all, problems. At first sight it would seem to be of no use in problems in which all its conditions of applicability are not met. But a deeper look shows the method of self-similarity can be used to get valuable information about asymptotic behavior in such problems. An excellent illustration is the clamped-flux problem in cylindrical geometry for the ordinary diffusion equation. Suppose at $t=0$ the cylindrical surface $r=R$ is suddenly caused to begin emitting a steady heat flux. What are the resulting temperature profiles?

The pde, boundary and initial conditions describing the problem, again suitably dimensionalized, are

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \tag{60a}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=1} = -1 \qquad t>0 \tag{60b}$$

$$T(r,0) = 0 \qquad r>1 \tag{60c}$$

$$T(\infty,t) = 0 \qquad t>0 \tag{60d}$$

The pde is invariant to the group

$$\left.\begin{aligned} T' &= \lambda^{\alpha} T \\ t' &= \lambda^{2} t \\ r' &= \lambda\, r \end{aligned}\right\} \qquad (0 < \lambda < \infty) \tag{61b}$$

but the boundary conditions (60b) and (60c) are not invariant because of the appearance of the value $r=1$.

(2) Very early, the heat form the source surface $r=1$ has not diffused far. When the thickness of the heated layer is still $<<1$, the curvature of the heated surface should not matter. So the problem goes over, for short times, to its plane analogue

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$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} \qquad (62a)$$

$$\left(\frac{\partial T}{\partial z}\right)_{z=0} = -1 \qquad t > 0 \qquad (62b)$$

$$T(z,0) = 0 \qquad z > 0 \qquad (62c)$$

$$T(\infty,t) = 0 \qquad t > 0$$

where $z = r-1$

All of these equations are invariant to the group (61). If we set

$$T = t^{1/2}\, y(x), \quad x = z/t^{1/2} \qquad (63)$$

we find

$$\ddot{y} = \frac{1}{2}(y - x\dot{y}) \qquad (64a)$$

$$\dot{y}(0) = -1 \qquad (64b)$$

$$y(\infty) = 0 \qquad (64c)$$

Eq.(64a) is linear, and in this cases it is easier to solve it using the properties of linear deqs than to exploit its group invariance. We can see immediately that $y = x$ is a special solution of (64a). To find a second, independent solution, we use the classical procedure of setting $y = wx$. Then we find the following separable equation for w:

$$\frac{\ddot{w}}{\dot{w}} = -\frac{2}{x} - \frac{x}{2} \qquad (65)$$

Thus

$$\dot{w} = -Cx^{-2}e^{-x^2/4} \qquad (66)$$

where C is a constant of integration. After a second integration we find

$$y = wx = Cx \int_x^\infty e^{-x^2/4} \frac{dx}{x^2} \qquad (67a)$$

$$= C\left[e^{-x^2/4} - \frac{x}{2}\int_x^\infty e^{-x^2/4}\, dx\right] \qquad (67b)$$

which satisfies (64c). In order to satisfy boundary condition (64b), we must take $C=\frac{2}{\sqrt{\pi}}$. This gives a complete solution for the temperature valid for short tines. We shall especially be interested in T(r = 1,t) gven by

$$T(r=1,t) = y(0)t^{1/2} = \frac{2}{\sqrt{\pi}} t^{1/2} \tag{68}$$

(3) The boundary condition (60b) is equivalent to the integral condition

$$\frac{d}{dt}\int_1^\infty T\ r\ dr = 1 \quad \text{or} \quad \int_1^\infty T\ r\ dr = t \tag{69}$$

which follows by integration of (60a) over the region $1<r<\infty$. Late in the history of the problem, when the tempereture profile has spread very far from r = 1, the value of the integral in (69) should be affected very little if the lower limit is extended to zero. If we replace (60b) by the condition

$$\int_0^\infty T\ r\ dr = t \tag{70}$$

we again have a totally invariant problem. Eq.(70) requires $\alpha = 0$, so we take

$$T = y(x),\ x = r/t^{1/2} \tag{71}$$

and obtain

$$\ddot{y} + \dot{y}\left(\frac{1}{x} + \frac{x}{2}\right) = 0 \tag{72a}$$

$$y(\infty) = 0 \tag{72b}$$

$$\int_0^\infty y\ x\ dx = 1 \tag{72c}$$

Eq.(72a) integrates at once to give

$$\dot{y} = \frac{C\ e^{-x^2/4}}{x} \quad ; \quad C = \text{constant of integration} \tag{73}$$

Integration by parts turns (72c) into

$$\int_0^\infty \dot{y}\ x^2\ dx = -2 \tag{74}$$

from which it follows at once that C = -1. Then

$$y = \int_x^\infty e^{-x^2/4}\ \frac{dx}{x} = \frac{1}{2} E_1\left(\frac{x^2}{4}\right) \tag{75}$$

where E_1 is the exponential integral discussed by Abramowitz and Stegun. Eq.(75) gives the late temperature profiles. The temperature of the heated surface is gven by

$$T(1,t) = \frac{1}{2} E_1\left(\frac{1}{4t}\right) \sim \ln\sqrt{t} + \ln 2 - \gamma + O\left(\frac{1}{t}\right) \tag{76}$$

when γ is the Euler-Mascheroni constant 0.57721•••. Fig.4 shows the limiting curves (68) and (76) and a graphical interpolation between them.

