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Dynamic Buckling Analysis of Liquid-Filled Shells with Imperfections

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DYNAMIC BUCKLING ANALYSIS OF LIQUID-FILLED SHELLS WITH IMPERFECTIONS*

Kazuyuki Tsukimori **

Abstract

There can be some imperfections in actual shell structures. In order to achieve high quality in manufacturing structures, the imperfections should be restricted within reasonable tolerance ranges through design or examinations.

In this paper, the governing equations to solve the dynamic buckling problems of shell structures, which consider dynamic fluid-structure interaction,modal coupling in both axial and circumferential directions, and circumferential imperfections,are derived by using Variationl Principle. Applying finite element method (FEM) to these equations, the matrix equation of motion is formulated to apply to computer analyses. Through the implementation of some examples, the influence of the amplitude and pattern of imperfections on the buckling strength is discussed. Also, the validity and applicability of this method to the design of shell structures is discussed.

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* This work was performed during the stay at Northwestern University in U.S.A.as a member of the research group of prof. W.K.Liu.

** Structural Engineering Section, Oarai Engineering Center

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形状不整を有する液体内包シェルの動的座屈解析

月森和之*

要　旨

実際のシェル構造では何らかの形状不整が存在する。構造の信頼性を確保するためには形状不整を合理的に押さえる必要がある。

液体を内包する円筒シェル構造の動的座屈問題について形状不整の影響を検討した例はない。本研究は周方向形状不整を有する不完全円筒シェルの動的座屈問題の解析に関するものである。まず、変分原理に基づき、動的な流体－構造連成、周方向と軸方向のモード連成および周方向形状不整を考慮した動的座屈問題を解くために基礎方程式を導いた。つぎに、これらに有限要素法を適用し、マトリックス形の方程式を導き解析プログラムを作成した。最後にいくつかの基本的な例題を通して形状不整の影響を検討した。その結果、完全円筒を仮定した解析では現われない不安定領域を捉えるとともに固有振動数の変化等についても確認された。

なお、本研究は筆者が、1990年9月から1991年8月にかけて米国ノースウエスタン大学に客員研究員として滞在中にW. K. Liu 教授のもとで行ったものである

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CONTENTS

1. Introduction

In the design of thin cylindrical tanks which are used for storage of fluid etc, not only static strength but also dynamic strength such as that against seismic loadings should be evaluated. Natsiavas and Babcock analized the buckling behavior of a tank model containing liquid statically using the maximum value of pressure induced by dynamic and static loadings[1]. Liu and Uras developed a further general analysis method of dynamic buckling of liquid-filled shells which included dynamic fluid-structure interaction and modal coupling in both axial and circumferential directions[2,3]. The numeical results based on this theory coincide well with the experimental results produced by Chiba et al[4,5] for major buckling modes.

However, the higher order buckling modes observed in experiments could not be seen in the analysis by Liu and Uras. It seems that this reason lies on the point that the theoretical analysis is derived based on the perfect cylindrical shells without imperfections.

In the actual manufacturing of cylindrical tanks it is impossible to eliminate imperfections perfectly. To evaluate the effect of imperfections on the dynamic stability is important, especially from the point of establishment of reasonable tolerances. In this paper the theory which consider circumferential imperfections in addition to dynamic fluid-structure interaction and modal coupling in both axial and circumferential directions is developed. Through the implementation of some examples, the influence of the amplitude and pattern of imperfection on the buckling strength is discussed comparing with the case of perfect cylindrical shells. Also, the validity and applicability of this method to the design of shell structures is discussed.

2. Derivation of Total Fluid Pressure

In order to introduce circumferential imperfections, we assume the radius of the cylindrical shell as the following function:

$$R = \bar{R}\,(1 + \nu\,\eta(\theta)) \qquad (2.1)$$

where R and $\bar{R}$ are the radius and mean radius of the cylindrical shell, respectively. ν is the first order small parameter which expresses the amplitude of the imperfections. $\eta(\theta)$, a function of circumferential coordinate θ, describes the circumferential imperfection pattern.

