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高レベル放射性廃棄物人工バリアシステムの
ウェーブレット離散化

(研究報告書)

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高レベル放射性廃棄物人工バリアシステムのウェーブレット離散化

（研究報告書）

Hesham Nasif*、梅木博之**

要　旨

高レベル放射性廃棄物処分（HLW）の核種移行モデル式に基づく非線形偏微分方程式のウェーブレットガラーキン解法のメリットについて、定性的かつ定量的な解析を行った。ウェーブレット関数は、スケール関数の拡張と移動によって作られる。また、ウェーブレット関数は、空間で局所化され、かつコンパクトにサポートされており、これらの物性は、極端に stiff な（硬い）微分方程式を解法するために利用することができる。

HLW の核種移行を予測するための数学モデルを定義し、第 2 次取りまとめのレファレンスケースとの比較を行った。本研究にて構築したモデルは、ウェーブレットで離散化したモデルであり、ウェーブレット次数やダイレーション次数の組み合わせの選択によって、効率的かつ正確な解が得られるよう工夫し、良好な結果が得られている。

なお本研究にあたっては、根山敦史氏（コンピューターソフト開発(株)）ならびに鈴木篤之教授（東京大学システム量子工学科）より多くの指導を受けた。

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Wavelet Discretization of the Engineered Barrier System

(Technical Report)

Hesham Nasif*、Hiroyuki Umeki**、

The relative merits of the wavelet-Galerkin solution of the nonlinear system of partial differential equations arising from a model formulation of migration of high-level radioactive waste (HLW) are quantitatively and qualitatively analyzed. Wavelet functions are generated by dilation and translation on a scaling function. The wavelet functions are localized in space and compactly supported, so these properties can be utilized to solve differential equations that have severe "**stiff**". A mathematical model for predicting the nuclide migration of (HLW) was formulated and compared with the results from the reference case. The model, which is wavelet-discretized model, is devised to be very reasonable and accurate by proper selection of wavelet order and dilation order pair.

We would like to thank Mr. Atsushi Neyama (Computer Software Development Co.,Ltd) and Prof. Atsuyuki Suzuki (Department of Quantum Engineering and Systems Science) for some productive discussions and their critical reviews of this study.

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1 Introduction

When discussing geological disposal in Japan, it is necessary to consider the specific geological conditions of the archipelago, which is located in a tectonically active zone. As a result, Japan has a high frequency of earthquakes, fault movement and volcanic activity. Thus, the Engineered Barrier System (EBS) is selected with a high performance margin, which reduces the requirements on the barrier functions of the geosphere. A detailed calculation is required to analyze the system showing that such natural phenomena will not affect the safety of geological disposal. The complexity of this situation makes it difficult to obtain analytical solutions for the radionuclide transport in the near field of the repository. Standard numerical methods require a detailed discretization at the interfaces between the different regions where a step radionuclide concentrations gradient exists. Therefore, nuclide release calculations would be very time consuming. Recent developments in wavelet techniques [1] have made the wavelet Galerkin procedure a viable option for the numerical solution of ordinary and partial equations with encouraging results [2,3,4].

Newly developed techniques for nuclide transport calculations uses the advantage of the wavelet discretization with a proper selection of wavelet order and dilation order pair of the system of equations that describe the nuclide migration. Radioactive decay/ingrowth, nuclide sorption in the solid, and precipitation are included in the model. The discretization of the geometrical system is very coarse and rather simple. Since the scaling functions are compactly supported, only a finite number of the connection coefficients are nonzero [1]. The resultant matrix has a block diagonal structure, which can be inverted easily using any iterative technique, which gives reasonable time, compared to conventional methods.

This work will show the applicability of wavelet Galerkin discretization to calculate the nuclide release from the nuclear waste repository based on Japanese concept. The calculations are made for various nuclides. Also, it presents a useful tool to calculate the transport of radionuclides in the near filed of the repository taking into account all the possible phenomena as diffusion, sorption and precipitation.

2 Objectives

As noted by JNC the requirements of fast and reliable code to solve the nuclide migration through the EBS system is urgently needed. There is always a need to correct the data and the available codes running time are too much to achieve the accurate solutions.

With this in mind, the objective of this work is mainly to develop a code that is fast and gives reasonable accuracy considering all the phenomena like, diffusion, precipitation, sorption, decay and ingrowth of radioactive isotopes.

3. Wavelet System Design

The term wavelets refer to sets of functions of the form

$$\psi_{ab}(x) = \mid a \mid^{-1/2} \psi\left(\frac{x-b}{a}\right) \tag{3.1}$$

Normalized by $\mid a \mid^{-1/2}$, i.e., sets of functions formed by the dilations, which are controlled by the positive real number $a \in \mathbf{R}^+$, and translations, which are controlled by the real number $b \in \mathbf{R}$, of a single function $\psi(x)$ often named the mother wavelet. Visually, the mother wavelet appears as a local oscillation, or wave, in which most of the energy of the oscillation is located in a narrow region in the physical space. This localization in the physical space limits the localization in the frequency or wavenumber domain due to the uncertainty principles. The dilation parameter a controls the width and rate of this local oscillation and intuitively can be thought of controlling the frequency of $\psi_{ab}(x)$. The translation parameter b simply moves the wavelet throughout the domain.

If the dilation and translation parameters a and b are chosen such that $a = 2^j$ and $b = 2^j k$, where j and k are integers, then there exist wavelets $\psi(x)$ such that the set of functions

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \tag{3.2}$$

Constitute an orthonormal basis of the space of functions or signals in $L^2(\mathbf{R})$, which have finite energy (Daubechies 1988 and Daubechies 1992), and as above, the two parameters j and k can be varied for analysis of local features of a given function. Note that as the wavelets is stretched by increasing j, say from $j = 0$ to $j = 1$, that the translation distance is, also accordingly increased so that a translation of size $20k$ when $j = 0$ becomes a translation of size $21k$ when $j = 1$. These two-degree of freedom, j and k give one the ability to resolve features at a variety of scales by adjusting j and at any location by adjusting k.

Wavelets are new families of basis function that yields the representation $f(x) = \sum b_{ij}\psi(2^j x - k)$. Their construction begins with the solution of the scaling function $\phi(x)$ to a dilation equation with coefficients a_k's, and then ψ comes from $\phi(x)$, and the basis comes by translation and dilation of ψ [59].

What makes the wavelet functions interesting is that, unlike sines and cosines, individual wavelet functions are quite localized in the space; simultaneously, like sines

and cosines, individual wavelet functions are quite localized in frequency or (more precisely) characteristic scale [56]. This particular kind of dual localization achieved by wavelet renders large classes of functions and operators sparse, or sparse to some high accuracy, when transformed into the wavelet domain. From this property, the wavelets are becoming a useful tool in many different areas of science and engineering.

3.1 The Fundamentals of Wavelets

We will now look more closely at the basic scaling function and wavelet to see when they exist and what their properties are. Using the same approach that is used in the theory of differential equations, we will examine the properties of $\phi(x)$ by considering the equation of which it is a solution. The basic recursion equation that comes from the multiresolution formulation is

$$\phi(x) = \sqrt{2}\sum_k a_k \phi(2x-k), \tag{3.3}$$

with a_k being the scaling coefficients and $\phi(x)$ being the scaling function which satisfies this equation which is sometimes called the *refinement equation*, the *dilation equation* or the *multiresolution analysis equation.*

In order to state the properties accurately some care has to be taken in specifying just what classes of functions are being considered or are allowed. We will attempt to walk a fine line to present enough detail to be correct but not so much as to obscure the main idea and results [60].

Refinement and transition Matrices

There are two matrices that are particularly important to determining the properties of wavelet systems. The first is the refinement matrix M, which is obtained from the basic recursion equation (3.3) by evaluating $\phi(x)$ at integers. This look like a convolution matrix with even (or odd) rows removed. For N = 6

$$\sqrt{2}\begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & 0 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & 0 & a_5 \end{bmatrix}\begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} \tag{3.4}$$

Which we write in matrix form

$$\mathbf{M}_0\phi = \phi$$

(3.5)

With $\mathbf{M}_0$ being a 6x6 matrix of the a_k and $\phi(x)$ being 6x1 vectors of integer samples of $\phi(x)$. In other words, the vector $\phi(x)$ is the eigenvector of $\mathbf{M}_0$ for an eigenvalue of unity.

The second matrix is a shifted version illustrated by

$$\sqrt{2}\begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & 0 & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} \phi_{1/2} \\ \phi_{3/2} \\ \phi_{5/2} \\ \phi_{7/2} \\ \phi_{9/2} \\ \phi_{11/2} \end{bmatrix}$$

(3.6)

With the matrix being denoted $\mathbf{M}_1$. The general refinement matrix M is the infinite matrix which $\mathbf{M}_0$ and $\mathbf{M}_1$ are partitions.

3.2 Further Properties of the Scaling Function and Wavelet

The scaling function and wavelet have some remarkable properties that should be examined in order to understand wavelet analysis and to gain some intuition for these systems. Likewise, the scaling and wavelet coefficients have important properties that should be considered.

We now look further at the properties of the scaling function and the wavelet in term of the basic definition equations and restrictions. We also consider the relationship of the scaling function and wavelet to the equation coefficients. A multiplicity or rank of two is used here and it is sufficient to our purpose, but the more general multiplicity case is easily derived from theses.

The basic recursive equation for the scaling function defined in (3.3) is homogeneous, so its solution is unique only within a normalization factor. In most cases, both the scaling function and wavelet are normalized to unity.

3.2.1 General Properties not requiring Orthogonality

There are several properties that are simply a result of the recursion equation (3.3) and, therefore, hold for orthogonal and biorthogonal systems [61].

The normalization of $\phi(x)$ is set to unity so that the basis functions are orthonormal and coefficients can easily be calculated with in ner products

$$\int \phi(x)dx = 1 \tag{3.7}$$

Not only can the scaling function be written as a weighted sum of functions in the next higher scale space as stated in the basic recursion equation (3.3) but it can also expressed in higher resolution space

$$\phi(x) = \sum_k a_k 2^{j/2} \phi(2^j x - k) \tag{3.8}$$

A formula for the sum of dyadic samples of $\phi(x)$

$$\sum_k \phi(\frac{k}{2^J}) = 2^J \tag{3.9}$$

A "partition of unity" follows from (3.9) for J=0

$$\sum_k \phi(k) = 1 \tag{3.10}$$

A generalized partition of unity exists if $\phi(x)$ is continuous

$$\sum_k \phi(x-k) = 1 \tag{3.11}$$

3.2.2 Properties that Depend on Orthogonality

The following properties depend on the Orthogonality of the scaling and wavelet functions.

The square of the integral of $\phi(x)$ *is equal to the integral of the square of* $\phi(x)$

$$\left[\int \phi(x)dx\right]^2 = \int \phi(x)^2 dx \tag{3.12}$$

The integral of wavelet is necessary zero

$$\int \psi(x)dx = 0 \tag{3.13}$$

Not only are integers translates of the wavelet orthogonal; different scales are also orthogonal

$$\int 2^{j/2}\psi(2^j x - k)2^{i/2}\psi(2^i x - l)dx = 0 \tag{3.14}$$

for all integers i,j,k,l where $i \geq j$.

3.3 Computation of the scaling coefficients

The coefficients a_k were computed by constructing a certain trigonometric polynomial $m_0(\zeta) = \frac{1}{2}\sum a_k \phi(2x-k)$, Daubechies [17]. However Strang pointed out [59] that the following relations could also solve them:

$$a_k = 0 \text{ For } k \notin \{0,1,.....,2p-1\} \tag{3.15}$$

From the normalization of the scaling function $\int \phi(x)dx = 1$, the first condition can be written as

$$\sum_k a_k = 2 \tag{3.16}$$

Smooth wavelet function requires the moment of the wavelet to be zero

$$\int x^m \psi(x)dx = 0$$

(3.17)

The formula (3.17) implies the second condition

$$\sum_k (-1)^k k^m a_k = 0 \qquad \text{For } 0 \le m \le p-1,\ (0^0 := 1)$$

(3.18)

The translation of $\phi(x)$ is required to be orthonormal

$$\int \phi(x-k)\phi(x-m)dx = \delta_{k,m}$$

(3.19)

This formula (3.17) implies the third condition

$$\sum_k a_k a_{k-2m} = 2\delta_{0m}$$

(3.20)

Here p is a given positive integer and it is wavelet order. This parameter p plays several roles in the theory of wavelets.

Note that (3.20) contributes exactly p-1 equations in the system, namely $1 \le m \le p-1$: The situation for $m \ge p$ and $m \le -p$ are consequences from (3.15); those for $1-p \le m \le -1$ are dual to the cases of $|m|$; and the case of m=0 can be derived from (3.16), the m=0 case of (3.18), and the $m \ne 0$ part of (3.20).

Therefore, we can write condition (3.15), (3.16), (3.18) and (3.20) into $2p$ equations in $\mathbf{x} = \{x_0, ..., x_{2p-1}\}^T$. Let it be a vector-valued function $F_p(\mathbf{x})$. Algebraic solutions for p=1,2 and 3 are known; but it is not likely to find those with higher p. Let $F_p(\mathbf{x})$ be the alternate system of equations. One can write down the Jacobean of F_p and perform the Newton's iteration to solve $F_p(\mathbf{x}) = 0$ (we use the roots of F_{p-1} appended with two zeros as the initial guess). In the following example a procedure of scaling coefficients calculation for p=4 is presented. The system of equations can be represented in the matrix form as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 2 & -3 & 4 & -5 & 6 & -7 \\ 1 & -1 & 4 & -9 & 16 & -25 & 36 & -49 \\ 1 & -1 & 8 & -27 & 64 & -125 & 216 & -343 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(3.21)

The initial guess is the scaling coefficients for p=3 appended with two zeros. Numerical solutions for F_p, $2 \le p \le 14$, are listed in Appendix A.

3.4 Calculating the basic scaling function and wavelet

There are two approaches that we will discuss. The first is a form of successive approximations that is used theoretically to prove existence and uniqueness of $\phi(x)$ and it can be used actually to calculate them. The second method solves for the exact values of $\phi(x)$ on the integers by solving a set of simultaneous equations. From these values, it is possible to then exactly calculate values at the half integers, then at the quarter integers and so on, giving values of $\phi(x)$ on what we are called the dyadic relations.

3.4.1 Successive Approximations or the Cascade Algorithm

In order to solve the basic recursion relation (3.3), we propose an iterative algorithm that will generate a successive approximation to $\phi(x)$. If the algorithm converges to fixed point, then that fixed point is a solution of (3.3) the iterations are defined by

$$\phi^{n+1}(x) = \sum_k a_k \phi^n(2x-k)$$

(3.22)

for the k^{th} iteration where an initial $\phi^{(0)}(x)$ must be given. Because this can be viewed as applying the same operation over and over to the output of the pervious application, it is sometimes called the cascade algorithm.

3.4.2 The Dyadic Expansion of the Scaling Function

The next method for evaluating the scaling function uses a completely different approach. It starts by calculating the values of the scaling function at integer values of x, which can be done exactly (within our ability to solve simultaneous linear equations). Consider the basic recursion equation (3.3) for integer values of $x=k$

$$\phi(x) = \sqrt{2}\sum_n a_n \phi(2k-n) \tag{3.23}$$

and assume $a_n \neq 0$ for $0 \leq n \leq N-1$.

This is a refinement matrix illustrated in (3.4) for N=6, which we write in matrix form as

$$\mathbf{M}_0 \phi = \phi \tag{3.24}$$

In other words the vector $\phi(k)$ is the eigenvector of $\mathbf{M}_0$ for the eigenvalue of unity. The simple sum of $\sum_k a_k = \sqrt{2}$ in Eq. (3.16) does not guarantee that $\mathbf{M}_0$ always has such an eigenvalue, but $\sum_k a(2k) = \sum_k a(2k+1)$ does guarantee a unity eigenvalue. This means that if $\sum_k a(2k) = \sum_k a(2k+1)$ is not satisfied, $\phi(x)$ is not defined on the dyadic relations.

Our problem now is to find the eigenvector. Note that $\phi(0) = \phi(N-1) = 0$. Therefore, the scaling function with N>2 and compact support will always be zero on the extremes of the support. This means that we can look for the eigenvector of the smaller 4 by 4 matrix obtained by eliminating the first and last rows and columns of $\mathbf{M}_0$.

From (3.24) we form $[M_0 - I]\phi = 0$ which shows that $[M_0 - I]$ is singular, meaning its rows are not independent. We remove the last row and assume the remaining rows are now independent. If that is not true, we remove another row. We next replace that row with a row of ones in order to implement the normalization equation.

This augmented matrix $[M_0 - I]$ with a row replaced by a row of ones, when multiplied by ϕ gives a vector of all zeros except for the one in the position of the replaced row. This equation should not be singular and solved for ϕ which gives $\phi(k)$, the scaling function evaluated at the integers.

From these values of $\phi(x)$ on the integers, we can find the values at the half integers using the recursive relation (3.3) or the modified form

$$\phi(k/2) = \sum_n a_n \sqrt{2}\phi(k-n)$$

(3.25)

this is illustrated with the matrix equation (3.6) as

$$M_1\phi = \phi_2$$

(3.26)

Here, the first and last columns and last row are not needed (because $\phi_0 = \phi_5 = \phi_{11/2} = 0$) and can be eliminated to save some arithmetic.