[(4) Fig.5 shows the results of constant-flux experments done with a 0.05-mm-dia wire in LN_2 by O. Tsukamoto and T. Uyemura [Adv. Cryo. Eng. 25: 475 (1980)]. The abscissa q is the constant heat flux; the ordinate t_d is the time at which the surface temperature rise ΔT of the wire reached 25K. This is the so-called "take-off" point, at which the rate of temperature rise suddenly increases strongly. Written in fully dimensional form, (68) and (76) are

$$\frac{k\Delta T}{qR} \mp \frac{1}{\sqrt{\pi}}\left(\frac{4\kappa t_d}{R^2}\right)^{\frac{1}{2}} \tag{77a}$$

and

$$\frac{k\Delta T}{qR} = \frac{1}{2} E_1\left(\frac{R^2}{4\kappa t_d}\right) \tag{77b}$$

where k is thermal conductivity, R is the radius of the wire, and $\kappa = k/\rho c_p$ is the thermal diffusivity (ρc_p = heat capacity per unit volume.) On a log plot with $k\Delta T/qR$ as ordinate and $4\kappa t_d/R^2$ as abscissa, the curves in Fig.4 and Fig.5 should be the same except for shifts of the axes. Shown in Fig.5 is the curve of Fig.4 shifted both vertically and horizontally until it agrees well with the experimental points. The value of $k\rho c_p$ implied by the fit of (77a) to the experimental points is much larger that the value calculated from the properties of saturated nitrogen at 1 atm ($k = 1.4 \times 10^{-3}$ W cm^{-1} K^{-1}, $c_p = 2.0$ J g^{-1} K^{-1}, $\rho = 0.80$ g cm^{-3}) Perhaps vapor is already forming before "takeoff", which would make the apparent specific heat higher. The formation of this vapor could also be accompanied by convection, which would increase the apparent thermal conductivity.]

(5) Reddi, Ray, Raghavan, and Narlikar[3] have studied the clamped-temperature problem in cylindrical geometry as part of an investigation

of the formation of A15 compounds like Nb_3Sn in multifilamentary composites. This problem, too, is described by Eq.(60a), (60c), and (60d), but (60b) is changed to

$$T(1,t) = 1 \qquad t > 0 \tag{78}$$

As before, the problem goes over to its plane analogue for short times. This latter problem is very easily soluble by the method under discussion and only the result will be quoted:

$$T(r,t) = \operatorname{erfc}\left(\frac{r-1}{2\sqrt{t}}\right) \qquad t \ll 1 \tag{79}$$

where erfc is the complementary error function. The flux out of the surface $r=1$ is then given by

$$-\left(\frac{\partial T}{\partial r}\right)_{r=1} = \frac{1}{\sqrt{\pi t}} \tag{80}$$

(6) For long times, we proceed by setting

$$T(r,t) = g(t)\, y(x), \quad x = \frac{r}{\sqrt{t}} \tag{81}$$

We find

$$\ddot{y} + \dot{y}\left(\frac{1}{x} + \frac{x}{2}\right) = \frac{\dot{g}t}{g}\, y \tag{82}$$

If g varies slowly enough with t, the rhs of (82) may be taken as zero. The lhs, the same as (71b), integrates to give $y = E_1\left(\frac{x^2}{4}\right)$ (any constant of integration may be subsumed in g). To satisfy (78) we now must have

$$g(t) = E_1\left(\frac{1}{4t}\right) \tag{83}$$

For long enough t, $g \sim \ln t$ so that $\dot{g}t/g = \frac{1}{\ln t}$, which eventually does become small as required.

Thus

$$T(r,t) = \frac{E_1\left(\frac{r^2}{4t}\right)}{E_1\left(\frac{1}{4t}\right)} \tag{84a}$$

and

$$-\left(\frac{\partial T}{\partial r}\right)_{r=1} = \frac{2\, e^{-1/4t}}{E_1(1/4t)} \tag{84b}$$

Fig.6 shows the limit (80) (marked slab) and the limit (84b) for $-\left(\frac{\partial T}{\partial r}\right)_{r=1}$ as well as the corresponding quantity for a spherical heated surface. The problem for a sphere is solved by introducing rT as a new unknown; the equations then reduce to those for a slab for which the solution is $rT = \mathrm{erfc}(\frac{r-1}{2\sqrt{t}})$.

(7) Reddi et al. calculated $-(\partial T/\partial r)_{r=1}$ by assuming a linear profile the slope of which changed with time. Fig.6 also shows their result. It departs substantially from the correct result for cylinders at about the same point that the cylinder and slab results begin to diverge. The main conclusion that Reddi et al. draw from their calculation is, in their words, the following: "When the diffusion of B atoms [in the formation of A_3B compounds] in the alloy matrix is rate controlling, the n values [in the growth law $R \sim t^n$ for the reaction layer thickness] derived from growth model range from 1/2 to 2/3." (brackets mine). Our more careful analysis shows that the range of exponents is from 1/2 to 1, but exponents >2/3 occur only for very long reaction times. The basic idea of Reddi et al., namely that the growth exponent varies because the diffusion occurs in a cylindrical geometry, is correct, but their supporting analysis is deficient.

IX. The Method of Assigned Level Curves

(1) In this section we apply the idea of self-similarity to a problem involving a partial differential equation in another way than in previous sections. The problem is an old one.

$$\nabla^2\phi = -1 \tag{85a}$$

in S

$$\phi(C) = 0, \tag{85b}$$

where S is a region of the (x,y)-plane bounded by the closed curve C(see Fig.7). Several different physical problems lead to the mathematical formulation (85a,b). In one, S is a thin flat, conducting plate being exposed to a slow ramped magnetic field in the z-direction. The function ϕ is then the stream function of the induced eddy currents. From it we can calculate the eddy current power being dissipated. In a second problem,

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S is the interior of a long, hollow, cylindrical pipe filled with a viscous fluid in steady laminar flow (Poiseuille's problem). The function ϕ is then the flow velocity for a given pressure gradient in the pipe.
A third problem is the torsion of a long cylindrical bar whose cross section is the area S. In this case, ϕ is the Airy stress function and can be used to find the torsional rigidity of the bar. A fourth problem is the distortion of an elastic film stretched over a wire rim having the shape of C by a uniform unbalanced pressure on one side. In this case ϕ is the displacement of the film from its original position.