The normal direction(ψ) and the radius of curvature(R_S) of shell midsurface in r-θ coordinates are illustrated in Fig. 2.1. Using eqn. (2.1), the following approximations are obtained.

$$\cos(\psi-\theta) \approx 1 \;,\; \sin(\psi-\theta) \approx -\nu\,\eta'(\theta) \qquad (2.2a,b)$$

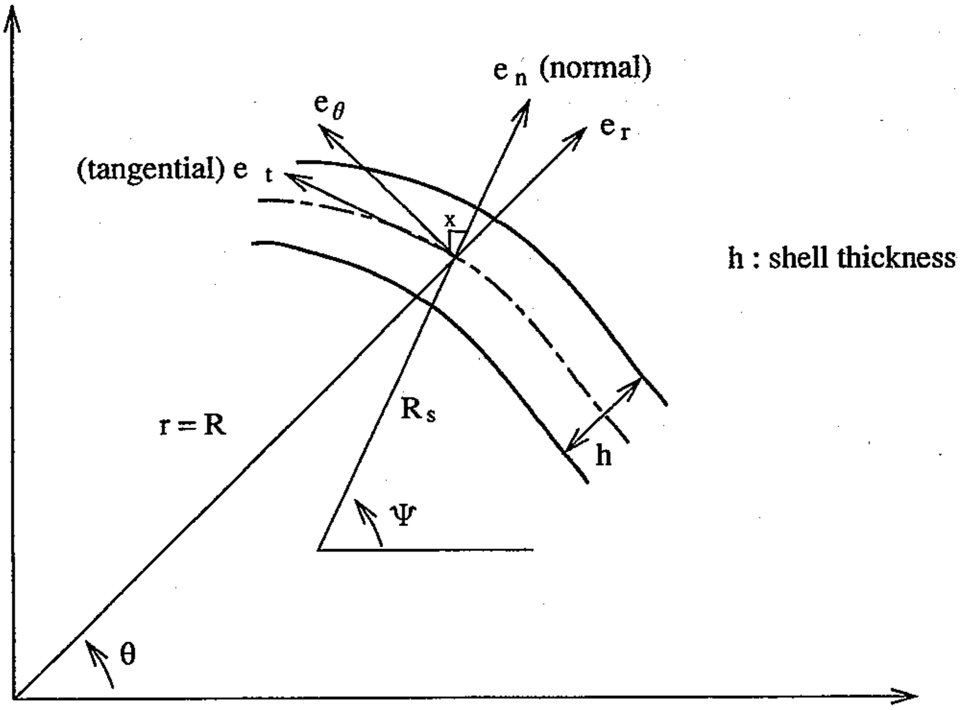


Fig.2.1 Schematic of imperfeation in r-θ coordinates

The governing equation of a compressible inviscid fluid is expressed as follows;

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{2.3}$$

where ϕ,c and t are the velocity potential,the velocity of sound and time,respectively. r,θ and z are used as spatial coordinates in eqn.(2.3).

The boundary conditions,which are needed to obtain the velocity potential,are specified as follows;

(i) Free surface boundary condition;

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \qquad \text{at } z=H \tag{2.4}$$

where g and H are the gravitational acceleration and the height of the fluid inside the tank, respectively.

(ii) Finite value condition;

$$\phi \text{ is bounded} \qquad \text{at } r=0 \tag{2.5}$$

(iii) Fluid-tank interface normal velocity compatibility;

$$\frac{\partial\phi}{\partial r}\cos(\psi\text{-}\theta) + \frac{1}{R}\frac{\partial\phi}{\partial\theta}\sin(\psi\text{-}\theta) = h(\theta,z,t) \qquad \text{at } r=R \tag{2.6}$$

where h(θ,z,t) is the normal velocity at the fluid-tank wall interface.

(iv) Fluid-base plate interface normal velocity compatibility;

$$\frac{\partial\phi}{\partial z} = f(r,\theta,t) \qquad \text{at } z=0 \tag{2.7}$$

Considering the effect of geometrical imperfections, we expand the velocity potential and the normal velocity at the fluid-tank wall interface as series of the imperfection amplitude;

$$\phi = \phi_0 + \nu\,\phi_1 + O(\nu^2) \tag{2.8}$$

and

$$h(\theta,z,t) = h_0(\theta,z,t) + \nu\,h_1(\theta,z,t) + O(\nu^2) \tag{2.9}$$

By applying eqns. (2.1), (2.8) and (2.9) to eqns. (2.3)-(2.7), the basic equations can be obtained separately in 0-th order and first order.