The procedure described her can be repeated to find a matrix that when multiplied by a vector of the scaling function evaluated at the odd integers divided by k will give the values at the odd integers divided by 2k. This modified matrix corresponds to convolving the sample of $\phi(x)$ by an up sampled a_n. In appendix B , graphs show scaling function and wavelet function for different wavelet order.

3.5 Vanishing Wavelet Moments

We will see in this section that the number of vanishing moments of $\psi(x)$ are related to the smoothness or differentiability of $\phi(x)$ and $\psi(x)$. Unfortunately, smoothness is difficult to determine directly because, unlike with differential equations, the defining recursion equation does not involve derivatives.

We also see that the representation and approximation of polynomials are related to the number of vanishing or minimized wavelet moments. Since polynomials are often a good model for certain functions, this property is both interesting and important. The number of zero scaling function moments is related to "goodness" of the approximation of scaling coefficients by samples of the function. They also affect the symmetry and concentration of the scaling function and wavelets.

We define the k^{th} moments of the $\phi(x)$ and $\psi(x)$ as

$$m(k) = \int x^k \phi(x) dx \tag{3.27}$$

$$m_1(k) = \int x^k \psi(x) dx \tag{3.28}$$

and the discrete k^{th} moments of $a(n)$ and $b(n)$ as

$$\mu(k) = \sum_n n^k a(n) \tag{3.29}$$

$$\mu(k) = \sum_n n^k b(n) \tag{3.30}$$

The partial moments of a_k are defined as

$$\nu(k,l) = \sum_n (2n+l)^k a(2n+l) \tag{3.31}$$

Note that $\mu(k) = \nu(k,0) + \nu(k,1)$

From these equations and the basic recursion equation we obtain

$$m(k) = \frac{1}{(2^k - 1)\sqrt{2}} \sum_{l=1}^{k} \binom{k}{l} \mu(l) m(k-l)$$

(3.32)

which can be derived by substituting (3.3) into (3.27), changing variables, and using (3.29). Similarly we obtain

$$m_1(k) = \frac{1}{2^k\sqrt{2}} \sum_{l=0}^{k} \binom{k}{l} \mu_1(l) m(k-l)$$

(3.33)

These equations exactly calculate the moments defined by the integrals in (3.27) and (3.28) with simple finite convolution of the discrete moments with the lower order continuous moments.

Requiring the moments of $\psi(x)$ to be zero has several interesting consequences. The following statements show a verity of equivalent characteristics for the K-regular scaling coefficients, which relate both to our desire for smooth scaling functions and wavelets as well as polynomial representation.

- *The scaling coefficients are K-regular if and only if the following equivalent statements are true:*

All moments of the wavelet coefficients are zero $\mu_1(k) = 0$, *for k=0,1,...,(K-1)*

All moments of the wavelets are zero $m_1(k) = 0$, *for k=0,1,...,(K-1)*

The partial moments of the scaling coefficient are equals for k=0,1,...,(K-1)

All polynomial sequence up to degree (K-1) can be expressed as a linear combination of shifted scaling coefficients.

All polynomials of degree up to (K-1) can be expressed as a linear combination of shifted functions at any scale.

This is a very powerful result. It not only ties the number of zero moments to the regularity but also to the degree of polynomials that can be exactly represented by a sum of weighted and shifted scaling functions.

- *If $\psi(x)$ are K-times differentiable and decays fast enough, then the first K-1 wavelet moments vanish; i.e.,*

$$\left| \frac{d^k}{dx^k} \psi(x) \right| < \infty, \qquad 0 \le k \le K$$

(3.34)

implies

$$m_1(k) = 0, \qquad 0 \le k \le K$$

(3.35)

Unfortunately, the converse of this sentence is not true. However, we can relate the differentiability of $\psi(x)$ to the vanishing moment by

- *There exists a finite positive integer L such that if zero* $m_1(k) = 0$, *for k=0,1,...,K- then*

$$\left| \frac{d^P}{dx^P} \psi(x) \right| < \infty$$

for LP>K.

for example, a three-times differentiable $\psi(x)$ must have three vanishing moments, but three vanishing moments results in only one-time differentiability.

4. The Evaluation of Connection Coefficients of Compactly Supported Wavelets

A connection coefficient is an integral of products of wavelet basis functions, their derivatives ad their translates. It is a functional that can be used to solve PDE's by the wavelet-Galerkin method. This chapter presents an exact method for evaluating connection coefficients. This is essential for the application of wavelets to the numerical solution of partial differential equations.

Any numerical method for solving a differential equation must be capable of approximating the derivatives and nonlinearities in the unknown function [62]. This is straightforward for Fourier-based spectral methods because the basis functions are eigenfunctions of the operator of differentiation, and because products of the basis elements are also basis elements. For compactly supported wavelets, however, this is not the case. The approximation of the derivatives and nonlinearities of a function in a wavelet basis reduces to the calculation of inner products of basis functions with products of differentiated basis functions. The required calculating integrals of the form

$$\int \phi_l^{d_1}(x)\phi_m^{d_2}(x)dx$$

For the diffusion term u_{xx} and

$$\int \phi_k^{d_1}(x)\phi_l^{d_2}(x)\phi_m^{d_3}(x)dx$$

For the nonlinear term uu_{xx}, these values (note that the superscript d_i refer to differentiation) are what we refer to as 2-term and 3-term scaling function connection coefficients. The numerical approximations of these integrals are difficult since these integrals are highly oscillatory. In fact, numerical approximation is unnecessary: as we shall show, these integrals can be explicitly evaluated in finite terms.

4.1 Definition of Terms

4.1.1 Compactly Supported Wavelets

Daubechies introduced the class of compactly supported wavelet bases in 1998[49]. They are an orthonormal basis for function in $L^2(\mathbf{R})$. A "wavelet system" consists of

the function $\phi(x)$, referred to as the scaling function and the function $\psi(x)$ referred to as the wavelet function.

By convention we define the translates of $\phi(x)$ as

$$\phi_i(x) := \phi_0(x-i) := \phi(x-i) \tag{4.1}$$

The scaling relation that defines $\phi(x)$ is

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x-k) = \sum_{k=0}^{N-1} a_k \phi_k(2x) \tag{4.2}$$

The scaling relation that define $\psi(x)$ is

$$\psi(x) = \sum_{k=-1}^{N-2} (-1)^k a_{k+1} \phi(2x+k) \tag{4.3}$$

From the scaling relation (4.2) above one sees that $\phi(x)$ is equal to a sum of scaled and shifted versions of itself. The scaling factor in this equation is 2 and hence we refer to $\phi(x)$ and $\psi(x)$ of this form as multiplier 2 systems. One can generalize wavelet systems to an arbitrary nonnegative integer multiplier. For higher order multipliers there are multiple function $\psi(x)$ with different sets of a_k coefficients [63]. For the purpose of this work we restrict ourselves to the multiplier 2 case with real valued a_k's.

The wavelet expansion of a function $f : \mathbf{R} \to \mathbf{C}$ is of the form

$$f(x) = \sum_{l \in Z} c_l \phi_l(x) + \sum_{j=0}^{\infty} \sum_{k \in Z} c_{jk} \psi_{jk}(x)$$

The indices k and j represent translation and scaling respectively.

$$\phi_{jl}(x) := 2^{j/2} \phi(2^j x - l) \quad , \quad \phi_l(x) := \phi_{0l}(x) \tag{4.4}$$

$$\psi_{jk}(x) := 2^{j/2} \phi(2^j x - k) \tag{4.5}$$

If $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only.

$$f(x) = \sum_{l \in Z} c_{Jl} \phi_{Jl}(x) \tag{4.6}$$

This simple but important relation can be found by repeated application of the scaling relation. Therefore, a finite wavelet expansion of $f(x)$ can be written solely in terms of translated scaling function [64]. For ease of notation we remove the J's from our equations and assume that the $\phi(x)$'s are all at the finest level of resolution.

The scaling functions we have used are those of Daubechies. In her work, Daubechies found and exploited the link between vanishing moments of the wavelet $\psi(x)$ and scaling function $\phi(x)$. We say that the wavelet $\psi(x)$ has **K** vanishing moments if

$$\int x^k \psi(x) dx = 0 \quad \text{for } 0 \leq k \leq K$$

A necessary and sufficient condition for this to hold is that integer translates of the scaling function $\phi(x)$ perfectly interpolate polynomials of degree up to **K**, that is, for each k, $0 \leq k \leq K$ there exist constants c_l such that

$$x^k = \sum c_l \phi_l(x)$$

Daubechies [49] introduced scaling function satisfying this property and distinguished by having the shortest possible support. The scaling function *DN* (where *N* is an even integer) will have support [0,*N*-1] and (*N*/2-1) vanishing wavelet moments. In [49] and with a refined analysis in [17] Daubechies showed that there exists $\lambda > 0$ such that *DN* has $\lambda(N/2-1)$ continuous derivatives, for small N, $\lambda \geq 0.55$.

4.1.2 Derivatives and Connection Coefficients

In order to solve PDE's we will need to evaluate the derivative of $f(x)$ in terms of $\phi(x)$. We first define the following shorthand for differentiation of a function:

$$\phi_l^n := \frac{d^n \phi_l(x)}{dx^n}$$

(4.7)

In general a superscript will represent differentiation. From equation (4.6) we derive the Galerkin approximation of a derivative of $f(x)$ as

$$f^d(x) = \sum_l c_l \phi_l^d(x)$$

(4.8)

Now $\phi_l^d(x)$ can be approximated in terms of $\phi(x)$

$$\phi_l^d(x) = \sum_m \lambda_m \phi_m(x)$$

(4.9)

where

$$\lambda_m = \int_{-\infty}^{\infty} \phi_l^d(x) \phi_m(x) dx$$

(4.10)

The coefficient λ_m is a 2-term connection coefficient, which in most generals from is defined as

$$\Lambda_{l_1 l_2}^{d_1 d_2} := \int_{-\infty}^{\infty} \phi_{l_1}^{d_1}(x) \phi_{l_2}^{d_2}(x)$$

(4.11)

There is an analogous connection coefficient for the function $\psi(x)$ but by use of equation (4.6) there is no necessity for calculating these values. It is however, possible to derive a connection coefficient which is an integral of products of differentiated $\phi(x)$'s and $\psi(x)$'s from just the scaling function connection coefficients. For the rest of this chapter the reference to connection coefficient will always imply scaling function connection coefficient. In general a connection coefficient is a coefficient in the Galerkin expansion (approximation) of a function of the form

$$f(x) := \prod_{i=1}^{n} \phi_{l_i}^{d_i}$$

(4.12)

The resulting connection coefficient is

$$\int_{-\infty}^{\infty} f(x)\phi_l(x)dx$$

(4.13)

In the most general case we allow ϕ_l to be differentiated which gives rise to the n-term connection coefficient:

$$\Lambda(l_1, l_2, ..., l_n, d_1, d_2, ..., d_n) := \Lambda_{l_1 l_2 ..... l_n}^{d_1 d_2 ... d_n} := \int_{-\infty}^{\infty} \prod_{i=1}^{n} \phi_{l_i}^{d_i}(x)dx$$

(4.14)

4.2 Outline of the General Method

4.2.1 Statement of Problem

In this chapter we will show how to evaluate integrals from equation (4.14) for n=2,3. Using the change of variables $u = x - l$, we can alter a doubly subscripted connection coefficient into a singly subscripted one, and triply subscripted connection coefficients into a doubly subscripted one. This transformation is useful because it simplifies the generation of the linear system of equations. We therefore define the two and three term connection coefficients as

$$\Lambda_l^{d_1 d_2} := \int_{-\infty}^{\infty} \phi^{d_1}(x)\phi_l^{d_2}(x)dx$$

(4.15) and,

$$\Lambda_{lm}^{d_1 d_2 d_3} := \int_{-\infty}^{\infty} \phi^{d_1}(x)\phi_l^{d_2}(x)\phi_m^{d_3}(x)dx$$

(4.16)

Where $d_i \geq 0$.

The method is general technique for evaluating integrals of products of arbitrary derivatives of the scaling function. The method for evaluating thee 3-term case will be presented in full and the appropriate equations for the 2-term case will be provided.

4.2.2 Consequences of the Scaling Relation and Moment Properties

Let us assume that the scaling function can be differentiated as many times as required for the following operations to be performed. Successive differentiation of the scaling relation and substitution for the integrand in equation (4.14) yields a system of homogeneous linear equations that the connection coefficients must satisfy. It can be shown that the solution space (eigenspace) of the homogeneous linear equations has some positive multiplicity, which depends on the number of terms in the connection coefficients.

In order to uniquely specify a solution (connection coefficient) we develop a set of linearly independent equations equal to the dimension of the solution space. One condition will be inhomogeneous equation that fixes the normalization. We expand the monomial x^k using scaling function and wavelets, and take the inner product of both sides with the scaling function $\phi(x)$. This yields an equation relating a moment of $\phi(x)$ to the connection coefficients. We repeat this process with different values of k until the solution is uniquely specified.

4.3 3-Term Connection Coefficients

4.3.1 The Scaling Equations

When N in equation (4.2) is a finite even positive integer the function $\phi(x)$ has compact support contained in [0,N-1]. For a fixed triple (d_1, d_2, d_3), only finitely many of the $\Lambda_{lm}^{d_1 d_2 d_3}$ are nonzero, namely those for which

$$2-N \le l \le N-2, \qquad 2-N \le m \le N-2, \quad \text{and} \quad |l-m| \le N-2$$

There are $M = 3N^2 - 9N + 7$ such pairs (l, m). We shall denote this set of M pairs (l, m) by S. Select a fixed but arbitrary bisection of the set S onto the integers $\{1, \ldots\ldots, M\}$ and let $\Lambda^{d_1 d_2 d_3}$ be an M-vector whose components are the numbers $\Lambda_{lm}^{d_1 d_2 d_3}$.

The strategy is to solve for the vector $\Lambda^{d_1 d_2 d_3}$ by creating a system of M or more linear equations that have the $\Lambda_{lm}^{d_1 d_2 d_3}$ as their unique solution. We assume that the triple (d_1, d_2, d_3) is fixed and we derive two classes of linear and affined relations for the vector $\Lambda^{d_1 d_2 d_3}$. Let us assume that $\phi(x)$ is d-times differentiable. Differentiating the scaling relation d times, we obtain

$$\phi^d(x) = 2^d \sum_{k=0}^{N-1} a_k \phi_k^d(2x)$$

Substituting the right side of the scaling relation for $\phi(x)$ into the definition of the 3-term connection coefficients $\Lambda_{lm}^{d_1 d_2 d_3}$ yields

$$\begin{aligned}
\Lambda_{lm}^{d_1 d_2 d_3} &= 2^{d_1+d_2+d_3} \int_{-\infty}^{\infty} \sum_p a_p \phi_p^{d_1}(2x) \sum_q a_q \phi_{2l+q}^{d_2}(2x) \sum_r a_r \phi_{2m+r}^{d_3}(2x) dx \\
&= 2^{d_1+d_2+d_3} \sum_{p,q,r} a_p a_q a_r \int_{-\infty}^{\infty} \phi_p^{d_1}(2x) \phi_{2l+q}^{d_2}(2x) \phi_{2m+r}^{d_3}(2x) dx \\
&= 2^{d_1+d_2+d_3} \sum_{p,q,r} a_p a_{q-2l+p} a_{r-2m+p} \int_{-\infty}^{\infty} \phi^{d_1}(x) \phi_q^{d_2}(x) \phi_r^{d_3}(x) dx
\end{aligned}$$

After simplification we find

$$A\Lambda^{d_1 d_2 d_3} = \frac{1}{2^{d-1}} \Lambda^{d_1 d_2 d_3}$$

where

$$d := d_1 + d_2 + d_3$$

and

$$A_{l,m:q,r} := \sum_p a_p a_{q-2l+p} a_{r-2m+p}$$

We refer to these equations as the *Scaling Equations*.

It is clear that the scaling equations cannot have a unique nonzero solution since they are homogeneous. The set of vectors, which satisfy the scaling equations from a vector space of nonzero dimension. By construction, the connection coefficients are solutions of the system and are in this vector space. We examine the number of linearly independent connection coefficients in this space and later show how to uniquely specify particular connection coefficients.

Since the coefficients in the scaling equations only depend on $d = d_1 + d_2 + d_3$ and not otherwise on the individual derivatives $d_1, d_2,$ and d_3 the vector $\Lambda^{c_1 c_2 c_3}$ will be the solution of the linear system of scaling equations whenever $c_1 + c_2 + c_3 = d$.