(2) The problem represented by Eqs.(85a,b) can be solved exactly when C is a circle, an ellipse, or an equilateral triangle. When it is a rectangle, a convenient solution exists in the form of a series. For other shapes, no such solutions are available. However, the functional

$$I[\phi] = \int_S [\phi - \frac{1}{2}(\nabla\phi)^2]\ dxdy \qquad (86)$$

can be calculated from approxmate solutions with a higher accuracy than that with which ϕ itself is known. That means, for example, if the approximate ϕ differs from the correct $\phi = \phi_*$ by about 10%, the approximate I may only differ from the exact $I = I_*$ by , say, 1%. The functional I represents each of the physical quantities mentioned earlier, namely, the eddy current dissipation in the plate, the total flow in the pipe, or the torsional rigidity of the bar. These important physical quantities can be estimated with very good accuracy using only a crude guess for ϕ. In carrying out this procedure we shall encounter some interesting problems involving self-similar families of functions.

(3) Suppose ϕ_* represents the correct solution of (85) and $\phi(x,y) = \phi_*(x,y) + \varepsilon\psi(x,y)$, $\varepsilon \ll 1$, represents some guess at ϕ_*. Then

$$I[\phi_* + \varepsilon\psi] = \int_S [-\frac{1}{2}(\nabla\phi_*)^2 - \varepsilon\nabla\phi_*\cdot\nabla\psi - \frac{1}{2}\varepsilon^2(\nabla\psi)^2 + \phi_* + \varepsilon\psi]\ dxdy \qquad (87a)$$

$$= I\ [\phi_*] - \varepsilon\psi\ \frac{\partial\phi^*}{\partial n}]_C + \varepsilon\int_S (\psi\nabla^2\phi_* + \psi)\ dxdy - \frac{\varepsilon^2}{2}\int_S (\nabla\psi)^2\ dxdy \qquad (87b)$$

Now if ϕ obeys the same boundary condition on C that ϕ_* does, then $\psi(C)$

= 0. So the second term in (87b) vanishes. Furthermore, since $\nabla^2\phi_* = -1$ the third term vanishes, too. Thus

$$I = I_* - \frac{\varepsilon^2}{2}\int_S (\nabla\psi)^2 \, dxdy \tag{87c}$$

Eq.(87c) shows that I differs from I_* only by terms of <u>second</u> order in ε. This is the origin of its high accuracy. Moreover, in this case, I_* is the largest value that I can attain for any choice of ϕ. So far these are well-known facts of the variational calculus.

(4) Polya and Szego[4] propose that we make the approximation that the level curves of ϕ are a self-similar family. By level curves we mean curves on which ϕ = constant. Their procedure is outlined in Fig.8. They select a point O as origin and specify the boundary curve C in polar coordinates $r = R(\theta)$. Then they choose the similar curves

$$r = \sigma R(\theta) \qquad\qquad 0 < \sigma < 1 \tag{88}$$

as the level curves of ϕ. This means they take $\phi = \phi(\sigma)$; the dependence of ϕ on σ will be chosen later to maximize $I[\phi(\sigma)]$. Next we must overcome some geometric problems to express $(\nabla\phi)^2$ in terms of σ. This we do with the aid of the sketch in Fig.9. If we advance along a radius OAB from A to B, ϕ changes by $d\phi = (d\phi/d\sigma)\, d\sigma_{AB}$ and r changes by $dr = R\, d\sigma_{AB}$. Thus $\left(\frac{\partial\phi}{\partial r}\right)_\theta = \frac{1}{R}\left(\frac{d\phi}{d\sigma}\right)$. If we advance along the arc of a circle AE from A to E, ϕ again changes by $d\phi = \frac{d\phi}{d\sigma}\, d\sigma_{AE}$. The point E has on the surface $\sigma + d\sigma$ given by the requirement $\sigma R(\theta) = r_A = r_E = (\sigma+d\sigma)\, R(\theta+d\theta) = \sigma R(\theta) + R d\sigma + \sigma\dot{R} d\theta$, so that $d\sigma_{AE} = -\frac{\sigma\dot{R}d\theta}{R}$. Therefore, $\left(\frac{\partial\phi}{\partial\theta}\right)_r = -\frac{\sigma\dot{R}}{R}\left(\frac{d\phi}{d\sigma}\right)$. Then

$$(\nabla\phi)^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 = \frac{R^2+\dot{R}^2}{R^4}\left(\frac{d\phi}{d\sigma}\right)^2 \tag{89}$$

Furthermore, $r dr\, d\theta = R^2 \sigma d\sigma d\theta$, so we have finally

$$I = \int_0^1 \sigma d\sigma \int_0^{2\pi} d\theta \left[-\frac{1}{2}\,\frac{R^2+\dot{R}^2}{R^2}\left(\frac{d\phi}{d\sigma}\right)^2 + R^2\phi\right] \tag{90}$$

(5) Polya and Szego's next step is to carry out the θ-integrals. The integral $\int_0^{2\pi} R^2 d\theta = 2A$ where A is the area of S, the interior of C. The integral $\int_0^{2\pi} \frac{\dot{R}^2}{R^2} d\theta = 2\pi a$, while a geometric quantity, has no such simple interpretation. The important point is that both A and are independent of σ. Now (90) becomes

$$I = \int_0^1 \sigma d\sigma \; [-\pi(1+a)\left(\frac{d\phi}{d\sigma}\right)^2 + 2A\phi] \tag{91}$$

A straightforward variational calculation shows that I will be a minimum when ϕ satisfies the Euler-Lagrange equation and boundary condition

$$\pi(1+a)\,\frac{d}{d\sigma}\left[\frac{(\sigma d\phi)}{d\sigma}\right] + \sigma A = 0 \tag{92a}$$

$$\phi(1) = 0 \tag{92b}$$

This equation is integrable and gives

$$\phi = \frac{(1-\sigma^2)A}{4\pi(1+a)} \tag{93}$$

Eq.(93) represents the best assignment possible of values of ϕ to the self-similar level curves $r = \sigma R(\theta)$. If we insert (93) into (91), we find

$$I = \frac{A^2}{16\pi(1+a)} \tag{94}$$

A convenient representation of (94) is to express it as a fraction f of I for a circle of the same area A as the figure S we are considering. Then, since $I_{\odot} = A^2/16\pi$ exactly

$$f = (1 + a)^{-1} \tag{95}$$

This fraction f gives a second-order estimate of (i) the eddy current power dissipated in a plate compared with a circular disk of the same area, (ii) the viscous flow through a pipe of irregular shape compared with a circular pipe of the same area, and (iii) the torsional rigidity of a bar of irregular shape compared with a circular bar of the same cross-sectional area.