0-th order

$$\nabla^2\phi_0 = \frac{1}{c^2}\frac{\partial^2\phi_0}{\partial t^2} \tag{2.10a}$$

$$\frac{\partial^2\phi_0}{\partial t^2} + g\frac{\partial\phi_0}{\partial z} = 0 \qquad \text{at} \quad z=H \tag{2.10b}$$

$$\phi_0 \text{ is bounded} \qquad \text{at} \quad r=0 \tag{2.10c}$$

$$\frac{\partial\phi_0}{\partial r} = h_0(\theta,z,t) \qquad \text{at} \quad r=\bar{R} \tag{2.10d}$$

$$\frac{\partial\phi_0}{\partial z} = f(r,\theta,t) \qquad \text{at} \quad z=0 \tag{2.10e}$$

1 order

$$\nabla^2\phi_1 = \frac{1}{c^2}\frac{\partial^2\phi_1}{\partial t^2} \tag{2.11a}$$

$$\frac{\partial^2\phi_1}{\partial t^2} + g\frac{\partial\phi_1}{\partial z} = 0 \qquad \text{at} \quad z=H \tag{2.11b}$$

$$\phi_1 \text{ is bounded} \qquad \text{at} \quad r=0 \tag{2.11c}$$

$$\frac{\partial\phi_1}{\partial r} = \frac{1}{\bar{R}}\eta'(\theta)\frac{\partial\phi_0}{\partial\theta} - \bar{R}\,\eta(\theta)\frac{\partial^2\phi_0}{\partial r^2} + h_1(\theta,z,t) \quad \text{at} \quad r=\bar{R} \tag{2.11d}$$

$$\frac{\partial\phi_1}{\partial z} = 0 \qquad \text{at} \quad z=0 \tag{2.11e}$$

The set of equations of 0-th order is the same as that without imperfections[2]. The 0-th order solution is given as follows. Detailed procedures are written in ref.[2].

$$\phi_0 = \sum_{n=-\infty}^{\infty} \bar{\phi}_{0n}(r,z,\omega)\, e^{jn\theta}\, e^{-j\omega t} \tag{2.12}$$

and

$$\bar{\phi}_{on}(r,z,\omega) = 2\sum_{i=1}^{\infty} \frac{I_n(\tilde{\mu}_i r)}{\{H - \frac{g}{\omega^2}\sin^2(\mu_i H)\}\tilde{\mu}_i\ I'_n(\tilde{\mu}_i \bar{R})}\cos(\mu_i z)\int_0^H \bar{h}_{0n}(\zeta,\omega)\cos(\mu_i \zeta)\,d\zeta$$

$$+\frac{2}{\bar{R}}\sum_{i=1}^{\infty}\left\{\sinh(\varepsilon_i z) - \cosh(\varepsilon_i z)\frac{\omega^2\sinh(\varepsilon_i H) - \varepsilon_i g\cosh(\varepsilon_i H)}{\omega^2\cosh(\varepsilon_i H) - \varepsilon_i g\sinh(\varepsilon_i H)}\right\}$$

$$* \frac{J_n(\tilde{\varepsilon}_i r)}{(\varepsilon_i \bar{R})\left(1 - \dfrac{n^2}{(\tilde{\varepsilon}_i R)^2}\right)J_n^2(\tilde{\varepsilon}_i \bar{R})}\int_0^{\bar{R}} \bar{f}_n(\zeta,\omega)J_n(\tilde{\varepsilon}_i \xi)\,\xi d\xi \qquad (2.13)$$

in eqn.(2.13) the following relations are satisfied.

$$\tilde{\mu}_i^2 = \mu_i^2 - \frac{\omega^2}{c^2} \qquad (2.14a)$$

$$-\omega^2\cos(\mu_i H) = \mu_i g\sin(\mu_i H) \qquad (2.14b)$$

$$\tilde{\varepsilon}_i^2 = \varepsilon_i^2 + \frac{\omega^2}{c^2} \qquad (2.14c)$$

$$J'_n(\tilde{\varepsilon}_i \bar{R}) = 0 \qquad (2.14d)$$

and

$$h_0(\theta,z,t) = \sum_{n=-\infty}^{\infty} \bar{h}_{0n}(z,\omega)\, e^{jn\theta}\, e^{-j\omega t} \qquad (2.15a)$$

$$f(r,\theta,t) = \sum_{n=-\infty}^{\infty} \bar{f}_n(r,\omega)\, e^{jn\theta}\, e^{-j\omega t} \qquad (2.15b)$$

where ω and J_n are radial frequency and the first kind Bessel function, respectively.

Employing a standard separation of variables technique to the first order velocity potential ϕ_1 in eqns. (2.11), the first order solution can be derived after some algedra.

$$\bar{\phi}_1(r,\theta,z,\omega) = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty} C_{nk}(\omega)\cos(\mu_k z)\cdot I_n(\tilde{\mu}_k r)\, e^{jn\theta} \qquad (2.16)$$

where I_n is the first kind modified Bessel function. And the coefficient $C_{nk}(\omega)$ is given in Appendix A.