The vectors $\Lambda^{c_1 c_2 c_3}$ are not all linearly independent. For all l and m, the function $\phi^{c_1 - 1}(x)\phi_l^{c_2}(x)\phi_m^{c_3}(x)$ has compact support, so we have the relation

$$0 = \int_{-\infty}^{\infty} (\phi^{c_1 - 1}(x)\phi_l^{c_2}(x)\phi_m^{c_3}(x))' dx$$

$$= \int_{-\infty}^{\infty} (\phi^{c_1}(x)\phi_l^{c_2}(x)\phi_m^{c_3}(x) + \phi^{c_1 - 1}(x)\phi_l^{c_2 + 1}(x)\phi_m^{c_3}(x) + \phi^{c_1 - 1}(x)\phi_l^{c_2}(x)\phi_m^{c_3 + 1}(x))dx$$

Whence

$$\Lambda^{c_1, c_2, c_3} + \Lambda^{c_1 - 1, c_2 + 1, c_3} + \Lambda^{c_1 - 1, c_2, c_3 + 1} = 0$$

This allows us to express the vector $\Lambda^{c_1 c_2 c_3}$ as a linear combination of other vectors $\Lambda^{b_1 b_2 b_3}$ with $b_1 < c_1$ and $b_1 + b_2 + b_3 = c_1 + c_2 + c_3 = d_1 + d_2 + d_3 = d$. Repeating this process, we see that the vector $\Lambda^{c_1 c_2 c_3}$ is a linear combination of vector form Λ^{0, b_2, b_3}. In fact, we can see this explicitly, via the formula

$$\Lambda^{c_1c_2c_3} = (-1)^{c_1} \sum_{i=0}^{c_1} \binom{c_1}{i} \Lambda^{0,c_2+i,c_3+c_1-i}$$

We show in section 4.3 that when the set of vectors $\{\Lambda^{0,b,d-b} \mid b = 0,1,\ldots\ldots,d\}$ exist, they are linearly independent. Intermediate results will not depend on this fact, and so without circular logic we have

The space spanned by the vectors

$$\{\Lambda^{c_1c_2c_3} \mid c_1 + c_2 + c_3 = d\}$$

is spanned by the $(d+1)$ *linearly independent vectors*

$$\{\Lambda^{0,b,d-b} \mid b = 0,1,\ldots\ldots,d\}$$

In order to uniquely determine the expressions $\Lambda^{0,b,d-b}$, we need to find at least $(d+1)$ additional independent conditions or equations which are still independent when restricted to the $(d+1)$ dimensional space spanned by the $\Lambda^{0,b,d-b}$. We will obtain d independent homogeneous linear equations and one inhomogeneous linear normalization equation involving the moments of $\phi(x)$. These will complete the system of equations and lead to a unique determination of the three term connection coefficients when the dimension of the solution space is d+1.

4.3.2 Moment Equations

Let K be the largest integer k such that x^k can be represented exactly by translations of the scaling function. That is, if, $0 \le k \le K$, then there exist c_l such that

$$x^k = \sum c_l \phi_l(x) \qquad (4.17)$$

The expansion coefficients c_l will be given by

$$c_l = M_l^k := \mathrm{Mom}(\phi_l(x)) = \int_{-\infty}^{\infty} x^k \phi_l(x)dx$$

(4.18)

It follows that the dimension of the solution space of the linear system of scaling equations is at least (d+1) when $d \le K$. Differentiating the scaling function expansion (4.17) k times, we obtain

$$k! = \sum_i M_i^k \phi_i^k(x)$$

Further differentiation yields

$$0 = \sum_i M_i^k \phi_i^k(x) \quad \text{whenever} \quad j > k$$

Multiplying both sides of the same equation by $\phi^{d_1}(x)\phi_m^{d_3}(x)$ and integrating over R, we obtain the relation

$$\sum_l M_l^k \Lambda_{lm}^{d_1 d_2 d_3} = \begin{cases} k! \int_{-\infty}^{\infty} \phi^{d_1}(x)\phi_m^{d_3}(x)dx & \text{for } k = d_2, \\ 0 & \text{for } k < d_2 \end{cases}$$

(4.19)

Similarly, multiplying both sides of this equation by $\phi^{d_1}(x)\phi_l^{d2}(x)$ and integrating over R, we obtain the relation

$$\sum_l M_m^k \Lambda_{lm}^{d_1 d_2 d_3} = \begin{cases} k! \int_{-\infty}^{\infty} \phi^{d_1}(x)\phi_l^{d_2}(x)dx & \text{for } k = d_3, \\ 0 & \text{for } k < d_3 \end{cases}$$

(4.20)

We refer to these equations (4.19) and (4.20) as the *moment equations*.

4.3.3 Calculation of Moments

In order to calculate the values of the scaling integrals $\Lambda_{l_1 l_2 \ldots l_n}^{d_1 d_2 \ldots d_n}$, it is first necessary to calculate the values of the moments of $\phi(x)$ and its translates,

$$M_i^k = \int_{-\infty}^{\infty} x^k \phi(x-i)dx$$

The techniques used to calculate these moments are a simpler version of the techniques used to calculate the scaling integrals themselves. In both cases, the techniques rely on the scaling relation to derive sufficient linear and affined relations among the quantities being calculated to determine them uniquely. This allows the rapid calculation of the desired quantities, as well as showing that they are relational functions of the scaling coefficients.

By definition, the scaling function $\phi(x)$ is normalized so that $M_0^0 = 1$, and a change of variables shows that $M_i^0 = 0$ for all i. We calculate the moments M_i^k by induction on k, so we assume that M_i^j has already been calculated for all i for all $j<k$.

The scaling relation yields the formula

$$M_0^j = \int_{-\infty}^{\infty} x^j \phi(x)dx = \int_{-\infty}^{\infty} x^j \sum_{i=0}^{N-1} a_i \phi(2x-i)dx$$

$$= 2^{-j-1} \sum_{i=0}^{N-1} a_i \int_{-\infty}^{\infty} (2x)^j \phi(2x-i)d(2x) = 2^{-j-1} \left(\sum_{i=0}^{N-1} a_i M_i^j \right)$$

To reduce the number of unknowns in this equation, we will eliminate M_i^j for $i \neq 0$.

The substitution $u = x - i$ implies

$$\begin{aligned} M_i^j &= \int_{-\infty}^{\infty} x^j \phi(x-i)dx \\ &= \int_{-\infty}^{\infty} (u+i)^j \phi(u)du \\ &= \sum_{k=0}^{j} \binom{i}{j} i^{j-k} \int_{-\infty}^{\infty} u^k \phi(u)du \\ &= \sum_{k=0}^{j} \binom{i}{j} i^{j-k} M_0^k \end{aligned}$$

Substituting this into the pervious equation, we obtain

$$
\begin{aligned}
M_0^j &= 2^{-j-1}\left(\sum_{i=0}^{N-1} a_i M_i^j\right) \\
&= 2^{-j-1}\left(\sum_{i=0}^{N-1} a_i \sum_{k=0}^{j}\binom{j}{k} i^{j-k} M_0^k\right) \\
&= 2^{-j-1}\sum_{k=0}^{j-1}\binom{j}{k} M_0^k\left(\sum_{i=0}^{N-1} a_i i^{j-k}\right) + 2^{-j-1} M_0^j\left(\sum_{i=0}^{N-1} a_i\right) \\
&= 2^{-j-1}\sum_{k=0}^{j-1}\binom{j}{k} M_0^k\left(\sum_{i=0}^{N-1} a_i i^{j-k}\right) + 2^{-j} M_0^j
\end{aligned}
$$

Solving for M_0^j we find the formula

$$
M_0^j = \frac{1}{2(2^j-1)}\sum_{k=0}^{j-1}\binom{j}{k} M_0^k\left(a_i i^{j-k}\right)
$$

Now that the jth moment of $\phi(x)$, M_0^j, has been determined, the moments M_i^j of the translates of $\phi(x)$ can be obtained using the formula derived above:

$$
M_i^j = \sum_{k=0}^{j}\binom{j}{k} i^{j-k} M_0^k
$$

This results in the explicit formula

$$
M_i^j = \frac{1}{2(2^j-1)}\sum_{k=0}^{j}\binom{j}{k} i^{j-k}\sum_{l=0}^{k-1}\binom{k}{l} M_0^l\left(\sum_{i=0}^{N-1} a_i i^{k-l}\right)
$$

We have just expressed the moments of the scaling function in terms of the "discrete" moments of the scaling coefficients.

4.3.4 Reduction by Moment Equations

We now show how to generate additional equations from the moment equations such that the new equations, the scaling equations, and the moment equations will all be linearly independent. Now

$$\sum_l M_l^0 \Lambda_{lm}^{0,0,d_3} = \int_{-\infty}^{\infty} \phi(x)\phi_m^{d_3}(x)dx$$

(4.21)

and we choose m such that

$$\int_{-\infty}^{\infty} \phi(x)\phi_m^{d_3}(x)dx \neq 0$$

then the homogeneous moment equation

$$\sum_l M_l^0 \Lambda_{lm}^{0,d_2,d_3} = 0$$

(4.22)

is not satisfied by $\Lambda^{0,0,d}$, though it is satisfied by $\Lambda^{0,b,d-b}$ for b=1,2,.....,d. The choice of m is determined by the fact that the values of $\int_{-\infty}^{\infty} \phi(x)\phi_m^{d_3}(x)dx$ are symmetric when d_3 is even and unsymmetric when d_3 is odd. The integral attains its maximum value when m=0(d_3 even), and m=1 (d_3 odd), so it is numerically advantageous to choose this value of m to make the computation better posed.

After adjoining the moment equation (4.21) to the scaling equations the solution space of the system is spanned by the vectors

$$\left\{\Lambda^{0,b,d-b} \mid b = 0,1,.....,d\right\}$$

and hence had dimension d. While $\Lambda^{0,0,d}$ is a solution to the homogeneous scaling equations, it yields a nonzero right hand side for the equation (4.21). Thus the moment equation (4.21) is linearly independent with respect to the scaling equation.

Similarly, the homogeneous moment equation

$$\sum_l M_l^1 \Lambda_{lm}^{0,d_2,d_3} = 0$$

is satisfied by $\Lambda^{0,b,d-b}$ for any b>1 but not by $\Lambda^{0,b1,d-1}$. We do not know whether this equation is satisfied by $\Lambda^{0,0,d}$, but this is unimportant. Since this information is

not needed to conclude that, the vectors span the space of vectors satisfying the moment equations as well as the scaling equations:

$$\left\{\Lambda^{0,b,d-b} \mid b = 2,3,\ldots\ldots,d\right\}$$

Proceeding inductively, we see that the space of vectors satisfying the scaling equations as well as the d_2 homogeneous moment equations.

$$\sum_l M_l^b \Lambda_{lm}^{0,d_2,d_3} = 0$$

where b varies from 0 to $d_2 - 1$ inclusive, and m is as described above, is spanned by the vectors

$$\left\{\Lambda^{0,b,d-b} \mid d_2 \le b \le d\right\}$$

The strategy we have employed is to adjoin additional moment equations that reduce the rank of the solution space by guaranteeing that they are not a linear combination of the scaling equations and the previously adjoined moment equations. So far we have reduced the rank deficiency of the system by d_2. Now we pick the appropriate equations to reduce it by d_3. If we also adjoin the homogeneous moment equations

$$\sum_m M_m^c A_{lm}^{0,d_2,d_3} = 0$$

where c varies from 0 to $d_3 - 1$ inclusive and l is chosen so that

$$\int_{-\infty}^{\infty} \phi(x)\phi_l^{d_2}(x)dx \neq 0$$

(for example, choosing l=0 if d_2 is even, and l=1 if d_2 is odd), then these additional equations eliminate the vectors $\Lambda^{0,b,d-b}$ with b> d_2 from the solution space, leaving a system with a one-dimensional solution space, all of those solutions are scalar multiples of $\Lambda^{0,d_2,d-d_2}$.

It remains to show, that when the vectors $\Lambda^{0,b,d-b}$ for b=0,1,....,d exist, they are linearly independent. Let us demonstrate this now. If we assume that there exist constants c_b such that

$$\Omega_{l,m}^{d} = \sum_{b=0}^{d} c_b \Lambda_{lm}^{0,b,d-b} = 0$$

then

$c_b = 0$

for $b = 0,1,.....,d$

For instance, examine

$$\sum_{l} M_l^0 \Omega_{l,m}^{d} = \sum_{l,b} c_b M_l^0 \Lambda_{lm}^{0,b,d-b} = 0$$

We have seen from section 4.2 that

$$\sum_{l} M_l^0 \Lambda_{lm}^{0,b,d-b} \neq 0$$

when b=0 and that

$$\sum_{l} M_l^0 \Lambda_{lm}^{0,b,d-b} = 0$$

when b>0. Therefore, in order that the summation over Ω_{lm}^{d} to be 0 we must have $c_0 = 0$. In a similar fashion one can show that for $k \leq K$

$$\sum_{l} M_l^k \Omega_{l,m}^{d} = 0$$

implies $c_k = 0$.

The above also demonstrates that equation (4.19) and equation (4.20) are independent of the scaling relations when they are restricted to the space

$$\left\{ \Lambda^{0,b,d-b} \mid b = 0,1,.....,d \right\}$$

That is, for each vector there is at least one moment equation that does not vanish when applied to the vector (is inhomogeneous).

4.3.5 Normalization of the Solution

To distinguish the desired solution vector $\Lambda^{0,d_2,d-d_2}$ from its scalar multiples, it suffices to find any single inhomogeneous equation satisfied by $\Lambda^{0,d_2,d-d_2}$, which by virtue of its inhomogeneity, will not be satisfied by any scalar multiple of $\Lambda^{0,d_2,d-d_2}$.

The following describes a method that uses moment equations to create a linearly independent inhomogeneous equation. Note that we will generalize the method to an n-term connection coefficient.

Assume that $d_1 = 0$, and that we seek to calculate the values of the connection coefficients $\Lambda_{l_1 l_2 \ldots . l_n}^{d_1 d_2 \ldots d_n}$ for fixed values of n and the $\{d_i\}$. We obtain such an equation in the following way. Assuming that $\psi(x)$ has at least d_i vanishing moments, we have the equation

$$x^{d_i} = \sum_j M_j^{d_i} \phi_j(x)$$

Differentiating d_1 times, we obtain

$$(d_i)! = \sum_j M_j^{d_i} \phi_j^{d_i}(x)$$

Taking this equation for each value of I from 2 through n, and multiplying them together, we obtain

$$\prod_{i=2}^{n} (d_i)! = \sum_{l_2, l_3, \ldots . l_n} \left(\prod_{i=2}^{n} M_{l_i}^{d_i} \right) \phi_{l_2}^{d_2}(x) \phi_{l_3}^{d_3}(x) \ldots\ldots\ldots \phi_{l_n}^{d_n}(x)$$

If we multiply both sides by $\phi(x)$, integrate, and use the fact that $\int \phi(x) dx = 1$, we obtain

$$\prod_{i=2}^{n} (d_i)! = \sum_{l_2, l_3, \ldots . l_n} \left(\prod_{i=2}^{n} M_{l_i}^{d_i} \right) \Lambda_{0, l_2, l_3, \ldots . l_n}^{0, d_2, d_3, \ldots . d_n}$$

The summation is extended over all values of the $\{l_i\}$, such that $(0,l_2,l_3,....,l_{nh}) \in S$. All other values of the l_i can be neglected, since they make no contribution to the value of the integral. The specific equation for the 3-term connection coefficient case is

$$d_2 d_3 != \sum_{lm} M_l^{d_2} M_m^{d_3} \Lambda_{lm}^{0,d_2,d_3}$$

The addition of this equation to the scaling equations and the d homogeneous moment equations described above yields a linear system with the desired unique solution $\Lambda^{0,d_2,d-d_2}$.

4.4 2-Term Connection Coefficient Equations

In this section we present the equations for the construction of the 2-term connection coefficients. In general they are a simplification of the 3-tem equations. For the 2-term case, however, the dimension of the solution space, when it exists, is always equal to 1. This can easily be seen from the fact that the integration by parts of a 2-term product produces an equation of the form

$$\Lambda_l^{d_1 d_2} = -\Lambda_l^{d_1-1,d_2+1}$$

We therefore only need one inhomogeneous independent equation to normalize the system of equations. For this we use a simple moment equation

From the derivation of the 3-term case one can see that

$$A\Lambda^{d_1 d_2} = \frac{1}{2^{d-1}} \Lambda^{d_1 d_2}$$

where

$$d := d_1 + d_2$$

and

$$A_{l;q} = \sum_p a_p a_{q-2l+p}$$

From the integration by parts we can always transform $\Lambda^{d_1 d_2}$ into $\Lambda^{0,d}$ where $d = d_1 + d_2$ and $\Lambda^{0,d}$ exists. This allows us to use the scaling function expansion of

x^k as an independent inhomogeneous equation. From the derivation of the 3-term case one can see that

$$d! = (-1)^d \sum_l M_l^d \Lambda_l^{0,d}$$

4.5. Solution and Numerical of the Connection Coefficients

We have created a system of $M+d+1$ equations in M unknowns. It has rank M, and therefore, a unique solution. The simplest way to solve our system of equations is to choose a subset of M of the equations that has a unique solution, and then solve the resulting square system, by means of LU decomposition. We introduce the solutions that we obtained in Appendix A.

5. Wavelet Discretization of the Engineered Barrier System (EBS)

The nuclide migration analysis for the EBS considers:

- Nuclide dissolution from the vitrified waste;
- Diffusion and sorption in the buffer material;
- Radioactive decay and ingrowth;
- Release into the surrounding rock.

In this section the modeling assumptions and corresponding mathematical models and data for the EBS in the reference case are described.