(6) The assignment of values of ϕ to level curves labelled by the various values of σ has been carried out in an optimal way. But we have said nothing so far about the origin O. The value of a depends on the choice of the origin O; and since the value of f given by (95) is a lower limit, we shall get the best (highest) lower limit by choosing O to make a as small as possible. So our variational problem has led us to an amusing geometric problem.

(7) If the bounding curve C is a convex polygon, we can evaluate f as follows (Fig.10): choose the altitude OP as the line $\theta = 0$ and measure

θ positive counter-clockwise. Then $R=h\sec\theta$ along the side AB, and $\dot{R}=h\sec\theta\times\tan\theta$. The contribution of the side AB to a is

$$\frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\tan^2\theta\, d\theta = \frac{1}{2\pi}(\tan\theta_2-\tan\theta_1)-\frac{1}{2\pi}(\theta_2-\theta_1) \tag{96a}$$

$$= \frac{1}{2\pi}\frac{s}{h}-\frac{\Delta\theta}{2\pi} \tag{96b}$$

where s is the length of the side AB, h is the length of the altitude OP and $\Delta\theta$ is the central angle ABO. Summing over all the sides, we find that

$$a = \frac{1}{2\pi}\sum_i\frac{s_i}{h_i}-1 \tag{97a}$$

so that

$$f = \left(\frac{1}{2\pi}\sum_i\frac{s_i}{h_i}\right)^{-1} \tag{97b}$$

(8) If a circle can be inscribed in the polygon, the best choice for O is its center Q (the incenter, so-called). For then, $d(f^{-1}) = -\frac{1}{2\pi}\sum_i\frac{s_i}{h_i^2}\times dh_i$ where dh_i is the increment in h_i when we move the origin O infinitesimally away from the incenter Q. Now $\Sigma s_i h_i = 2A$, where A is the area of the polygon. Therefore $\Sigma s_i\, dh_i = 0$. Since $h_i = R$, the radius of the inscribed circle, when O = Q, $d(f^{-1})]_Q = 0$. Thus Q gives an extremum (and clearly a minimum) of f^{-1}. Furthermore, $R = 2A/P$, where P is the perimeter, so

$$f = \frac{4\pi A}{P^2} \tag{98}$$

for any polygon in which a circle can be inscribed.

If the polygon has reflection symmetry, it can be shown that the best choice for the origin must lie on the axis of symmetry. Thus the best origin for a rectangle is the intersection of the diagonals. Since any closed curve can be approximated by a polygon, this symmetry theorem is true for any curve. The method of assigned level lines has been applied by the author to bound the quantity f from below for triangles, rectangles, rhombuses, and L-shaped plates. Other methods have been used to bound f from above so that estimates with bounded error have been obtained.[5] We break off the discussion here, however, since we have already achieved our stated goal of showing how the notion of self-similarity can be used in variational problems.

X. The Specific Heat of Type-II Superconductors

The calculation of the specific heat of type-II superconductors presents an interesting example of the use of self-similarity not involving differential equations. The Gibbs free energy of long cylinder of a magnetic material in a paraxial magnetic field, $G = U - TS - (\mu_0/\rho_m)HM$, can be written as

$$G_s(T,H) = G_n(T) + \frac{\mu_0}{\rho_m}\int_H^{H_{c2}(T)} MdH \qquad (99)$$

where the subscript s refers to the superconducting state and the subscript n refers to the normal state. Eq.(99) is purely a consequence of the laws of thermodynamics-so far no hypotheses have been framed regarding the behavior of type-II superconductors (save that their magnetization may be treated as reversible.)

(2) Measured magnetization curves of soft (reversible) type-II superconductors look like the curves sketched in Fig.11. The upper and lower critical fields vary with temperature as shown in Fig.12.
With good accuracy, we can write

$$H_{c_i}(T) = H_{c_i}(0)\ [1 - (T/T_c)^2] \qquad i = 1,2 \qquad (100)$$

for both critical fields. According to (100), the ratio $H_{c_1}(T)/H_{c_2}(T)$ is independent of temperature. Hence as the temperature changes from T to T', both the abscissas and ordinates of the point Q and the points on the line segment OP of the magnetization curve scale as $(1-T'^2/T_c^2)/(1-T^2/T_c^2)$. It is tempting to apply this factor to the abscissa and ordinate of every point on the magnetization curve, i.e., to assume the magnetization curves form a self-similar family.

If this is so, then

$$\int_0^{H_{c_2}(T)} MdH = -a\left(1 - \frac{T^2}{T_c^2}\right)^2 \qquad (101a)$$

where

$$a = \left| \int_0^{H_{c_2}} M(T=0)\ dH \right| \qquad (101b)$$

Thus

$$G_s(T,0) = G_n(T) - \frac{\mu_o a}{\rho_m}\left(1 - \frac{T^2}{T_c^{\,2}}\right) \tag{102a}$$

So that at zero applied field

$$C(T) = -T\left(\frac{\partial^2 G_s}{\partial T^2}\right) = C_n(T) - \frac{4\mu_o a}{\rho_m}\left(\frac{T}{T_c^{\,2}}\right)\left(1 - \frac{3T^2}{T_c^{\,2}}\right) \tag{103}$$

The constant a can be determined from the empirical observation that the specific heat of the superconducting phase at zero field vanishes faster than linearly with vanishing temperature. Since this is so, the linear term $4\mu_0 aT/\rho_m T_c^{\,2}$ must cancel the linear tern in $C_n = \gamma T + \beta T^3$.
Thus $a = \rho_m \gamma T_c^2 / 4\mu_0$ and

$$C_s(T,H = 0) = \left(\beta + \frac{3\gamma}{T_c^{\,2}}\right) T^3 \tag{104}$$

(3) As it happens, formula (104) compares well with experiment. It was derived long ago on the basis of the Görter-Casimir two-fluid model. But since we see it to be a consequence of a fairly non-specific hypothesis, we may feel now it is not as supportive of the details of the two-fluid model as we formerly thought. It any case, it represents an interesting application of the idea of self-similarity.