The fluid dynamic pressure P_f is given by

$$P_f = -\rho_F \frac{\partial \phi}{\partial t} = \bar{P}_f e^{-j\omega t} \tag{2.17}$$

Applying eqns. (2.12) and (2.16) to eqn. (2.17),

$$\tilde{P}_f = j\,\omega\,\rho_F \sum_{n=-\infty}^{\infty} \{\, \bar{\phi}_{0n}(r,z,\omega) + \nu \sum_{k=1}^{\infty} C_{nk}(\omega) \cos(\mu_k z)\, I_n(\tilde{\mu}_k r) \,\}\, e^{jn\theta} \tag{2.18}$$

where ρ_F is the density of the fluid.

The tank wall boundary condition can be interpreted as

$$\dot{h}(\theta,z,t) = \ddot{w}^t(\theta,z,t) = \ddot{w} + \ddot{w}_h + \ddot{w}_{rm} \tag{2.19}$$

where $\ddot{w}^t$ is the total tank wall acceleration, and $\ddot{w}_h$ and $\ddot{w}_{rm}$ are defined below:

(a) The relative wall acceleration is expressed as

$$\ddot{w}(\theta,z,t) = \sum_{n=-\infty}^{\infty} \ddot{w}_n(z,t)\, e^{jn\theta} \tag{2.20}$$

where w is the tank normal displacement.

(b) The rigid body acceleration due to horizontal seismic excitation is given as

$$\ddot{w}_h(\theta,z,t) \approx G_h(t)\ (\cos\theta + \nu\, \eta'(\theta) \sin\theta) \tag{2.21}$$

where $G_h(t)$ is the horizontal ground acceleration. It should be noted that the geometrical imperfection effect appears in the second term in eqn. (2.21).

(c) The rigid body acceleration induced by a rocking motion is given as:

$$\ddot{w}_{rm}(\theta,z,t) \approx z\, \ddot{q}(t)\ (\cos\theta + \nu\, \eta'(\theta) \sin\theta) \tag{2.22}$$

where q(t) is the angle of rotation. The geometrical imperfection effect appears in eqn. (2.22) again.

The base plate boundary condition can be treated similarly. However,the geometrical imperfection effect doesn't appear, since the base plate is assumed to have no imperfection.

$$\dot{f}(r,\theta,t) = \ddot{W}^t(r,\theta,t) = \ddot{W} + \ddot{W}_{rm} \tag{2.23}$$

where $\ddot{W}^t$ is the total base plate acceleration.

(d) The relative plate acceleration is given by:

$$\ddot{W}(r,\theta,t) = \sum_{n=-\infty}^{\infty} \ddot{W}_n(r,t)\, e^{jn\theta} \tag{2.24}$$

where W is the transverse plate deformation.

(e) The rigid body acceleration due to the rocking motion is

$$\ddot{W}_{rm}(r,\theta,t) = - r\, \ddot{q}(t) \cos\theta \tag{2.25}$$

The boundary conditions for the fluid equations are summarized in Table 2.1.

Table 2.1 Boundary conditions for the fluid equation

	structural vibration due to	rigid body motion due to	
boundary excitation	relative deformation	horizontal excitation	rocking motion
at the tank wall $\dot{h}(\theta,z,t)$	$\ddot{w} = \sum_{n=-\infty}^{\infty} \ddot{w}_n(z,t)\, e^{jn\theta}$	(0th order) $G_h(t) \cos\theta$ (1st order) $G_h(t)\, \eta'(\theta) \sin\theta$	(0th order) $z\, \ddot{q}(t) \cos\theta$ (1st order) $z\, \ddot{q}(t)\eta'(\theta) \sin\theta$
at the base plate $\dot{f}(r,\theta,t)$	$\ddot{W} = \sum_{n=-\infty}^{\infty} \ddot{W}_n(r,t)\, e^{jn\theta}$		$- r\, \ddot{q}(t) \cos\theta$

note:

w	:	shell deformation
W	:	plate deformation
G_h	:	horizontal ground acceleration
q	:	angle of tank rotation

A superposed "." denotes the time derivative.