5.1 Nuclide migration analysis for the EBS

5.1.1 conceptual model

The following specific assumption are made for the conceptual model:

- The overpack is designed to have a minimum lifetime of 1,000 years. Subsequent to the first 1,000 years, it will fail mechanically due to decreased strength caused by corrosion;
- At the time of overpack failure, the heat generated by the vitrified waste will have decayed to a negligible level and the temperature in the repository will approach that of the surrounding host rock. The buffer material will completely water saturated, providing homogeneous low permeability barrier around the overpack;
- The surface area of the vitrified waste after disposal will exceed its original geometric surface area due to cracking during cooling in the vitrification process and fragmentation due to corrosion expansion of the overpack;
- After the overpack fails, the porewater in the buffer material will come into contact with the vitrified waste. This will cause dissolution of the glass at long-term dissolution rate. Nuclide will be released from the vitrified waste as the glass matrix dissolves. The rate of glass matrix dissolution defines the release rate of nuclides, which are mixed homogeneously in the vitrified waste (congruent release).
- Retardation of nuclide migration in cracks in the vitrified waste, glass decomposition products and overpack corrosion products is conservatively ignored.
- Near-field porewater chemistry is determined predominantly by chemical reactions between groundwater, buffer material and overpack corrosion products, and will be in a reducing state. Solubility is determined based on this porewater composition.
- The nuclide concentration in the vicinity of the vitrified waste and in the buffer

material will be limited by solubility that is portioned among stable and radioactive isotopes in the waste. Dissolution and precipitation of nuclides occur much faster than their (diffusive) transport and achieve local instantaneous equilibrium. Precipitated solid phases may re-dissolve later to maintain saturation. Dilution by isotopes derived from the groundwater or the buffer material is conservatively ignored.

- Nuclides are transported through the buffer material by diffusion and are retarded by linear, reversible and instantaneous sorption onto the Bentonite.
- Colloid migration will be prevented by filtration in the microporous structure of the buffer. Hydrogen generated by corrosion of the overpack, radiolysis or microbial degradation of organic materials will not affect nuclide migration.
- Nuclides released from the buffer are mixed instantly and completely with groundwater flowing through EDZ. Sorption of nuclides in the EDZ is not considered and all nuclides are transported to fracture in the host rock.
- Nuclides leached from upstream water package will increase the nuclide concentrations in the vicinity of downstream packages. This will decrease nuclide release rates from downstream packages because of the lower concentration gradient in the buffer material. This effect is conservatively ignored.

5.1.2.Mathematical Formulation

The mathematical models for the nuclide migration in the EBS are developed based on the above assumptions. The following additional assumptions have been included for mathematical formulation:

- The decrease of the glass surface area with the time due to dissolution is conservatively ignored.
- All the overpacks are assumed to fail simultaneously 1,000 year after disposal and then present no transport resistance.
- Nuclides in the vitrified waste dissolve homogeneously into a hypothetical water-
- filled region around the vitrified waste. Retardation of nuclide migration in cracks in the vitrified waste is ignored.
- Diffusion in the buffer material is modeled using a cylindrical radial co-ordinate system. The nuclide concentrations in the buffer material are assumed not to have any axial or angular gradients, but to have radial gradient only.

- The four-actinide decay chains are considered simultaneously.

The geometry of the EBS assumed in the analysis is based on the design geometry, but with the following simplifications:

- Nuclide migration in the EBS is calculated for a single waste package. Because the modeling of diffusion is one-dimensional it does not account for transport through the buffer in the axial direction (Figure 5.1-1 (a)). The length of buffer material hypothetically considered in the analysis is defined to avoid nuclide mass loss at the interface between the overpack and the buffer material, assuming homogenous release of nuclide from overpack surface (Figure 5.1-1 (b)).
- The internal surface area of the buffer is defined as the total surface area of the overpack in order to take account of the additional surface area provided by the top and the bottom ends of the overpack.
- The physical resistance of the overpack is ignored, so that the dissolved nuclides reach the outer boundary of the overpack instantaneously.

The mathematical model for nuclide migration in EBS as shown in Fig. 5.1, consists of a series of equations describing various processes related to:

- Dissolution of vitrified waste to determine the inner boundary condition, dissolution of nuclides to a hypothetical region in the vicinity of the vitrified waste, precipitation/dissolution, decay/ingrowth and release to the buffer material
- Nuclide diffusion in the buffer material, sorption onto the buffer material, precipitation/dissolution and decay/ingrowth.
- Instantaneous mixing of nuclides reaching the boundary between the buffer and host rock. Ground water flow through the EDZ, and nuclide release from the EDZ to the surrounding host rock.

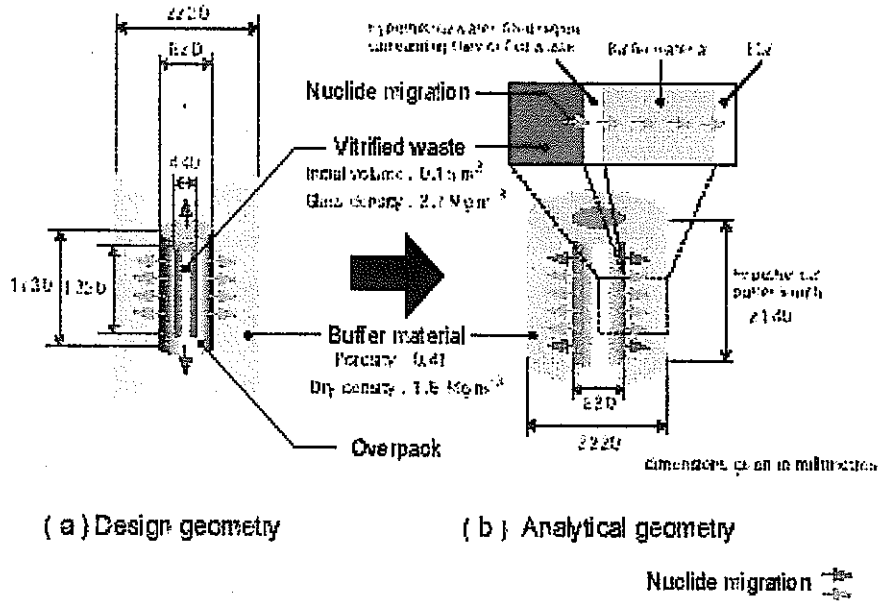

Fig. 5.1 Components and Geometry of EBS

The governing equations are formulated in terms of the following variables: concentration of dissolved nuclide (C_{ij}), concentration of sorbed nuclide (S_{ij}) and concentration of precipitate nuclide (P_{ij}), where "ij" represents isotope j of element i.

The four regions considered are the vitrified waste (G), a hypothetical water-filled region around the vitrified waste (R), the buffer material (B) and the Excavation Distributed Zone EDZ (M).

1) Glass dissolution

Dissolution of glass vitrified waste, which, expressed in terms of long-term glass dissolution rate. The concentration in vicinity of the vitrified waste are described by the mass balance equation as follows:

$$V^R \frac{\partial A_{ij}^R}{\partial t} = 2\pi r_{in} L \varepsilon^B D_{Pi} \frac{\partial C_{ij}^B}{\partial r}\Big|_{r=r_{in}} + M_{ij}^G g_{Si} - V^R \lambda_{ij} A_{ij}^R + V^R \lambda_{IJ} A_{IJ}^R$$

(5.1)

where V^R volume of the hypothetical water-filled region $[m^3]$, M_{ij}^G inventory of isotope j of element i in the vitrified waste [mole], g_{si} fractional rate of decrease of the glass volume $[y^{-1}]$, D_{pi} diffusion coefficient in the porewater $[m^2 y^{-1}]$, ε^B porosity of the buffer material, r_{in} inner radius of the buffer material $[m]$, λ_{ij} nuclide decay constant $[y^{-1}]$. The total amount of nuclide in a unit volume of the hypothetical water-filled region around the vitrified waste A_{ij}^R and the fractional rate of decrease of the glass volume are given by the following equations:

$$A_{ij}^R = C_{ij}^R + P_{ij}^R$$

(5.2)

$$g_{si} = \frac{a^G}{\rho^G V^G} k$$

(5.3)

where, a^G surface area of the vitrified waste $[m^2]$, ρ^G density of the vitrified waste $[kg/m^3]$, V^G volume of the vitrified waste $[m^3]$, and k glass dissolution rate $[kg/m^2 y]$.

The time-dependent amounts of nuclides in the vitrified waste are given by:

$$\frac{dM_{ij}^G}{dt} = -M_{ij}^G g_{si} - \lambda_{ij} M_{ij}^G + \lambda_{IJ} M_{IJ}^G$$

(5.4)

Nuclide migration in the buffer material region, which represents nuclide diffusion, sorption, precipitation and decay/ingrowth, is expressed:

$$\frac{\partial A_{ij}^B}{\partial t} = \varepsilon^B D_{pi} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial C_{ij}^B}{\partial r} \right) - \lambda_{ij} A_{ij}^B + \lambda_{IJ} A_{IJ}^B$$

(5.5)

where A_{ij}^B represents the total amount of nuclide in a unit volume of buffer material , which can be expressed as follows:

$$A_{ij}^B = \varepsilon^B C_{ij}^B + (1 - \varepsilon^B) S_{ij}^B + P_{ij}^B$$

(5.6)

The concentration of sorbed nuclide in a unit volume of the solid part of the buffer material S_{ij}^B is given by,

$$S_{ij}^B = \rho^B kd_i C_{ij}^B$$

(5.7)

where ρ^B is the density of the buffer material, and kd_i is the distribution coefficient of the element $i \left[m^3 / kg \right]$. Precipitation occurs when

$$\varepsilon^B C_i^* + (1 - \varepsilon^B) \rho^B kd_i C_i^* < \sum_j A_{ij}$$

(5.8)

where C_i^* is the elemental solubility $\left[mol / m^3 \right]$. The concentration of precipitate in a unit volume of the buffer material, P_{ij}^B, is derived by,

$$P_{ij}^B = A_{ij}^B - (\varepsilon^B C_i^* + (1 - \varepsilon^B) \rho^B kd_i C_i^*) \frac{A_{ij}^B}{\sum_j A_{ij}^B}$$

(5.9)

where the concentration of dissolved nuclides in a unit volume of porewater in the buffer material , C_{ij}^B, are given by,

$$C_{ij}^{B} = C_i^{*} \frac{A_{ij}^{B}}{\sum_j A_{ij}^{B}}$$

(5.10)

This equation accounts for the limited elemental solubility (C_i^{*}), which is partitioned among stable and radioactive isotopes.

The nuclides reaching the boundary between buffer and host rock are mixed instantaneously with the ground water flowing through the Excavation Distributed Zone (EDZ). The nuclide concentrations in the EDZ region are described by the following equation, which includes solubility limits with precipitation/dissolution, diffusion from the buffer, sorption and radioactive decay/ingrowth:

$$V^{M} \frac{\partial A_{ij}^{M}}{\partial t} = -2\pi r_{out} L \varepsilon^{B} D_{pi} \left. \frac{\partial C_{ij}^{B}}{\partial r} \right|_{r=r_{out}} - V^{M} \lambda_{ij} A_{ij}^{M} + V^{M} \lambda_{IJ} A_{IJ}^{M} - Q C_{ij}^{M}$$

(5.11)

where V^{M} is the volume of the EDZ $[m^3]$, Q is the groundwater flow rate $[m^3/y]$, and A_{ij}^{M} represents the total amount of nuclide in a unit volume of EDZ.

Nuclide migration in the EBS is evaluated by solving the above equations and the release rate of nuclide into the surrounding host rock $f_{ij}^{buffer \rightarrow hostrock}$ is derived by,

$$f_{ij}^{buffer \rightarrow hostrock} = Q C_{ij}^{M}$$

(5.12)

Application of the leap-frog scheme to equations (5.1), (5.5), and (5.11) yields the semi-discrete system

$$[A_{ij}^{R}]^{n+1} = [A_{ij}^{R}]^{n-1} + \frac{2\Delta t}{V^{R}} \left[2\pi r_{in} L \varepsilon^{B} D_{Pi} (C_{ij}^{B})_{r} \Big|_{r=r_{in}} + M_{ij}^{G} g_{Si} - V^{R} \lambda_{ij} A_{ij}^{R} + V^{R} \lambda_{IJ} A_{ij}^{R} \right]^{n}$$

(5.13)

$$[A_{ij}^{B}]^{n+1} = [A_{ij}^{B}]^{n-1} + 2\Delta t \left[\varepsilon^{B} D_{Pi} (C_{ij}^{B})_{rr} + \frac{\varepsilon^{B} D_{Pi}}{r} (C_{ij}^{B})_{r} - \lambda_{ij} A_{ij}^{B} + \lambda_{IJ} A_{IJ}^{B} \right]^{n}$$

(5.14)

$$[A_{ij}^M]^{n+1} = [A_{ij}^M]^{n-1} - \frac{2\Delta t}{V^M}\left[2\pi r_{wall} L\varepsilon^B D_{Pi}(C_{ij}^B)_r\Big|_{r=r_{wall}} + V^M \lambda_{ij} A_{ij}^M - V^M \lambda_{ij} A_{ij}^M + QC_{ij}^M\right]^n$$

(5.15)

5.2 Wavelet Galerkin Discretization

Our Galerkin procedure uses a class of compactly supported scaling functions introduced by Daubechies [5]. The scaling function are determined by a wavelet order $2N$ and a set of scaling parameters $\{c_k : 0 \le k \le 2N\}$ that define a generator function $\phi(x)$ through the scaling relation

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x-k)$$

(5.16)

For each $0 \le j$ we set

$$\phi_k^j(x) = 2^{j/2}\phi(2^j x - k), \quad for\ 0 \le k \le 2^j$$

(5.17)

If one sets $V^j = span\{\phi_k^j : 0 \le k \le 2^j\}$, in [13] it is shown that $\{\phi_k\}$ can be periodized and made to form an orthonormal basis for $V^j \in L^2[0,1]$, with $\overline{\cup V^j} = L^2[0,1]$ and $\cap V^j = 0$. Moreover the subspaces V^j are nested, so that $V^j \subset V^{j+1}$. If one lets W^j denote the orthogonal complement of V^j in V^{j+1}, it is shown in [13] that W^j is spanned by an orthonormal set of wavelet functions $\psi_k^j = 2^{j/2}\psi(2^j x - k)$, where the generator wavelet $\psi(x)$ defined by

$$\psi(x) = \sum_{k=-1}^{N-2} (-1)^k c_{k+1} \phi(2x+k)$$

(5.18)

The base generators $\phi(x)$ and $\psi(x)$ have support $[0, N-1]$ and every polynomial of degree $K \le N/2$ lies in the space V^0, which is equivalent to $\psi(x)$ having N/2 vanishing moments. The Daubechies class is distinguished by having this interpolation property and the smallest possible support. Thus, from interpolation property, we see that $\phi(x)$ has at least $N/2$ continuous derivatives.

Consider a set $\{\phi_k^p\}$ that spans the space $V^p[0,1] \subset L^2[0,1]$. A multiresolution is effected by noting that the space $V^p \supset V^{p-1} ..... \supset V^1 \supset V^0$. For the Galerkin approximation

of the problem, the field variables are projected into the space of trail functions belonging to V^p. When we use test functions from the same space, a system of differential equations in time for the coefficients of the field variable results when the inner products $\langle .,. \rangle$ are evaluated and orthogonality among the elements of V^p is used. In this work the evolution equations are solved at scale p determined by the resolution of the space V^p. If, at any time a multiresolution is desired, this can be performed as a post-processing step or as an adjunct calculation.

In what follows, we project the semi-discrete real variable to V^p so that

$$f^n(x_j) = \sum_{l=0}^{N-1} f_l^n \phi_l(x_j)$$

(5.19)

The weak formulation of the semi-discrete system is obtained by substituting Equation (5.19) into Equations (5.13), (5.14), and (5.15) multiplying by the test function $\phi_k \in V^p$, and integrating:

$$\left\langle \left[A_{ij}^R\right]^{n+1}, \phi_k \right\rangle = \left\langle \left[A_{ij}^R\right]^{n-1}, \phi_k \right\rangle + 2\Delta t \begin{bmatrix} \frac{a_{im} D_e}{V^R} \left\langle \left[C_{ij}^B\right]_r^n \Big|_{r=r_m}, \phi_k \right\rangle + \frac{g_{si}}{V^R} \left\langle \left[M_{ij}^G\right]^n, \phi_k \right\rangle \\ -\lambda_{ij} \left\langle \left[A_{ij}^R\right]^n, \phi_k \right\rangle + \lambda_{IJ} \left\langle \left[A_{IJ}^R\right]^n, \phi_k \right\rangle \end{bmatrix}$$

(5.20)

$$\left\langle \left[A_{ij}^B\right]^{n+1}, \phi_k \right\rangle = \left\langle \left[A_{ij}^B\right]^{n-1}, \phi_k \right\rangle + 2\Delta t \begin{bmatrix} D_e \left\langle \left[C_{ij}^B\right]_{rr}^n, \phi_k \right\rangle + \frac{D_e}{r} \left\langle \left[C_{ij}^B\right]_r^n, \phi_k \right\rangle - \\ \lambda_{ij} \left\langle \left[A_{ij}^B\right]^n, \phi_k \right\rangle + \lambda_{IJ} \left\langle \left[A_{IJ}^B\right]^n, \phi_k \right\rangle \end{bmatrix}$$

(5.21)

$$\left\langle \left[A_{ij}^M\right]^{n+1}, \phi_k \right\rangle = \left\langle \left[A_{ij}^M\right]^{n-1}, \phi_k \right\rangle - 2\Delta t \begin{bmatrix} \frac{a_{im} D_e}{V^M} \left\langle \left[C_{ij}^B\right]_r^n \Big|_{r=r_m}, \phi_k \right\rangle + \frac{Q}{V^M} \left\langle \left[C_{ij}^M\right]^n, \phi_k \right\rangle - \\ \lambda_{ij} \left\langle \left[A_{ij}^M\right]^n, \phi_k \right\rangle + \lambda_{IJ} \left\langle \left[A_{IJ}^M\right]^n, \phi_k \right\rangle \end{bmatrix}$$

(5.22)

Following the convention in chapter 4, we refer to the inner products as connection coefficients:

$$\Omega_{k,l}^{0,1} = \left\langle \phi_k, \phi_l' \right\rangle$$

$$\Omega_{k,l}^{1,1} = \left\langle \phi_k', \phi_l' \right\rangle$$

$$\Omega_{k,j,l}^{0,1,1} = \left\langle \phi_k, \phi_j' \phi_l' \right\rangle$$

$$\Omega_{k,j,l}^{1,0,0} = \left\langle \phi_k', \phi_j \phi_l \right\rangle$$

$$\Omega_{k,j,l}^{1,0,1} = \left\langle \phi_k', \phi_j \phi_l' \right\rangle$$

(5.23)

The most expedient strategy available for the evaluation of these connection coefficients is given in chapter 4. The connection coefficients should be precomputed. The resulting tables are then read in the time marching procedure.