XI. Concluding Remarks

The aspects of self-similarity dealt with in this paper all concern the self-similarity of one-parameter families of curves. But self-similarity occurs widely in other ways as well in physics, mathematics, and engineering. Drop an altitude onto the hypotenuse of a right triangle--it divides the triangle into two smaller triangles both similar to their parent. As Barenblatt has shown,[6] the additivity of areas gives us a one-line proof of the Pythagorean theorem. Join the midpoints of the sides of any triangle with lines. These lines divide the triangle into four equal triangles similar to their parent. With this construction done we can easily complete a proof of the theorem that the medians of a triangle all intersect in a point two-thirds of the way along each from its vertex. Divide a long solenoid in half--it becomes two long solenoids. Immediately we see that the on-axis field at end of a long solenoid is half as great as the central field. Remember the schoolboy's algebra puzzle: if

$$x^{(x^{(x^{(\cdot^{\cdot^{\cdot}}}}} = 2$$

how great is x? To realize that the foregoing equation is equivalent to $x^2 = 2$ brings us close to the idea of self-similarity. Kepler's law that the cube of the major axis of a planetary orbit is proportional to the square of the period is a direct consequence of the invariance of the equation of motion in an inverse square field to the transformation $r'=\lambda r$, $\theta' = \theta$, $\phi' = \phi$, $t' = \lambda^{3/2}t$. Thus elliptical orbits of the same eccentricity are **dynamically self-similar:** their orbital velocities at the same angular position are in the inverse ratio of the square roots of their sizes. Finally, formulas written in terms of dimensionless numbers, like the correlations of heat transfer and fluid flow, represent self-similar families of phenomena any one of which is sufficient to specify the others by change of scale.

By this time I hope the reader is convinced of the immense utility of the concept of self-similarity. It represents a kind of inner symmetry that is widespread among the objects that interest us. That these objects should often be endowed with self-similar aspects perhaps is best thought of as part of the "wonderful gift which we neither understand nor deserve" that E. P. Wigner speaks of in his essay "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."[7]

Abbreviations Used in the Text

deq	differential equation
ode	ordinary differential equation
pde	partial differential equations
rhs	right-hand side
lhs	left-hand side
wrt	with respect to

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References

1. An excellent treatment of Lie's theory can be found in "An Introduction to the Lie Theory of One-Parameter Groups," A. Cohen, G. E. Stechert & Co., New York, 1931. All the results attributed to Lie in this paper can be found in this book.

2. "Superconducting Magnet Systems," H. Brechna, Springer-Verlag, 1973, sec. 4.1.2.

3. B. V. Reddi, S. Ray, V. Raghavan, and A. V. Narlikar, Phil. Mag. A38 (5): 559 (1978).

4. "Isoperimetric Inequalities in Mathematical Physics," G. Polya and G. Szego, Princeton University Press, Princeton, NJ, 1951.

5. "Eddy Current Heating of Irregularly Shaped Plates by Slow Ramped Fields," L. Dresner, Eighth Symposium on the Engineering Problems of Fusion Research, San Francisco, CA, Nov. 13-16, 1979, pp. 89-93, IEEE Pub. No. CH1441-5/79.

6. "Similarity, Self-Similarity, and Intermediate Asymptotics," G. I. Barenblatt, (N. Stein, translator; M. Van Dyke, translation editor), Consultants Bureau, NY, 1979.

7. E. P. Wigner, Comm. Pure Appl. Math. 13 1-14 (1960).

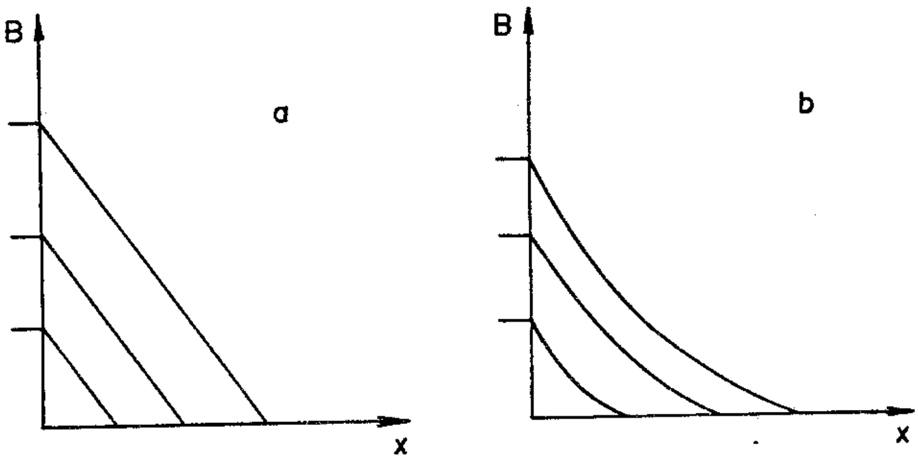

Fig. 1 Sketches illustrating self-similar profiles of magnetic induction penetrating a superconducting slab. a: field-independent J_c. b: field-dependent J_c.