In the frequency domain the general form of the tank wall and the base plate boundary excitations can be rewritten as

$$-j\omega\bar{h}_0(\theta,z,\omega) = -\omega^2 \sum_{n=-\infty}^{\infty} \bar{w}_n(z,\omega)\, e^{jn\theta} + \bar{G}_h(\omega)\cos\theta - z\,\omega^2\bar{q}(\omega)\cos\theta \tag{2.26a}$$

$$-j\omega\bar{h}_1(\theta,z,\omega) = \eta'(\theta)\sin\theta\,[\bar{G}_h(\omega) - z\,\omega^2\bar{q}(\omega)\,] \tag{2.26b}$$

and

$$-j\omega\bar{f}(r,\theta,\omega) = -\omega^2 \sum_{n=-\infty}^{\infty} \bar{W}_n(r,\omega)\, e^{jn\theta} + r\,\omega^2\bar{q}(\omega)\cos\theta \tag{2.27}$$

P_f is decomposed into four components from the point of view of physical meaning.

$$P_f = P_d + P_b + P_h + P_{rm} \tag{2.28}$$

where P_d and P_b denote the shell and base plate vibration pressures, P_h and P_{rm} are the pressures due to horizontal ground excitation and rocking motion under the rigid tank assumption, respectively.

Each component can be obtained by substituting eqns. (2.26) and (2.27) into eqn. (2.18). Furthermore, eqn. (2.28) can be rewritten as follows to make clear the effect of geometrical imperfections.

$$\begin{aligned} P_f &= P_{f0} + \nu\, P_{f1} + O(\nu^2) \\ &=(P_{d0} + P_{b0} + P_{h0} + P_{rm0}) + \nu\,(P_{d1} + P_{b1} + p_{h1} + P_{rm1}) + O(\nu^2) \end{aligned} \tag{2.29}$$

The total pressure in the fluid is determined by superposing the pressure due to the induced fluid motion P_f with the pressure due to vertical ground acceleration P_v and the hydrostatic pressure P_L.

$$P = P_f + P_v + P_L \tag{2.30}$$

where

$$P_v = -\,\rho_F\, G_v(t)\,[\,H - z\,] \tag{2.31}$$

and

$$P_L = \rho_F\, g\,[\,H - z\,] \tag{2.32}$$

in eqn.(2.31) $G_v(t)$ is the vertical ground acceleration.The expressions for the pressure components are summarized in Table 2.2(1) and Table 2.2(2).

Table 2.2(1) 0-th order fluid pressure components

Pressure component due to	Expression
vibration of the shell P_{d0}	$\sum_{n=-\infty}^{\infty}\sum_{i=1}^{\infty} 2\omega^2 F_{in}(r,z,t)\, e^{jn\theta}$ $* \int_0^H \bar{w}_n(\zeta,\omega) \cos(\mu_i\zeta)\, d\zeta$
vibration of the plate P_{b0}	$\sum_{n=-\infty}^{\infty}\sum_{i=1}^{\infty} \frac{2\omega^2}{\bar{R}} Q_{in}(r,z,t)\, e^{jn\theta}$ $* \int_0^{\bar{R}} \bar{W}_n(\zeta,\omega) J_n(\tilde{\varepsilon}_i\zeta)\, \zeta\, d\zeta$
horizontal ground motion P_{ho}	$- \sum_{i=1}^{\infty} 2H\bar{G}_h(\omega) \frac{\sin(\mu_iH)\cdot F_{i1}(r,z,t)}{\mu_iH} \cos\theta$
rocking motion P_{rm0}	$\sum_{i=1}^{\infty} 2\omega^2H^2\bar{q}(\omega)\cos\theta$ $* \left[\frac{\{\mu_iH\sin(\mu_iH)+\cos(\mu_iH) - 1\}F_{i1}(r,z,t)}{(\mu_iH)^2} - \frac{\bar{R}^2J_2(\tilde{\varepsilon}_i\bar{R})\cdot Q_{i1}(r,z,t)}{H^2(\tilde{\varepsilon}_i\bar{R})} \right]$
vertical ground motion P_V	$- \rho_F G_v(t) [H - z]$
weight of the fluid P_L	$\rho_F g [H - z]$

note : $F_{in}(r,z,t)$ and $Q_{in}(r,z,t)$ are given in Appendix B.