The dimension of the matrix T in the heterogeneous case is $2 \times (I(2^j + N) - I + 1)$, I is the number of nodes. Because the scaling function is compact supported, only a finite number Ω are nonzero. Then, the resultant matrix T becomes $2 * (I(N-2)+1)$. The matrix T has the block diagonal structure. Such matrix can be inverted easily using any efficient matrix inversion method.

5.3. Problem Solution

The radionuclides to be considered in the nuclide migration analysis were selected by the following procedure:

- Nuclide whose ratio of calculated concentration in well water to maximum permissible concentration in the water is greater than 10^{-3} are include in the safety assessment
- Daughter nuclides with half-lives of less than one year are excluded from the nuclide migration calculation on the assumption that they are in the equilibrium with their parent nuclides.

Table 1, shows the safety relevant radionuclides considered in the analysis of nuclide migration in the Engineered Barrier System [6].

To give a picture of the capability of WGM to calculate the radionuclide transport, release calculations for several fission nuclides and radioactive chains are discussed. Results obtained running Sn-126 isotope case are shown in Fig. 2 for several wavelet-dilation orders pair. At long times, a good accuracy is found regardless of the choice of wavelet-dilation order pair. The choice of wavelet-order pair is only important at early time. Better accuracy is obtained by increasing wavelet order and dilation order.

The results from our WGM and MESHNOTE a finite difference code [7] are compared. The comparison shows a well agreement between both of them. The running time using WGM was less than the one using FD code with fine discretization.

Nuclides released from the waste form precipitate when their concentrations in the porewater result in their elemental solubility limits the release rates to the surrounding rock as shown in Fig. 3 for Np-237 isotope.

The following observations are made regarding the time-dependent nuclide release rates from the EBS:

During the period of 5×10^5 years following the overpack failure, Cs-135 has the highest release rate for any nuclide, in terms of Bq per year as shown in Fig. 4. This is due to its high solubility, relatively small distribution coefficient and relatively long half-life. After $7. \times 10^4$ years, the release rate of Cs-135 decreases sharply due to the glass having completely dissolved by this time. Nb-93m shows the highest release rate after 5×10^5 years. Nb-93m is in radioactive equilibrium with its parent (Zr-93), due to its relatively short (several decades) half-life. Nb-93m is mobile in the EBS due to its relatively low distribution coefficient and its release rate is higher than that of Zr-93. Precipitation of a nuclide in the decay series under radioactive equilibrium may result in a release rate for the nuclide different from that of its parent and daughters (e.g. Pb-210, Ra-226 and Th-230) as well as a difference in sorption behavior as shown in Figures, 5, 6, 7, and 8. Nuclides with large inventories and low solubility, e.g. Np-237, and Tc-99, precipitate in the vicinity of the vitrified waste. This results in a low and pseudo-steady-state release rate for a prolonged period. Relatively short-lived (half-life shorter than several tens of thousands of years) of highly sportive elements, e.g. Pu-240 and Am-241, decay significantly within the buffer material and their peak release rate extremely small.

Table 5.1. Safety-relevant Radionuclides

Fission Products	Sm-151, Cs-135, Sn-126, Pd-107, Tc-99, Nb-94, Zr-93, Nb-93m, Se-79
4N Series	Pu-240→U-236→Th-232
4N+1 Series	Cm-245→Pu-241→Am-241→Np-237 →U-233→Th-229
4N+2 Series	Cm-246→Pu-242→U-238→U-234→ Th-230→Ra-226→Pb-210→Am-242m →Pu-238
4N+3 Series	Am-243→Pu-239→U-235→Pa-231 →Ac-227

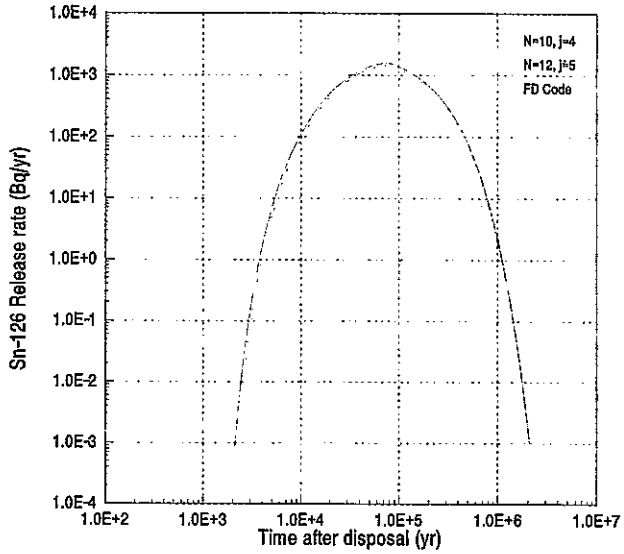

Fig. 5.2 Sn-126 release rate using various wavelet orders-dilation orders pairs.