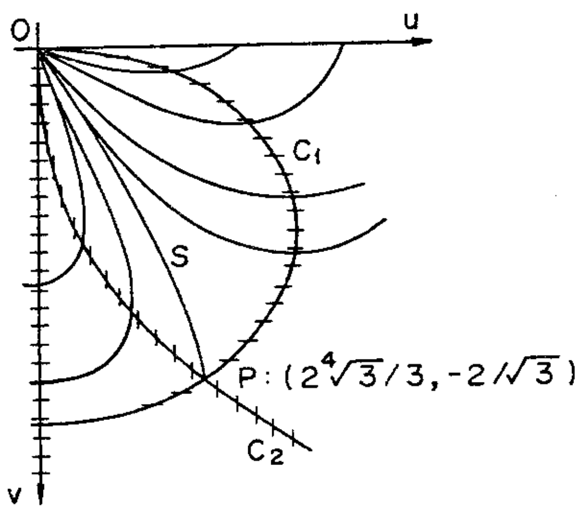

Fig. 2 The direction field of eq.(45) in the fourth quadrant.

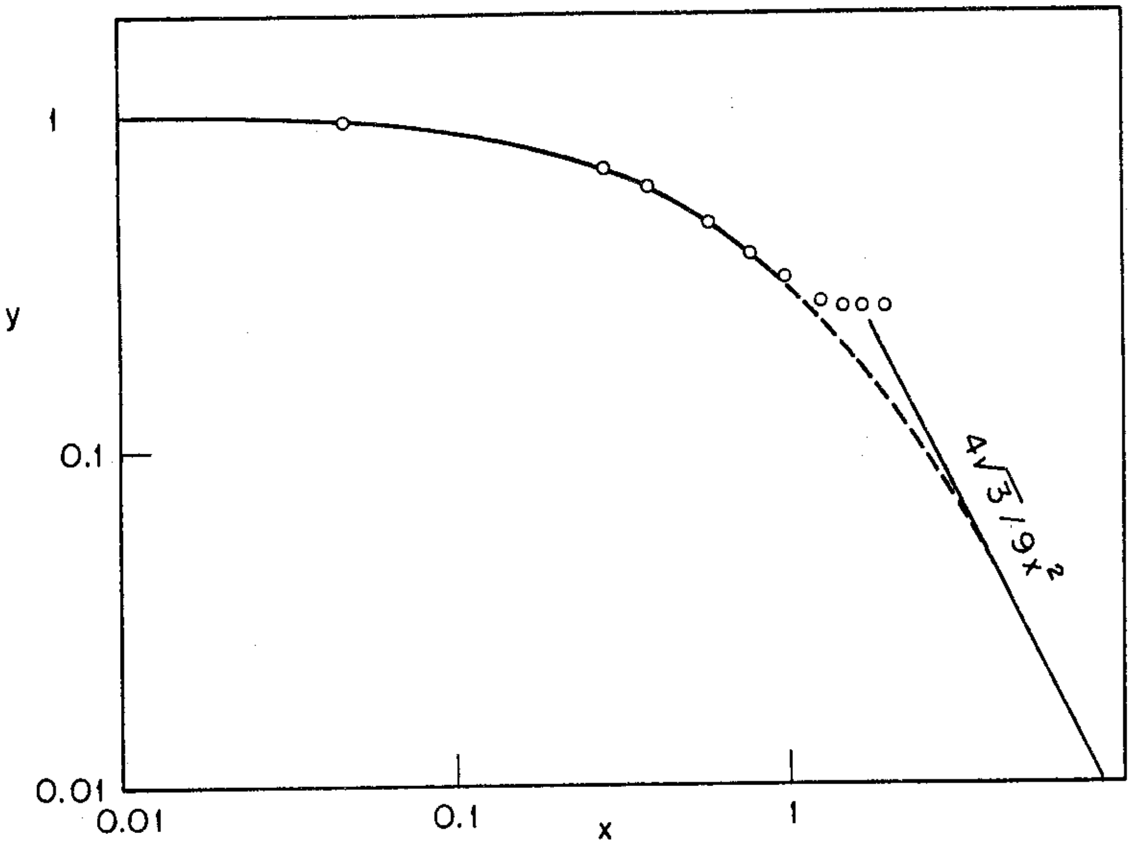

Fig. 3 The integral curve y(x) corresponding to the separatrix for which y(0) = 1.

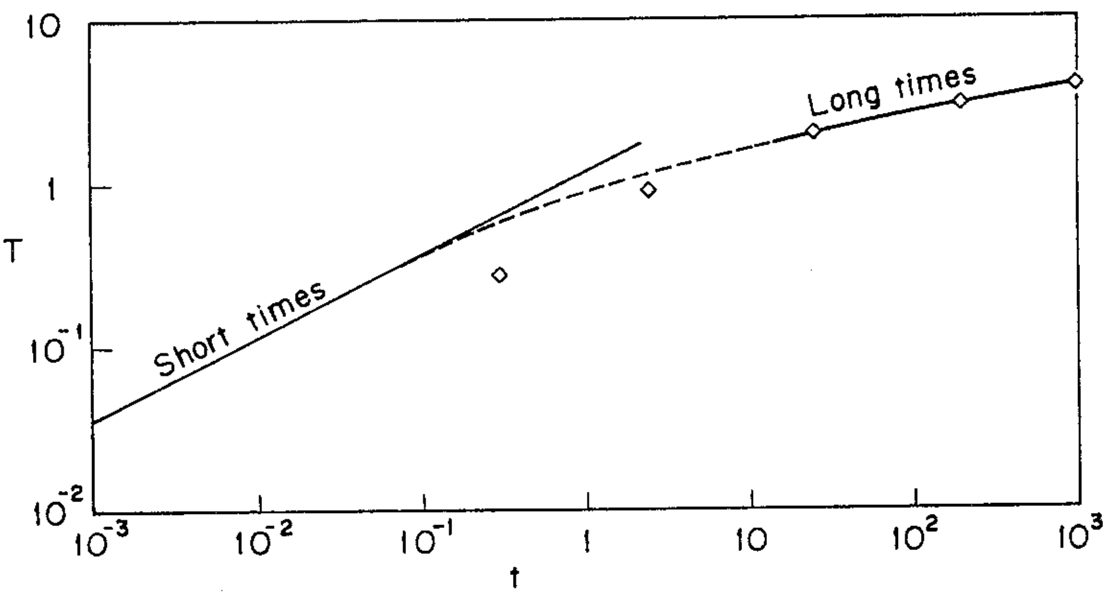

Fig. 4 The limiting curves (68) and (76) of wall temperature vs. time for constant wall flux in cylindrical geometry.