Table 2.2(2) 1st order fluid pressure components

pressure component due to	expression
vibration of the shell P_{d1}	$\displaystyle\sum_{n=-\infty}^{\infty}\sum_{i=1}^{\infty}\sum_{m=-\infty}^{\infty}\left\{j\omega^2\,\hat{\hat{A}}{}^{m}_{in}(r,z,t)\int_0^{2\pi}\eta'(\theta)\,e^{j(m-n)\theta}\,d\theta\right.$ $\displaystyle\left.-\,\omega^2\hat{A}^{m}_{in}(r,z,t)\int_0^{2\pi}\eta(\theta)\,e^{j(m-n)\theta}\,d\theta\right\}\int_0^{H}\bar{w}_m(\zeta,\omega)\cos(\mu_i\zeta)\,d\zeta\cdot e^{jn\theta}$
vibration of the plate P_{b1}	$\displaystyle\sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{\infty}\left\{j\omega^2\,\hat{\hat{B}}{}^{im}_{kn}(r,z,t)\int_0^{2\pi}\eta'(\theta)\,e^{j(m-n)\theta}\,d\theta\right.$ $\displaystyle\left.-\,\omega^2\hat{B}^{im}_{kn}(r,z,t)\int_0^{2\pi}\eta(\theta)\,e^{j(m-n)\theta}\,d\theta\right\}\int_0^{\bar{R}}\bar{W}_m(\zeta,\omega)J_m(\tilde{\varepsilon}_i\zeta)\,\zeta\,d\zeta\cdot e^{jn\theta}$
horizontal ground motion P_{h1}	$\displaystyle\sum_{n=-\infty}^{\infty}\sum_{i=1}^{\infty}\frac{\bar{G}_k(\omega)}{\pi}\cdot\frac{\sin(\mu_i H)\,F_{in}(r,z,t)}{\mu_i}\,\tilde{C}_{in}\,e^{jn\theta}$
rocking motion P_{rm1}	$\displaystyle-\sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{\omega^2\bar{q}(\omega)}{\pi}F_{kn}(r,z,t)\,e^{jn\theta}$ $\displaystyle*\left\{\frac{\mu_k H\sin(\mu_k H)+\cos(\mu_k H)-1}{\mu_k^2}\tilde{C}_{kn}\right.$ $\displaystyle\left.+\sum_{i=1}^{\infty}\frac{2J_2(\tilde{\varepsilon}_i\bar{R})}{\tilde{\varepsilon}_i\bar{R}\left(1-\dfrac{1}{(\tilde{\varepsilon}_i\bar{R})^2}\right)J_1(\tilde{\varepsilon}_i\bar{R})(\varepsilon_i^2+\mu_k^2)}\tilde{D}_{in}\right\}$
Note : F_{in}, $\hat{\hat{A}}{}^{m}_{in}$,$\hat{A}^{m}_{in}$,$\hat{\hat{B}}{}^{im}_{kn}$,$\hat{B}^{im}_{kn}$,$\tilde{C}_{in}$ and $\tilde{D}_{in}$ are given in Appendix B.	

For most liquids the compressibility effect is negligible. If the compressibility is neglected;

$$\tilde{\mu}_i \approx \mu_i \quad , \quad \tilde{\varepsilon}_i \approx \varepsilon_i \tag{2.33a,b}$$

The sloshing frequencies constitute the lowest range of the FSI spectrum. Therefore, the contribution from sloshing is insignificant, that is,

$$\frac{1}{\tilde{\omega}} = \frac{g}{\omega^2 H} << 1 \qquad (2.34)$$

The root of eqn.(2.14b) can be approximated by following equation by using eqn. (2.34);

$$\tilde{\mu}_i H \approx \frac{2i-1}{2} \pi \qquad (2.35)$$

The pressures at the tank wall can be written as follows assuming that the base plate is rigid.

$$P_d|_R = P_{d0}|_{\bar{R}} + \nu \left\{ \frac{\partial P_{d0}}{\partial r}\Big|_{\bar{R}} . \bar{R}\eta(\theta) + P_{d1}|_{\bar{R}} \right\} + O(\nu^2) \qquad (2.36a)$$

$$P_h|_R = P_{h0}|_{\bar{R}} + \nu \left\{ \frac{\partial P_{h0}}{\partial r}\Big|_{\bar{R}} . \bar{R}\eta(\theta) + P_{h1}|_{\bar{R}} \right\} + O(\nu^2) \qquad (2.36b)$$

$$P_{rm}|_R = P_{rm0}|_{\bar{R}} + \nu \left\{ \frac{\partial P_{rm0}}{\partial r}\Big|_{\bar{R}} . \bar{R}\eta(\theta) + P_{rm1}|_{\bar{R}} \right\} + O(\nu^2) \qquad (2.36c)$$