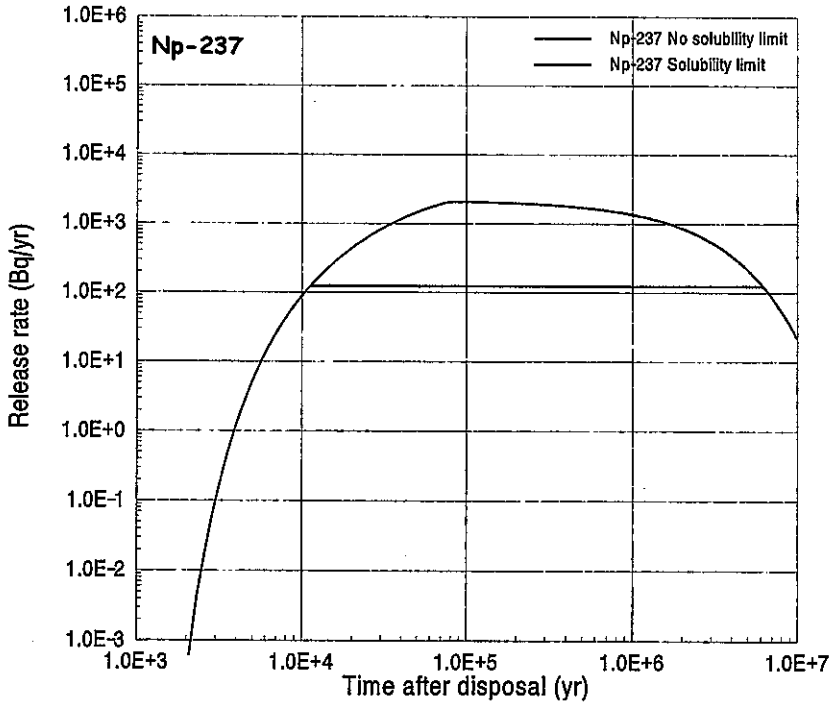

Fig. 5.3 Np-237 Precipitation calculation and its effect on nuclide release

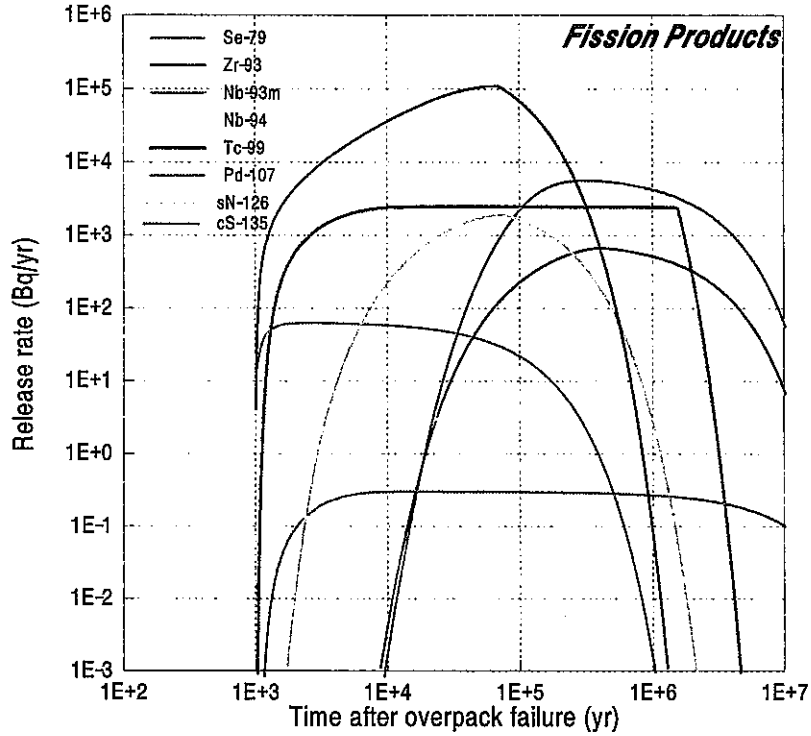

Fig. 5.4 Release rate of fission products

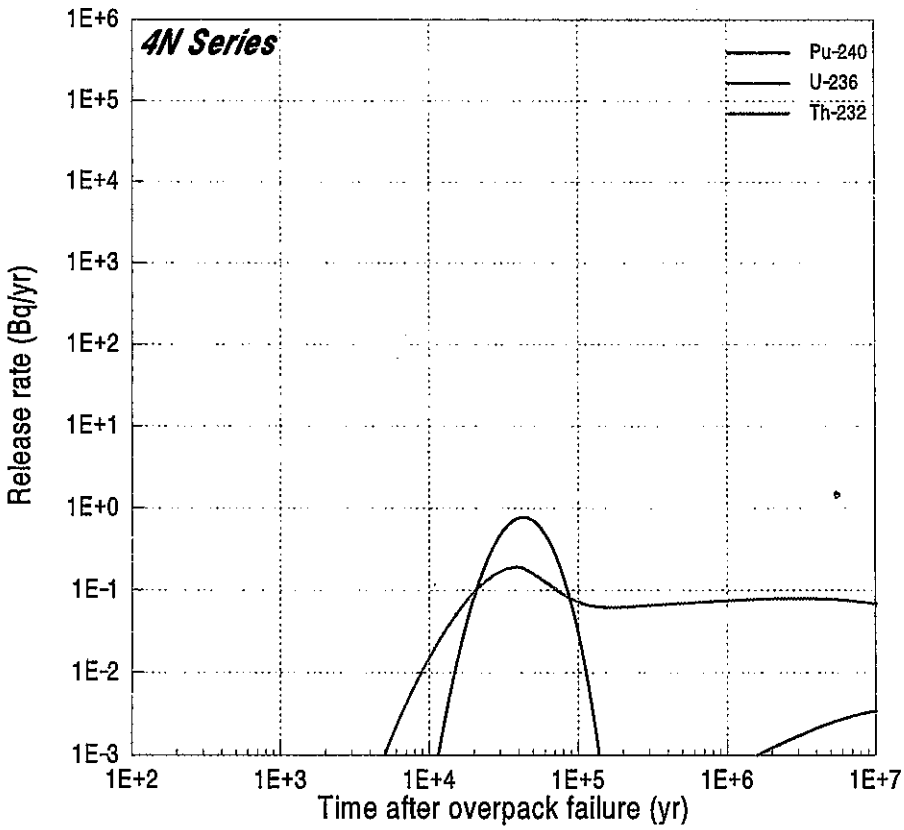

Fig. 5.5 Release rate of 4N series

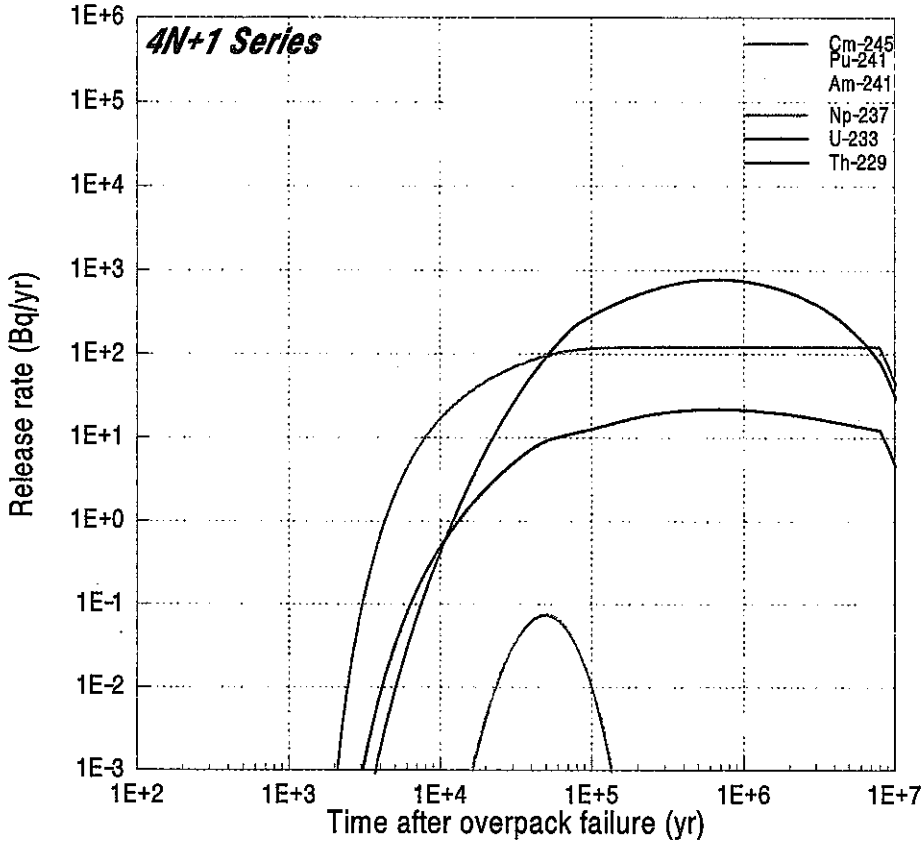

Fig. 5.6 Release rate of 4N+1 series

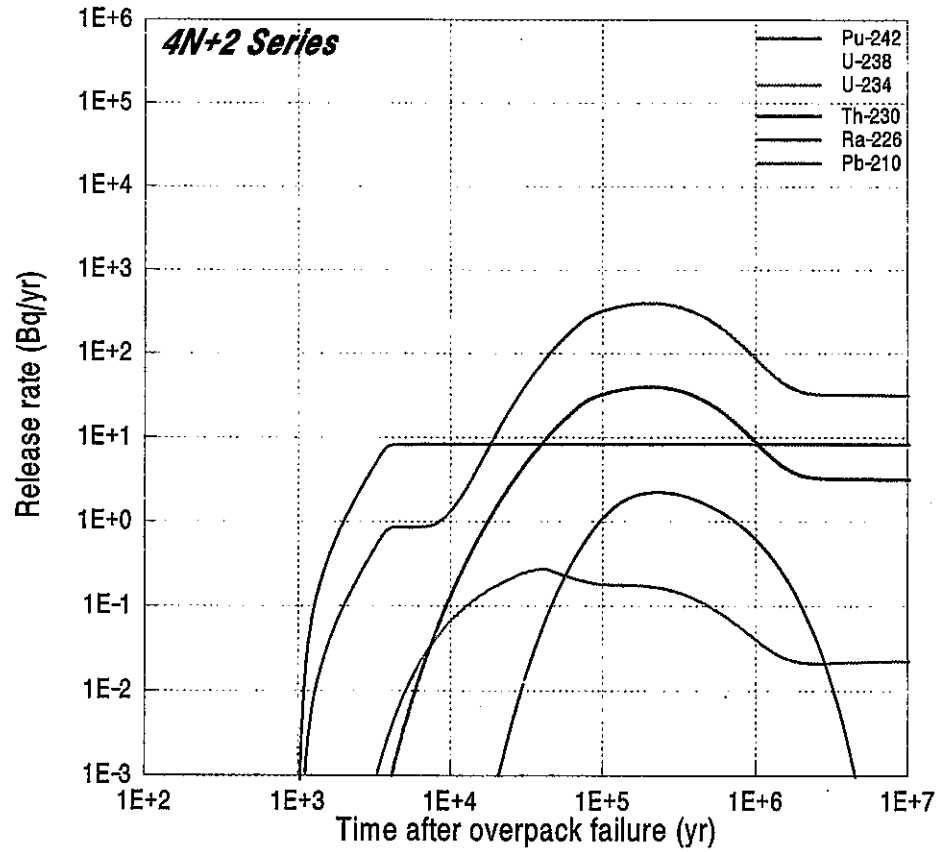

Fig.5. 7. Release rate of 4N+2 series

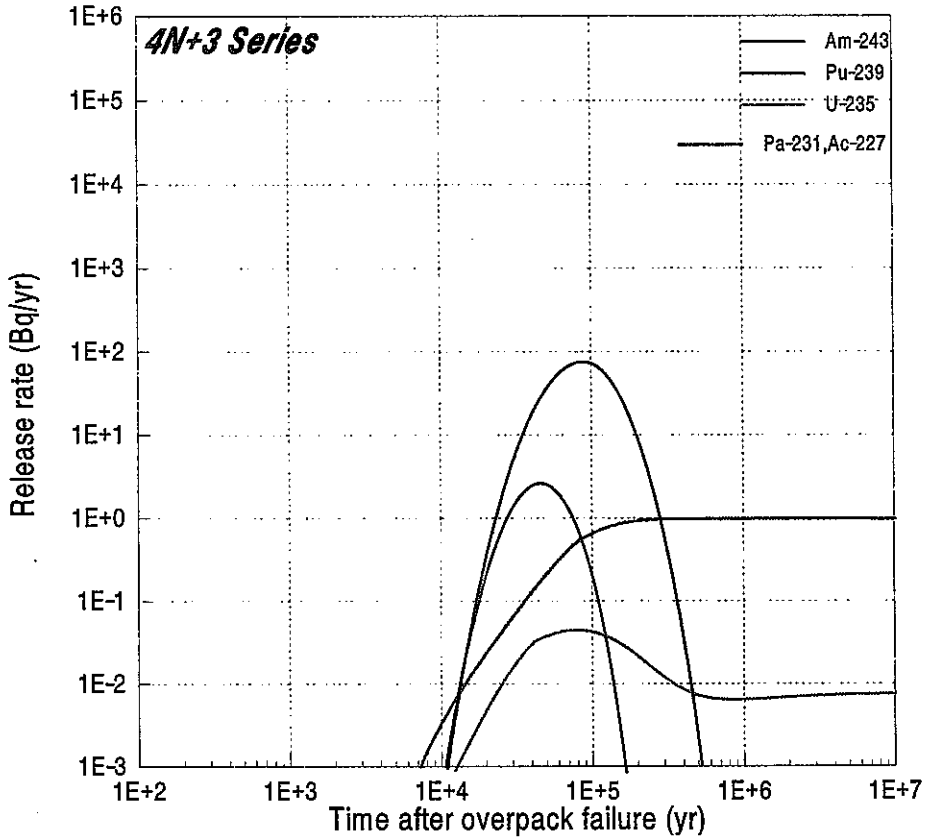

Fig. 5.8 Release rate of 4N+3 series

The capability of WGM to calculate radionuclide transport was shown by applying the model to the very complex transport situation at the Japanese repository.

The wavelet Galerkin method has been shown to be a powerful tool for the fast and accurate solution of the nonlinear system of partial differential equations arising from a model formulation of migration of high-level radioactive waste (HLW). A large Daubechies order gives more smoothness of the scaling function, and a large dilation order provides finer solution in the node, so wavelet function has strong ability to treat such a kind of stiff problems.

In order to verify our approach, we compared the results obtained using WGM with the results obtained by other codes. The results were very accurate and matched well with the results obtained with the conventional methods like finite difference. The accuracy improves as the dilation order j increase for a Daubechies order N chosen large enough to ensure wavelet function smoothness. The selection of a proper value of

dilation order j becomes very important especially near the boundaries for a highly soluble element like Cs-135.

Finally, this technique gives us the opportunity to overcome many problems encountered. The analytic solution of the connection coefficients makes the system with any Daubechies' order N and dilation order j unconditionally stable. The computation time is reasonably good compared to the other numerical methods for two reasons. First, those connection coefficients are computed once and stored for later use. Second, the matrix arising from the WGM can be inverted easily.

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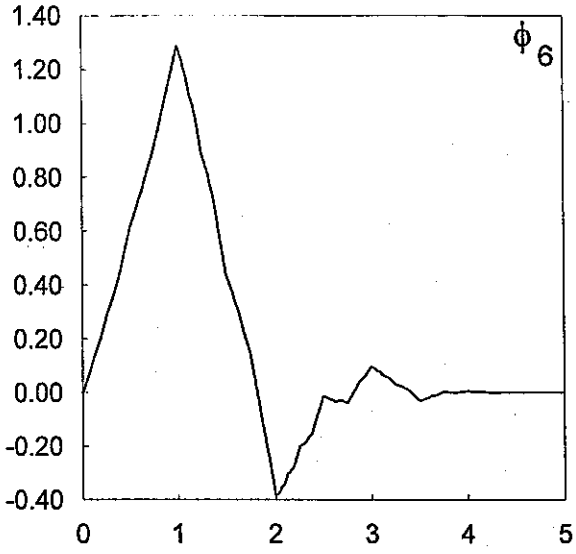
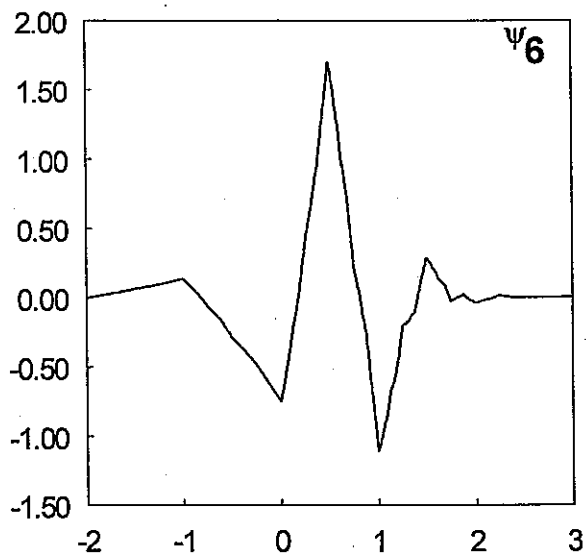
Appendix A: List of the wavelet coefficients for the compactly supported wavelets. The a_k's are normalized so that $\sum_k a_k = 2$

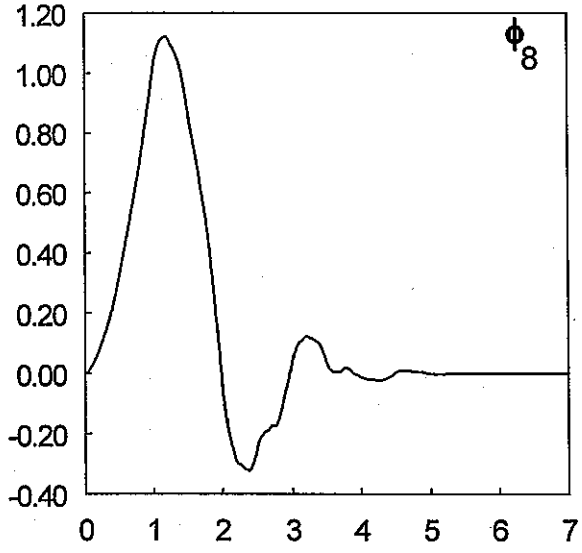
	K	a_k		k	a_k
N=4	0	0.4829629131445341	N=16	0	0.0544158422431072
	1	0.8365163037378077		1	0.3128715909143166
	2	0.2241438680420134		2	0.6756307362973195
	3	-0.1294095225512603		3	0.5853546836542159
N=6	0	0.3326705529500825		4	-0.0158291052563823
	1	0.8068915093110924		5	-0.2840155429615824
	2	0.4598775021184914		6	0.0004724845739124
	3	-0.1350110200102546		7	0.1287474266204893
	4	-0.0854412738820267		8	-0.0173693010018090
	5	0.0352262918857095		9	-0.0440882539307971
N=8	0	0.2303778133088964		10	0.0139810279174001
	1	0.7148465705529154		11	0.0087460940474065
	2	0.6308807679398587		12	-0.0048703529934520
	3	-0.0279837694168599		13	-0.0003917403733770
	4	-0.1870348117190931		14	0.0006754494064506
	5	0.0308413818355607		15	-0.0001174767841248
	6	0.0328830116668852	N=18	0	0.0380779473638778
	7	-0.0105974017850690		1	0.2438346746125858
N=10	0	0.1601023979741929		2	0.6048231236900955
	1	0.6038292697971895		3	0.6572880780512736
	2	0.7243085284377726		4	0.1331973858249883
	3	0.1384281459013203		5	-0.2932737832791663
	4	-0.2422948870663823		6	-0.0968407832229492
	5	-0.0322448695846381		7	0.1485407493381256
	6	0.0775714938400459		8	0.0307256814793385
	7	-0.0062414902127983		9	-0.0676328290613279
	8	-0.0125807519990820		10	0.0002509471148340
	9	0.0033357252855738		11	0.0223616621236798
N=12	0	0.1115407433501095		12	-0.0047232047577518
	1	0.4946238903984533		13	-0.0042815036824635
	2	0.7511339080210959		14	0.0018476468830563
	3	0.3152503517091982		15	0.0002303857635232
	4	-0.2262646939654400		16	-0.0002519631889427
	5	-0.1297668675672625		17	0.0000393473203163
	6	0.0975016055873225	N=20	0	0.0266700579005473
	7	0.0275228655303053		1	0.1881768000776347
	8	-0.0315820393174862		2	0.5272011889315757
	9	0.0005538422011614		3	0.6884590394534363
	10	0.0047772575109455		4	0.2811723436605715
	11	-0.0010773010853085		5	-0.2498464243271589
N=14	0	0.0778520540850037		6	-0.1959462743772862
	1	0.3965393194818912		7	0.1273693403357541
	2	0.7291320908461957		8	0.0930573646035547
	3	0.4697822874051889		9	-0.0713941471663501
	4	-0.1439060039285212		10	-0.0294575368218399
	5	-0.2240361849938412		11	0.0332126740593612
	6	0.0713092192668272		12	0.0036065535669870
	7	0.0806126091510774		13	-0.0107331754833007
	8	-0.0380299369350104		14	0.0013953517470688
	9	-0.0165745416306655		15	0.0019924052951925
	10	0.0125509985560986		16	-0.0006858566949564
	11	0.0004295779729214		17	-0.0001164668551285
	12	-0.0018016407040373		18	0.0000935886703202
	13	0.0003537137999745		19	-0.0000132642028945

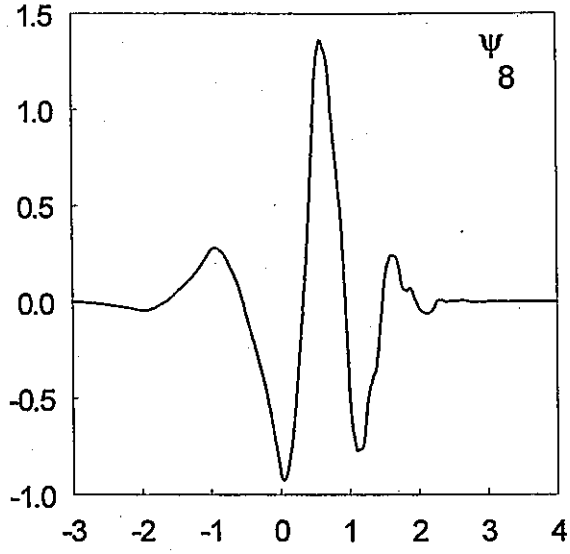
	K	a_k		k	a_k
N=22	0	0.0186942978722789	N=24	0	0.0131122573079307
	1	0.1440670213928020		1	0.1095662750146482
	2	0.4498997797368737		2	0.3773551489402250
	3	0.6856867833957955		3	0.6571987169629179
	4	0.4119643707697337		4	0.5158864873457728
	5	-0.1622752502029178		5	-0.0447638833121931
	6	-0.2742308511330729		6	-0.3161784504176317
	7	0.0660435900493206		7	-0.0237792571982025
	8	0.1498120161664735		8	0.1824786159911302
	9	-0.0464799554695869		9	0.0053595697465734
	10	-0.0664387852427391		10	-0.0964321158334711
	11	0.0313350913181445		11	0.0108491302521221
	12	0.0208409040434795		12	0.0415462769930380
	13	-0.0153648206512867		13	-0.0122186492890905
	14	-0.0033408588507246		14	-0.0128408250180754
	15	0.0049284176937469		15	0.0067114990597337
	16	-0.0003085928657264		16	0.0022486073018605
	17	-0.0008930232313170		17	-0.0021795035924598
	18	0.0002491525149587		18	0.0000065451285220
	19	0.0000544390756608		19	0.0003886530476400
	20	-0.0000346349830823		20	-0.0000885041116233
	21	0.0000044942742951		21	-0.0000242415455441
N=28	0	0.0064611533650241		22	0.0000127769519219
	1	0.0623647602301987		23	-0.0000015290716956
	2	0.2548502761214902	N=26	0	0.0092021334867151
	3	0.5543056156932723		1	0.0828612415447362
	4	0.6311878315512681		2	0.3119963240070169
	5	0.2186706837898898		3	0.6110558414510090
	6	-0.2716885532308958		4	0.5888895494136744
	7	-0.2180335288075264		5	0.0869857268716562
	8	0.1383952146215697		6	-0.3149729030362503
	9	0.1399890187714335		7	-0.1245767275978333
	10	-0.0867484137727595		8	0.1794760745032141
	11	-0.0715489536474273		9	0.0729489349777016
	12	0.0552371275544153		10	-0.1059490428146250
	13	0.0269814087948442		11	-0.0264884065711001
	14	-0.0301853516325476		12	0.0561394757759653
	15	-0.0056150494542673		13	0.0023799722785877
	16	0.0127894927907672		14	-0.0238314218257688
	17	-0.0007462190078724		15	0.0039239414020821
	18	-0.0038496388130632		16	0.0072555891907735
	19	0.0010616910501086		17	-0.0027619113003327
	20	0.0007080211215036		18	-0.0013156738861358
	21	-0.0003868319662803		19	0.0009323261553702
	22	-0.0000417772455507		20	0.0000492515250660
	23	0.0000687550409530		21	-0.0001651289858214
	24	-0.0000103372091560		22	0.0000306785377137
	25	0.0000043897049516		23	0.0000104419305949
	26	0.0000017249946354		24	-0.0000047004164116
	27	-0.0000001787140030		25	0.0000005220035200

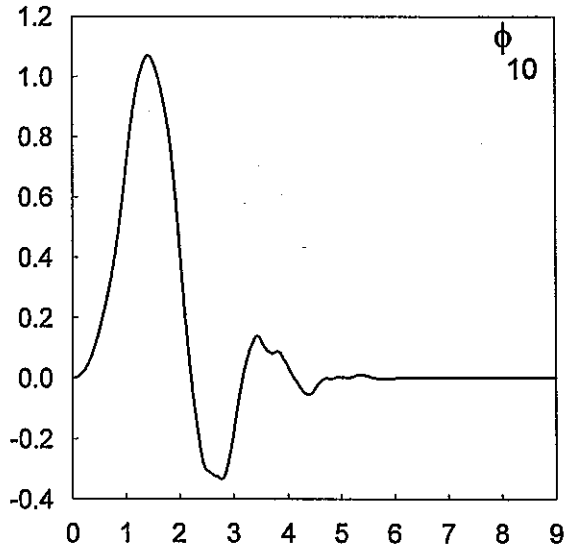
Appendix B: Scaling functions and Wavelet Functions for different wavelet orders