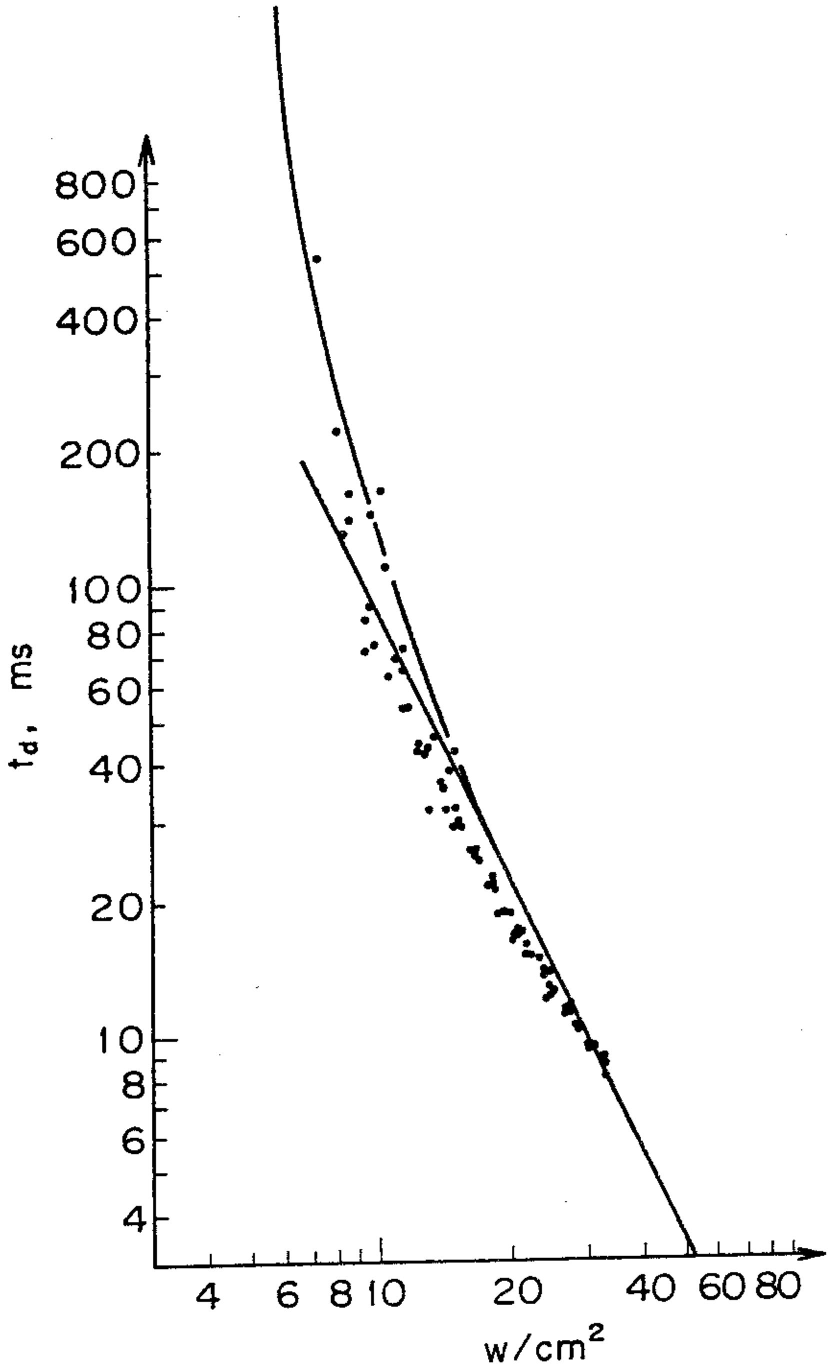

Fig. 5 Time when vapor surrounds the wire, t_d, vs. the heating power per unit surface area of the wire. From O. Tsukamoto and T. Uyemura, Adv. Cryo. Eng. 25: 475 (1980).