The pressure components in eqns. (2.36) can be derived by applying eqns. (2.33) and (2.35) to the equations in Table 2.2(1) and 2.2(2) and specifying $\bar{R}$ for r. Each pressure component is summarized in Table 2.3(1) through Table 2.3(3). In the equations in Table 2.3(1) to 2.3(3), the relative deformation of the tank wall is modified to the following expression,

$$\sum_{n=-\infty}^{\infty} \bar{w}_n(z,\omega)\, e^{jn\theta} = \sum_{n=0}^{\infty} \{\bar{w}_n^s(z,\omega) \sin n\theta + \bar{w}_n^c(z,\omega) \cos n\theta\} \qquad (2.37)$$

Table 2.3(1) Expressions of pressures acting upon the shell wall (0-th order components)

pressure components due to	Expressions
vibration of the shell $P_{d0}\|_{\bar{R}}$	$\sum_{n=0}^{\infty} \sum_{i=1}^{\infty} 2\omega^2 F_{in}(\bar{R},z,t)$ $* \{ \sin n\theta \int_0^H \bar{w}_n^s(\zeta,\omega) \cos(\mu_i \zeta)\, d\zeta$ $+ \cos n\theta \int_0^H \bar{w}_n^c(\zeta,\omega) \cos(\mu_i \zeta)\, d\zeta \}$
horizontal ground motion $P_{h0}\|_{\bar{R}}$	$-\sum_{i=1}^{\infty} 2\, \bar{G}_h(\omega) \cdot \frac{(-1)^{i+1}}{\mu_i} F_{i1}(\bar{R},z,t) \cos\theta$
rocking motion $P_{rm0}\|_{\bar{R}}$	$\sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \cos\theta$ $* \left\{ \frac{\mu_i H (-1)^{i+1} - 1}{(\mu_i H)^2} F_{i1}(\bar{R},z,t) - \frac{\bar{R}^2 J_2(\varepsilon_i \bar{R})}{(\varepsilon_i \bar{R}) H^2} Q_{i1}(\bar{R},z,t) \right\}$
vertical ground motion P_v	$- \rho_F G_v(t) (H - z)$
weight of the fluid P_L	$\rho_F g (H - z)$

where

$$F_{in}(\bar{R},z,t) \approx \frac{\rho_F I_n(\mu_i \bar{R}) \cos(\mu_i z)\, e^{-j\omega t}}{(\mu_i H)\, I_n'(\mu_i \bar{R})}$$

$$Q_{in}(\bar{R},z,t) \approx \frac{\rho_F}{(\varepsilon_i \bar{R}) \left(1 - \frac{n^2}{(\varepsilon_i \bar{R})^2} \right) J_n(\varepsilon_i \bar{R})} \{ \sinh(\varepsilon_i z) - \tanh(\varepsilon_i H) \cosh(\varepsilon_i z) \} \cdot e^{-j\omega t}$$

Table 2.3(2) Expressions of pressures acting upon the shell wall (1st order components (I))

pressure components due to	Expressions
vibration of the shell $\frac{\partial P_{d0}}{\partial r}\Big\vert_{\bar{R}}\,\bar{R}\eta(\theta)$	$\sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\frac{2\rho_F\omega^2\bar{R}}{H}\left\{\sin n\theta\int_0^H \bar{w}_n^s(\zeta,\omega)\cos(\mu_i\zeta)\,d\zeta\right.$ $\left.+\cos n\theta\int_0^H \bar{w}_n^c(\zeta,\omega)\cos(\mu_i\zeta)\,d\zeta\right\}\cos(\mu_i z)\cdot\eta(\theta)\,e^{-j\omega t}$
horizontal ground motion $\frac{\partial P_{h0}}{\partial r}\Big\vert_{\bar{R}}\cdot\bar{R}\eta(\theta)$	$\sum_{i=1}^{\infty}\frac{(-1)^i\cdot 2\rho_F\bar{G}_h(\omega)\bar{R}}{\mu_i H}\cdot\cos(\mu_i z)\cdot\eta(\theta)\cdot\cos\theta\cdot e^{-j\omega t}$ $=-\sum_{i=1}^{\infty}2\,\bar{G}_h(\omega)\frac{(-1)^{i+1}}{\mu_i\,l_{1i}^{\langle 1\rangle}}F_{i1}(\bar{R},z,t)\,\eta(\theta)\cos\theta$
rocking motion $\cdot\bar{R}\eta(\theta)\frac{\partial P_{rm0}}{\partial r}\Big\vert_{\bar{R}}$	$\sum_{i=1}^{\infty}\frac{2\rho_F\omega^2\,\bar{q}(\omega)\bar{R}\,\{(-1)^{i+1}\mu_i H-1\}}{\mu_i^2 H}\cdot\cos(\mu_i z)\cdot\eta(\theta)\cdot\cos\theta\cdot e^{-j\omega t}$ $=\sum_{i=1}^{\infty}2\omega^2\,H^2\,\bar{q}(\omega)\cdot\,\eta(\theta)\cos\theta\cdot\frac{\mu_i H(-1)^{i+1}-1}{(\mu_i^2 H^2)\,l_{1i}^{\langle 1\rangle}}F_{i1}(\bar{R},z,t)$