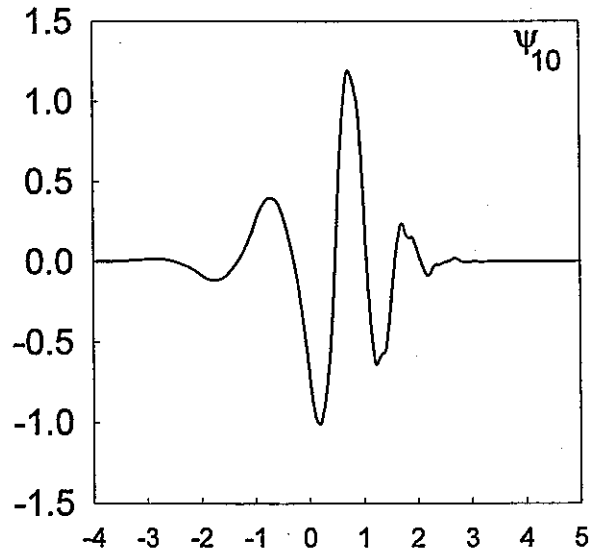

ϕ_6
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
0 1 2 3 4 5

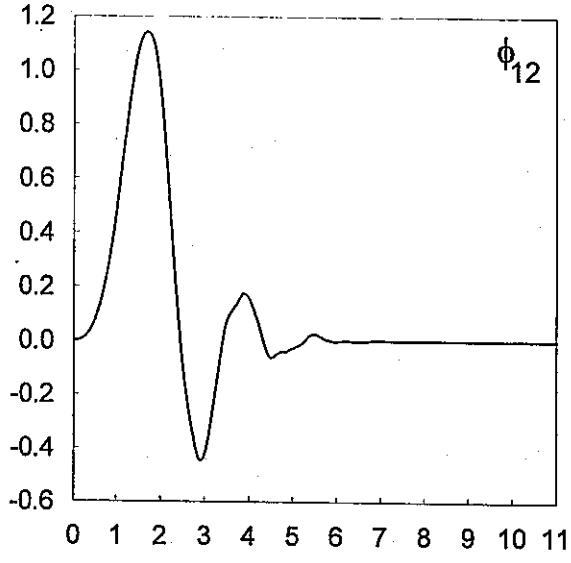

ψ_6
2.00
1.50
1.00
0.50
0.00
-0.50
-1.00
-1.50
-2 -1 0 1 2 3

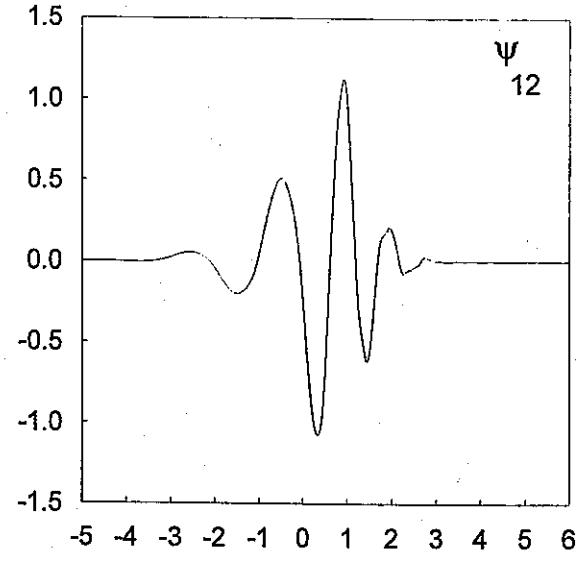

ϕ_8
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
0 1 2 3 4 5 6 7

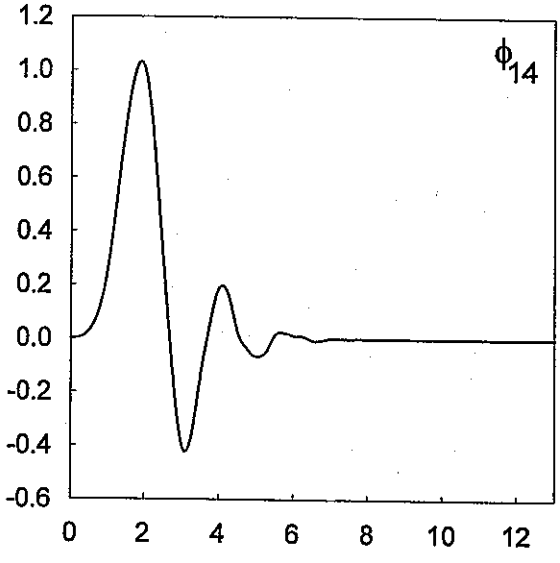

ψ_8
1.5
1.0
0.5
0.0
-0.5
-1.0
-3 -2 -1 0 1 2 3 4

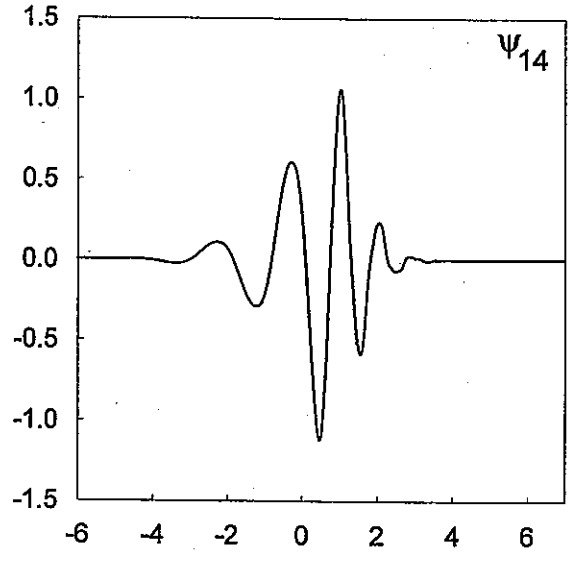

ϕ_{10}
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0 1 2 3 4 5 6 7 8 9

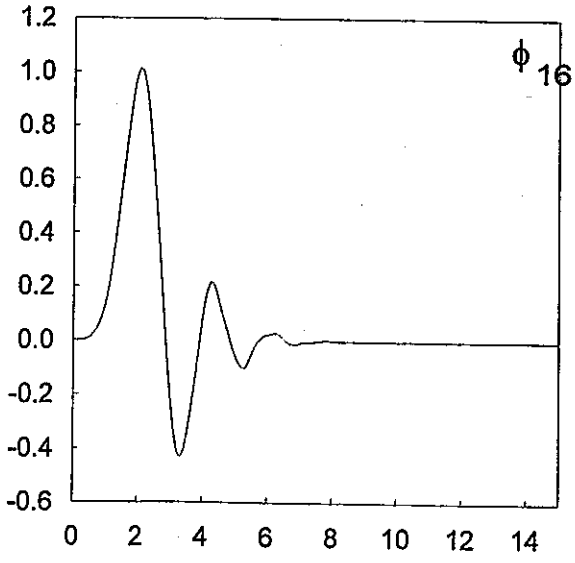

ψ_{10}
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-4 -3 -2 -1 0 1 2 3 4 5

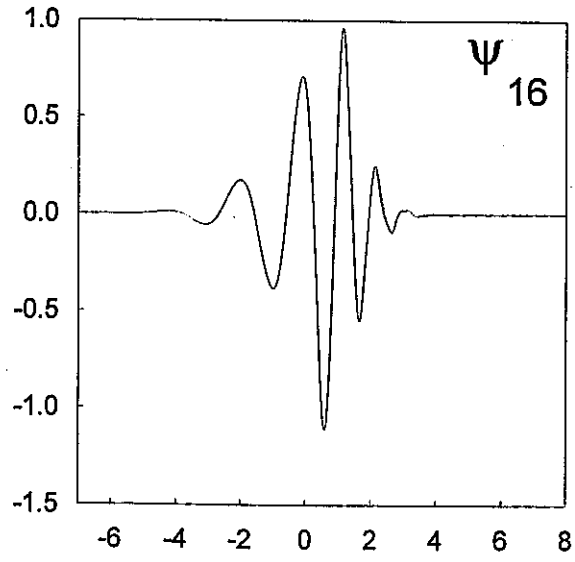

ϕ_{12}
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 1 2 3 4 5 6 7 8 9 10 11

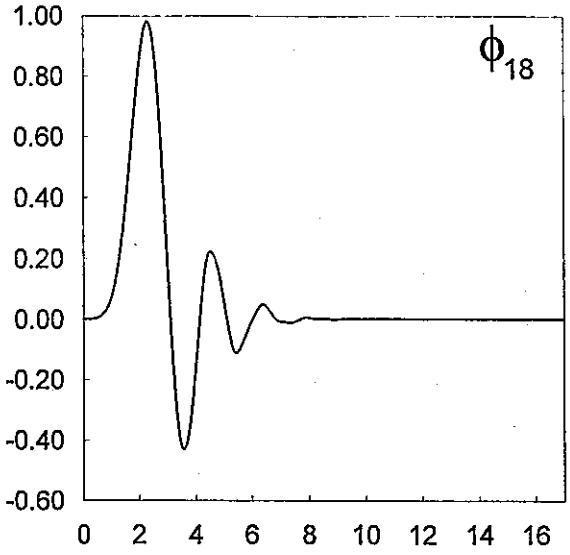

ψ_{12}
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-5 -4 -3 -2 -1 0 1 2 3 4 5 6

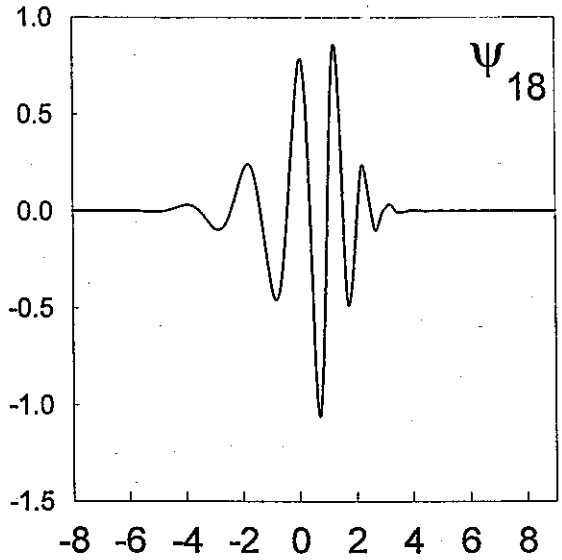

ϕ_{14}
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 2 4 6 8 10 12

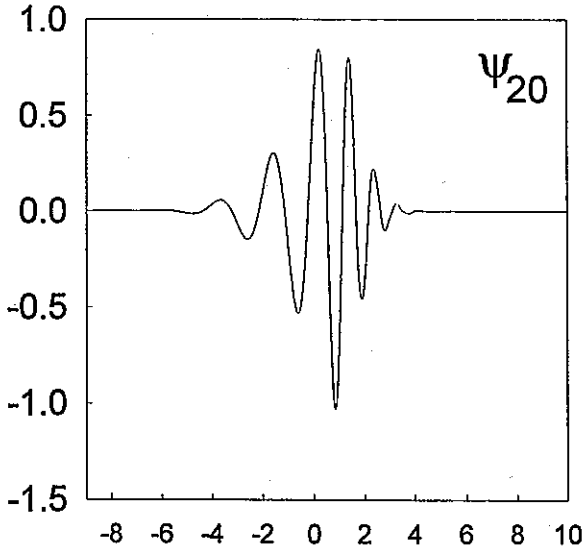

ψ_{14}
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-6 -4 -2 0 2 4 6

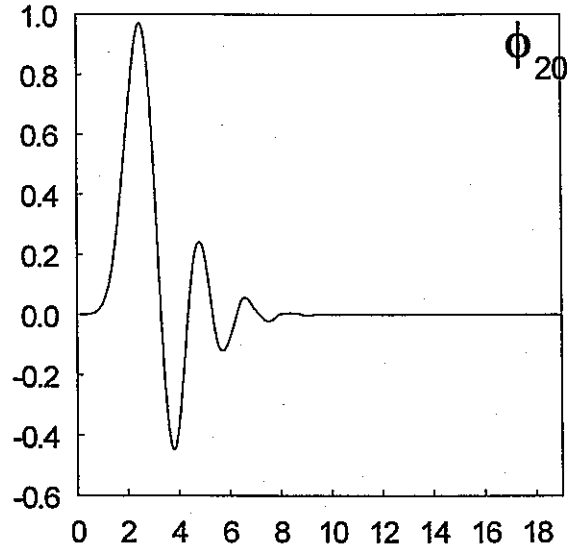

ϕ_{16}
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 2 4 6 8 10 12 14

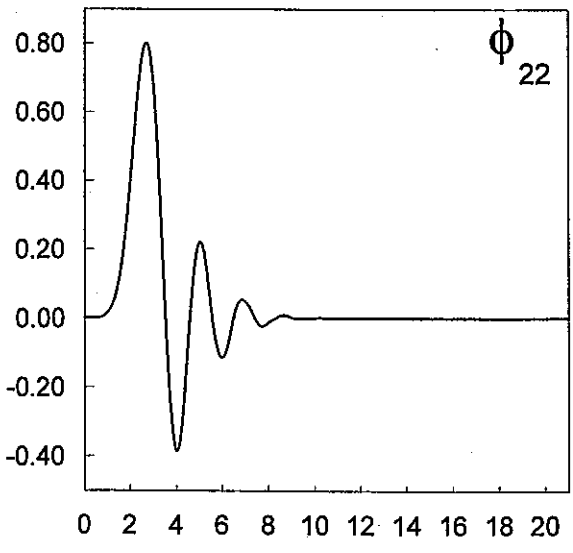

ψ_{16}
1.0
0.5
0.0
-0.5
-1.0
-1.5
-6 -4 -2 0 2 4 6 8

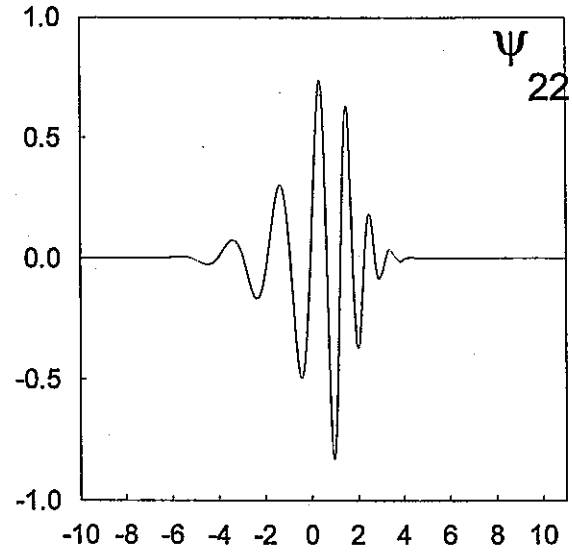

ϕ_{18}
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
-0.60
0 2 4 6 8 10 12 14 16


ψ_{18}
1.0
0.5
0.0
-0.5
-1.0
-1.5
-8 -6 -4 -2 0 2 4 6 8


ψ_{20}
1.0
0.5
0.0
-0.5
-1.0
-1.5
-8 -6 -4 -2 0 2 4 6 8 10


ϕ_{20}
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 2 4 6 8 10 12 14 16 18


ϕ_{22}
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
0 2 4 6 8 10 12 14 16 18 20


ψ_{22}
1.0
0.5
0.0
-0.5
-1.0
-10 -8 -6 -4 -2 0 2 4 6 8 10

Appendix C: List of some useful connection coefficients

4.5.1 Results of 2-Term Connection Coefficients

The results of

$$\Omega_{0,l}^{1,0} = \int_{-\infty}^{\infty} \phi^1(x)\phi(x-l)dx$$

$$\Omega_{0,l}^{2,0} = \int_{-\infty}^{\infty} \phi^2(x)\phi(x-l)dx$$

are listed as follow

N=6

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-4	3.42466e-04	5.35714e-03	4	-3.42466e-04	5.35714e-03
-3	1.46119e-02	1.14286e-01	3	-1.46119e-02	1.14286e-01
-2	-1.45205e-01	-8.76190e-01	2	1.45205e-01	-8.76190e-01
-1	7.45205e-01	3.39048e+00	1	-7.45205e-01	3.39048e+00
0	6.66134e-16	-5.26786e+00			

N=8

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-6	8.40851e-07	1.59216e-05	6	-8.40851e-07	1.59216e-05
-5	-1.72206e-04	-1.63038e-03	5	1.72206e-04	-1.63038e-03
-4	-2.22405e-03	-1.05727e-02	4	2.22405e-03	-1.05727e-02
-3	3.35802e-02	1.50973e-01	3	-3.35802e-02	1.50973e-01
-2	-1.91999e-01	-6.97869e-01	2	1.91999e-01	-6.97869e-01
-1	7.93010e-01	2.64207e+00	1	-7.93010e-01	2.64207e+00
0	3.33067e-16	-4.16597e+00			

N=10

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-8	2.69605e-10	3.53876e-09	8	-2.69605e-10	3.53876e-09
-7	2.52412e-07	1.65654e-06	7	-2.52412e-07	1.65654e-06
-6	5.40473e-05	3.67145e-04	6	-5.40473e-05	3.67145e-04
-5	2.39236e-04	7.94621e-04	5	-2.39236e-04	7.94621e-04
-4	-7.46140e-03	-2.99080e-02	4	7.46140e-03	-2.99080e-02
-3	5.33526e-02	1.80954e-01	3	-5.33526e-02	1.80954e-01
-2	-2.28820e-01	-6.49502e-01	2	2.28820e-01	-6.49502e-01
-1	8.25906e-01	2.41479e+00	1	-8.25906e-01	2.41479e+00
0	7.43205e-12	-3.83499e+00			

N=12

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-10	-6.96825e-13	-1.26410e-11	10	6.96825e-13	-1.26410e-11
-9	2.89967e-09	2.62998e-08	9	-2.89967e-09	2.62998e-08
-8	-4.20691e-07	-3.46609e-06	8	4.20691e-07	-3.46609e-06
-7	-1.20266e-05	-5.43634e-05	7	1.20266e-05	-5.43634e-05
-6	-4.29689e-06	-6.56963e-05	6	4.29689e-06	-6.56963e-05
-5	1.58856e-03	6.47806e-03	5	-1.58856e-03	6.47806e-03
-4	-1.45455e-02	-4.93616e-02	4	1.45455e-02	-4.93616e-02
-3	7.24406e-02	2.04905e-01	3	-7.24406e-02	2.04905e-01
-2	-2.58553e-01	-6.30733e-01	2	2.58553e-01	-6.30733e-01
-1	8.50137e-01	2.31187e+00	1	-8.50137e-01	2.31187e+00
0	-1.55431e-15	-3.68606e+00			