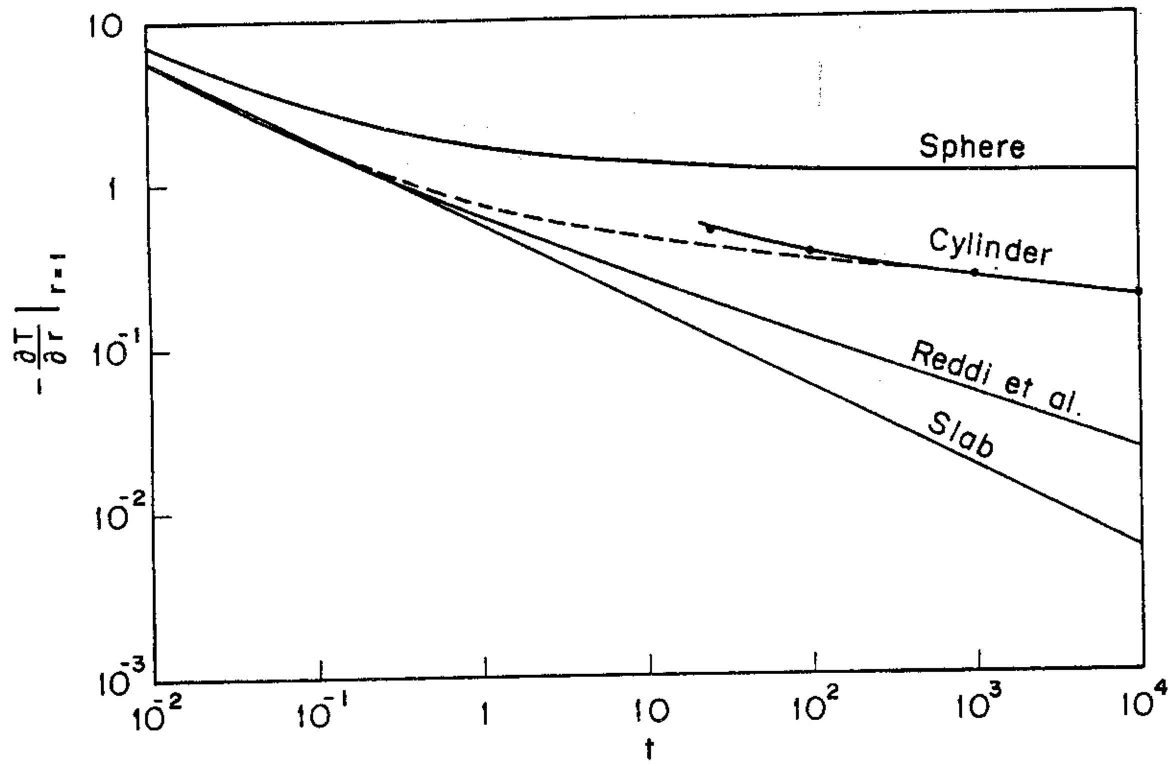

Fig. 6 The wall flux vs. time for spheres, cylinders, and slabs for constant wall temperature. The approximation of Reddi et al. (ref. 3) intended for use in cylindrical geometry is also shown.

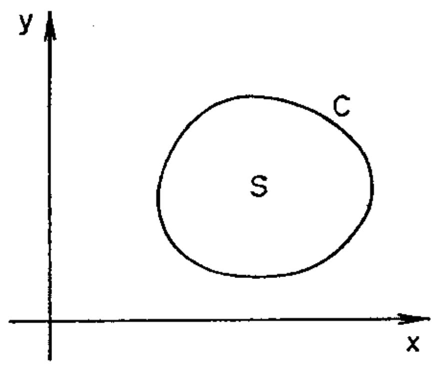

Fig. 7 Sketch of the region S bounded by the curve C.

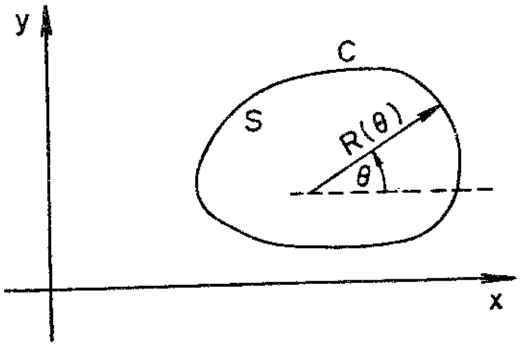

Fig. 8 Sketch showing how Polya and Szego specify the self-similar curves they assign as the approximate level curves of ϕ.

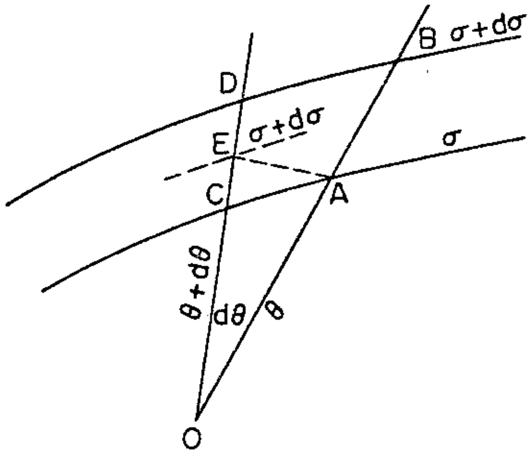

Fig. 9 Sketch to aid in the calculation of $(\nabla\phi)^2$.

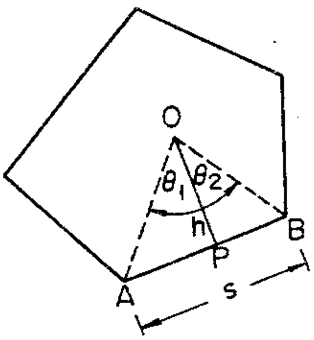

Fig. 10 Sketch aid in the evaluation of f for polygons.

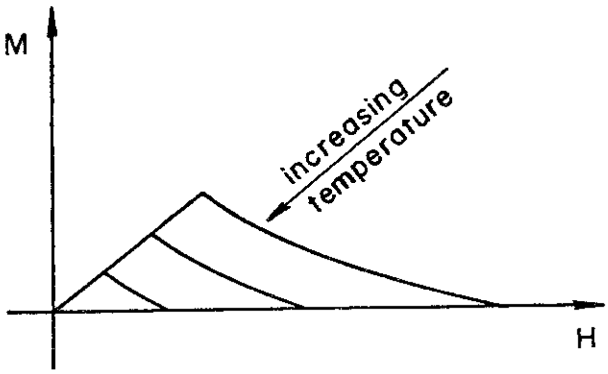

Fig. 11 Sketch of the magnetization curves of a soft type-II superconductor.

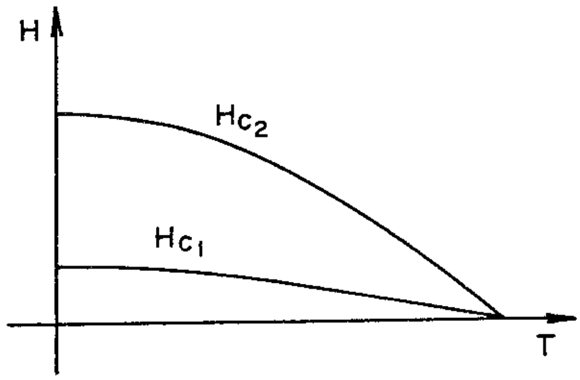

Fig. 12 Variation of the upper and lower critical fields with temperature.