N=14

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-12	5.86491e-16	1.79462e-14	12	-5.86491e-16	1.79462e-14
-11	1.20353e-11	1.63426e-10	11	-1.20353e-11	1.63426e-10
-10	-4.18305e-10	2.06549e-09	10	4.18305e-10	2.06549e-09
-9	2.18711e-07	1.48618e-06	9	-2.18711e-07	1.48618e-06
-8	1.65017e-06	1.40112e-06	8	-1.65017e-06	1.40112e-06
-7	-4.23639e-06	4.15664e-05	7	4.23639e-06	4.15664e-05
-6	-3.37344e-04	-1.60524e-03	6	3.37344e-04	-1.60524e-03
-5	3.88145e-03	1.39637e-02	5	-3.88145e-03	1.39637e-02
-4	-2.26874e-02	-6.70524e-02	4	2.26874e-02	-6.70524e-02
-3	9.01891e-02	2.23359e-01	3	-9.01891e-02	2.23359e-01
-2	-2.82965e-01	-6.21498e-01	2	2.82965e-01	-6.21498e-01
-1	8.68744e-01	2.25505e+00	1	-8.68744e-01	2.25505e+00
0	2.66454e-15	-3.60452e+00			

N=16

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-14	-1.58546e-19	-6.92447e-18	14	1.58546e-19	-6.92447e-18
-13	1.24008e-14	2.70800e-13	13	-1.24008e-14	2.70800e-13
-12	-7.25221e-13	-5.81388e-11	12	7.25221e-13	-5.81388e-11
-11	-9.69718e-10	-1.05857e-08	11	9.69718e-10	-1.05857e-08
-10	-7.20795e-08	-3.72308e-07	10	7.20795e-08	-3.72308e-07
-9	3.99381e-08	2.09042e-06	9	-3.99381e-08	2.09042e-06
-8	2.45199e-07	-2.39823e-05	8	-2.45199e-07	-2.39823e-05
-7	7.66771e-05	4.51679e-04	7	-7.66771e-05	4.51679e-04
-6	-1.03153e-03	-4.09766e-03	6	1.03153e-03	-4.09766e-03
-5	6.95838e-03	2.20703e-02	5	-6.95838e-03	2.20703e-02
-4	-3.12901e-02	-8.22664e-02	4	3.12901e-02	-8.22664e-02
-3	1.06364e-01	2.37178e-01	3	-1.06364e-01	2.37178e-01
-2	-3.03259e-01	-6.15614e-01	2	3.03259e-01	-6.15614e-01
-1	8.83446e-01	2.21915e+00	1	-8.83446e-01	2.21915e+00
0	7.30838e-12	-3.55369e+00			

N=18

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-16	4.85075e-24	1.76991e-21	16	-4.85075e-24	1.76991e-21
-15	1.61879e-18	2.95326e-16	15	-1.61879e-18	2.95326e-16
-14	4.78140e-14	2.79314e-13	14	-4.78140e-14	2.79314e-13
-13	5.40252e-13	4.92837e-11	13	-5.40252e-13	4.92837e-11
-12	8.13741e-10	6.37052e-09	12	-8.13741e-10	6.37052e-09
-11	1.59669e-08	4.75558e-08	11	-1.59669e-08	4.75558e-08
-10	-1.03612e-07	-7.34347e-07	10	1.03612e-07	-7.34347e-07
-9	4.87825e-07	9.13093e-06	9	-4.87825e-07	9.13093e-06
-8	-1.96204e-05	-1.33610e-04	8	1.96204e-05	-1.33610e-04
-7	2.78121e-04	1.23050e-03	7	-2.78121e-04	1.23050e-03
-6	-2.10340e-03	-7.24848e-03	6	2.10340e-03	-7.24848e-03
-5	1.06169e-02	3.00686e-02	5	-1.06169e-02	3.00686e-02
-4	-3.99527e-02	-9.49708e-02	4	3.99527e-02	-9.49708e-02
-3	1.20954e-01	2.47332e-01	3	-1.20954e-01	2.47332e-01
-2	-3.20312e-01	-6.10929e-01	2	3.20312e-01	-6.10929e-01
-1	8.95316e-01	2.19407e+00	1	-8.95316e-01	2.19407e+00
0	-3.10862e-15	-3.51886e+00			

N=20

l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$	l	$\Omega_{0,l}^{1,0}$	$\Omega_{0,l}^{2,0}$
-18	-1.48752e-26	-4.79929e-25	18	1.48752e-26	-4.79929e-25
-17	2.10246e-20	3.39173e-19	17	-2.10246e-20	3.39173e-19
-16	2.61057e-16	3.13406e-15	16	-2.61057e-16	3.13406e-15
-15	-2.97158e-14	-2.39686e-13	15	2.97158e-14	-2.39686e-13
-14	-3.31960e-12	-5.01585e-11	14	3.31960e-12	-5.01585e-11
-13	-3.69609e-10	-2.21993e-09	13	3.69609e-10	-2.21993e-09
-12	-1.70163e-09	6.11426e-09	12	1.70163e-09	6.11426e-09
-11	3.41579e-08	1.22297e-07	11	-3.41579e-08	1.22297e-07
-10	-2.67262e-07	-2.57930e-06	10	2.67262e-07	-2.57930e-06
-9	5.49729e-06	3.85245e-05	9	-5.49729e-06	3.85245e-05
-8	-7.66835e-05	-3.69388e-04	8	7.66835e-05	-3.69388e-04
-7	6.31220e-04	2.35727e-03	7	-6.31220e-04	2.35727e-03
-6	-3.52603e-03	-1.07807e-02	6	3.52603e-03	-1.07807e-02
-5	1.46694e-02	3.75800e-02	5	-1.46694e-02	3.75800e-02
-4	-4.84273e-02	-1.05430e-01	4	4.84273e-02	-1.05430e-01
-3	1.34055e-01	2.54697e-01	3	-1.34055e-01	2.54697e-01
-2	-3.34784e-01	-6.06689e-01	2	3.34784e-01	-6.06689e-01
-1	9.05071e-01	2.17522e+00	1	-9.05071e-01	2.17522e+00
0	-1.80729e-11	-3.49324e+00			

4.5.2 Results of 3-Term Connection Coefficients

The results of

$$\Lambda_{0,l_1,l_2}^{1,0,0} = \int_{-\infty}^{\infty} \phi^1(x)\phi(x-l_1)\phi(x-l_2)dx$$

$$\Lambda_{0,l_1,l_2}^{2,0,0} = \int_{-\infty}^{\infty} \phi^2(x)\phi(x-l_1)\phi(x-l_2)dx$$

are listed as follow.

N=6

l_1, l_2	$\Lambda^{1,0,0}_{0,l_1,l_2}$	$\Lambda^{2,0,0}_{0,l_1,l_2}$	l_1, l_2	$\Lambda^{1,0,0}_{0,l_1,l_2}$	$\Lambda^{2,0,0}_{0,l_1,l_2}$
-4,-4	1.524127e-06	1.910001e-05	4,4	2.459265e-06	1.038635e-03
-4,-3	3.290197e-05	4.774341e-04	4,3	-4.456481e-04	5.273187e-03
-4,-2	-1.358901e-04	-1.988491e-03	4,2	1.339407e-04	-1.150802e-03
-4,-1	4.451594e-04	6.682145e-03	4,1	-3.245557e-05	2.271402e-04
-4,0	-1.229632e-06	1.669547e-04	4,0	-7.620636e-07	-3.101756e-05
-3,-4	3.290197e-05	4.774341e-04	3,4	-4.456481e-04	5.273187e-03
-3,-3	1.452297e-03	1.033653e-02	3,3	2.376364e-03	1.792559e-02
-3,-2	-6.177322e-03	-4.603734e-02	3,2	-2.074151e-02	9.395534e-02
-3,-1	2.049169e-02	1.580535e-01	3,1	4.925522e-03	9.858749e-03
-3,0	-1.188182e-03	-8.074606e-03	3,0	-7.261486e-04	-1.265325e-02
-3,1	4.887386e-07	-4.698465e-04	3,-1	-4.463991e-07	-7.390202e-05
-2,-4	-1.358901e-04	-1.988491e-03	2,4	1.339407e-04	-1.150802e-03
-2,-3	-6.177322e-03	-4.603734e-02	2,3	-2.074151e-02	9.395534e-02
-2,-2	3.290269e-02	1.604630e-01	2,2	9.354837e-02	-1.150604e+00
-2,-1	-1.252725e-01	-5.043032e-01	2,1	8.746228e-02	3.260455e-01
-2,0	-4.677419e-02	-4.686360e-01	2,0	-1.645134e-02	-1.747595e-01
-2,1	2.498264e-04	-1.609619e-02	2,-1	1.251801e-03	2.991476e-02
-2,2	1.949357e-06	4.078114e-04	2,-2	1.949357e-06	4.078114e-04
-1,-4	4.451594e-04	6.682145e-03	1,4	-3.245557e-05	2.271402e-04
-1,-3	2.049169e-02	1.580535e-01	1,3	4.925522e-03	9.858749e-03
-1,-2	-1.252725e-01	-5.043032e-01	1,2	8.746228e-02	3.260455e-01
-1,-1	4.968918e-01	1.546996e+00	1,1	-6.271755e-01	4.142284e+00
-1,0	3.135878e-01	1.674584e+00	1,0	-2.484459e-01	-1.549995e+00
-1,1	3.781027e-02	4.786225e-01	1,-1	3.781027e-02	4.786225e-01
-1,2	1.251801e-03	2.991476e-02	1,-2	2.498264e-04	-1.609619e-02
-1,3	-4.463991e-07	-7.390202e-05	1,-3	4.887386e-07	-4.698465e-04
0,-4	-1.229632e-06	1.669547e-04	0,4	-7.620636e-07	-3.101756e-05
0,-3	-1.188182e-03	-8.074606e-03	0,3	-7.261486e-04	-1.265325e-02
0,-2	-4.677419e-02	-4.686360e-01	0,2	-1.645134e-02	-1.747595e-01
0,-1	3.135878e-01	1.674584e+00	0,1	-2.484459e-01	-1.549995e+00
0,0	7.452775e-16	-4.728459e+00			

N=8

l_1,l_2	$\Lambda^{1,0,0}_{0,l_1,l_2}$	$\Lambda^{2,0,0}_{0,l_1,l_2}$	l_1,l_2	$\Lambda^{1,0,0}_{0,l_1,l_2}$	$\Lambda^{2,0,0}_{0,l_1,l_2}$
-6,-6	4.499515e-11	5.877465e-10	6,6	2.602599e-08	7.309988e-07
-6,-5	-2.848327e-09	-3.729767e-08	6,5	-9.348390e-07	1.783390e-05
-6,-4	-1.277917e-08	-2.495275e-07	6,4	1.275694e-07	-3.913499e-06
-6,-3	6.444029e-08	1.193528e-06	6,3	-7.694401e-08	1.583062e-06
-6,-2	-9.551220e-08	-2.056306e-06	6,2	1.385662e-08	-2.819516e-07
-6,-1	9.005179e-07	1.686738e-05	6,1	3.502927e-09	-2.945293e-08
-6,0	-1.301299e-08	2.032880e-07	6,0	-2.249757e-11	-1.406428e-09
-5,-6	-2.848327e-09	-3.729767e-08	5,6	-9.348390e-07	1.783390e-05
-5,-5	5.966180e-07	3.776884e-06	5,5	-1.072828e-05	-3.451976e-04
-5,-4	4.622343e-06	2.896113e-05	5,4	2.150517e-04	-1.607901e-03
-5,-3	-2.242969e-05	-1.464299e-04	5,3	-5.180543e-05	3.787688e-04
-5,-2	3.887053e-05	2.897235e-04	5,2	2.717976e-05	-1.026101e-04
-5,-1	-1.992616e-04	-1.690475e-03	5,1	-6.257722e-06	3.826252e-05
-5,0	5.364139e-06	-1.151055e-04	5,0	-2.983090e-07	-9.570031e-06
-5,1	3.432101e-08	-7.904597e-07	5,-1	-6.546007e-10	3.621156e-08
-4,-6	-1.277917e-08	-2.495275e-07	4,6	1.275694e-07	-3.913499e-06
-4,-5	4.622343e-06	2.896113e-05	4,5	2.150517e-04	-1.607901e-03
-4,-4	1.227288e-04	3.324244e-04	4,4	-1.694283e-03	1.154833e-02
-4,-3	-5.190097e-04	-1.843446e-03	4,3	4.705793e-03	-2.951791e-02
-4,-2	9.144273e-04	4.353071e-03	4,2	-1.352025e-03	7.879713e-03
-4,-1	-3.578125e-03	-1.919794e-02	4,1	4.091152e-04	2.749472e-03
-4,0	8.471415e-04	5.464799e-03	4,0	-6.136439e-05	-1.605073e-03
-4,1	-1.579006e-05	2.885272e-04	4,-1	1.635379e-06	-1.552824e-05
-4,2	-3.205719e-08	1.119659e-06	4,-2	-1.077451e-09	8.039925e-08
-3,-6	6.444029e-08	1.193528e-06	3,6	-7.694401e-08	1.583062e-06
-3,-5	-2.242969e-05	-1.464299e-04	3,5	-5.180543e-05	3.787688e-04
-3,-4	-5.190097e-04	-1.843446e-03	3,4	4.705793e-03	-2.951791e-02
-3,-3	3.215877e-03	9.084209e-03	3,3	-1.096890e-02	2.031136e-01
-3,-2	-9.842045e-03	-2.090269e-02	3,2	-3.527921e-02	-4.558761e-02
-3,-1	3.637802e-02	9.886252e-02	3,1	9.516776e-03	4.100513e-02
-3,0	5.484452e-03	7.099116e-02	3,0	-1.607939e-03	-2.279622e-02
-3,1	-1.127668e-03	-4.739006e-03	3,-1	1.098945e-04	4.207427e-03
-3,2	1.293489e-05	-3.339641e-04	3,-2	-4.750068e-06	1.688028e-04
-3,3	1.250372e-08	-6.476917e-07	3,-3	1.250372e-08	-6.476917e-07

N=8					
l_1, l_2	$\Lambda^{1,0,0}_{0,l_1,l_2}$	$\Lambda^{2,0,0}_{0,l_1,l_2}$	l_1, l_2	$\Lambda^{1,0,0}_{0,l_1,l_2}$	$\Lambda^{2,0,0}_{0,l_1,l_2}$
-2,-6	-9.551220e-08	-2.056306e-06	2,6	1.385662e-08	-2.819516e-07
-2,-5	3.887053e-05	2.897235e-04	2,5	2.717976e-05	-1.026101e-04
-2,-4	9.144273e-04	4.353071e-03	2,4	-1.352025e-03	7.879713e-03
-2,-3	-9.842045e-03	-2.090269e-02	2,3	-3.527921e-02	-4.558761e-02
-2,-2	4.067303e-02	5.306867e-02	2,2	1.619503e-01	-1.089269e+00
-2,-1	-1.421420e-01	-2.419033e-01	2,1	8.621344e-02	6.450648e-01
-2,0	-8.097516e-02	-3.881157e-01	2,0	-2.033652e-02	-2.850633e-01
-2,1	-1.098810e-03	-1.031521e-01	2,-1	3.252690e-04	7.121577e-02
-2,2	4.375975e-04	-1.673597e-03	2,-2	4.375975e-04	-1.673597e-03
-2,3	-4.750068e-06	1.688028e-04	2,-3	1.293489e-05	-3.339641e-04
-2,4	-1.077451e-09	8.039925e-08	2,-4	-3.205719e-08	1.119659e-06
-1,-6	9.005179e-07	1.686738e-05	1,6	3.502927e-09	-2.945293e-08
-1,-5	-1.992616e-04	-1.690475e-03	1,5	-6.257722e-06	3.826252e-05
-1,-4	-3.578125e-03	-1.919794e-02	1,4	4.091152e-04	2.749472e-03
-1,-3	3.637802e-02	9.886252e-02	1,3	9.516776e-03	4.100513e-02
-1,-2	-1.421420e-01	-2.419033e-01	1,2	8.621344e-02	6.450648e-01
-1,-1	4.996935e-01	1.196987e+00	1,1	-6.929822e-01	2.856451e+00
-1,0	3.464911e-01	1.012737e+00	1,0	-2.498468e-01	-1.316486e+00
-1,1	5.592860e-02	5.208504e-01	1,-1	5.592860e-02	5.208504e-01
-1,2	3.252690e-04	7.121577e-02	1,-2	-1.098810e-03	-1.031521e-01
-1,3	1.098945e-04	4.207427e-03	1,-3	-1.127668e-03	-4.739006e-03
-1,4	1.635379e-06	-1.552824e-05	1,-4	-1.579006e-05	2.885272e-04
-1,5	-6.546007e-10	3.621156e-08	1,-5	3.432101e-08	-7.904597e-07
0,-6	-1.301299e-08	2.032880e-07	0,6	-2.249757e-11	-1.406428e-09
0,-5	5.364139e-06	-1.151055e-04	0,5	-2.983090e-07	-9.570031e-06
0,-4	8.471415e-04	5.464799e-03	0,4	-6.136439e-05	-1.605073e-03
0,-3	5.484452e-03	7.099116e-02	0,3	-1.607939e-03	-2.279622e-02
0,-2	-8.097516e-02	-3.881157e-01	0,2	-2.033652e-02	-2.850633e-01
0,-1	3.464911e-01	1.012737e+00	0,1	-2.498468e-01	-1.316486e+00
0,0	1.513246e-15	-3.240976e+00			