Table 2.3(3) Expressions of pressures acting upon the shell wall (1st order components (II))

Pressure component due to	Expressions
vibration of the shell $P_{d1}\vert_{\tilde{R}}$	$\frac{1}{2}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\omega^2 F_{0k}(z,t)\cdot\mu_k$ $*\{m\,I_{mk}^{<1>}(H_{m0}^{'cc}\bar{W}_{mk}^{s}-H_{m0}^{'sc}\bar{W}_{mk}^{c})-I_{mk}^{<2>}(H_{m0}^{cc}\bar{W}_{mk}^{c}+H_{m0}^{sc}\bar{W}_{mk}^{s})\}$ $+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\omega^2 F_{nk}(z,t)\cdot\mu_k$ $*[m\,I_{mk}^{<1>}\{(H_{mn}^{'cc}\bar{W}_{mk}^{s}-H_{mn}^{'sc}\bar{W}_{mk}^{c})\cos n\theta$ $+(H_{mn}^{'cs}\bar{W}_{mk}^{s}-H_{mn}^{'ss}\bar{W}_{mk}^{c})\sin n\theta\}$ $-I_{mk}^{<2>}\{(H_{mn}^{cc}\bar{W}_{mk}^{c}+H_{mn}^{sc}\bar{W}_{mk}^{s})\cos n\theta$ $+(H_{mn}^{cs}\bar{W}_{mk}^{c}+H_{mn}^{ss}\bar{W}_{mk}^{s})\sin n\theta\}]$
horizontal ground motion $P_{h1}\vert_{\tilde{R}}$	$\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\bar{G}_h(\omega)F_{0k}(z,t)\{(I_{1k}^{<1>}-1)H_{10}^{'sc}+I_{1k}^{<2>}H_{10}^{cc}\}$ $+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+1}\bar{G}_h(\omega)F_{nk}(z,t)$ $*\{(I_{1k}^{<1>}-1)(H_{1n}^{'sc}\cos n\theta+H_{1n}^{'ss}\sin n\theta)$ $+I_{1k}^{<2>}(H_{1n}^{cc}\cos n\theta+H_{1n}^{cs}\sin n\theta)\}$
rocking motion $P_{rm1}\vert_{\tilde{R}}$	$-\frac{1}{2}\sum_{k=1}^{\infty}\omega^2\bar{q}(\omega)\mu_k F_{0k}(z,t)$ $*[\frac{(-1)^{k+1}\mu_k H-1}{\mu_k^2}\{(I_{1k}^{<1>}-1)H_{10}^{'sc}+I_{1k}^{<2>}H_{10}^{cc}\}$ $+\sum_{i=1}^{\infty}\frac{J_i^{<2>}}{\varepsilon_i^2+\mu_k^2}(H_{10}^{'sc}+J_i^{<1>}H_{10}^{cc})]$ $-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\omega^2\bar{q}(\omega)\mu_k F_{nk}(z,t)$ $*[\frac{(-1)^{k+1}\mu_k H-1}{\mu_k^2}\{(I_{1k}^{<1>}-1)(H_{1n}^{'sc}\cos n\theta+H_{1n}^{'ss}\sin n\theta)$ $+I_{1k}^{<2>}(H_{1n}^{cc}\cos n\theta+H_{1n}^{cs}\sin n\theta)\}$ $+\sum_{i=1}^{\infty}\frac{J_i^{<2>}}{\varepsilon_i^2+\mu_k^2}\{(H_{1n}^{'sc}\cos n\theta+H_{1n}^{'ss}\sin n\theta)$ $+J_i^{<1>}(H_{1n}^{cc}\cos n\theta+H_{1n}^{cs}\sin n\theta)\}]$

where $F_{nk}(z,t)$, $I_{mk}^{<1>}$, $I_{mk}^{<2>}$, $J_{i}^{<1>}$, $J_{i}^{<2>}$, $H_{mn}^{'cc}$, $H_{mn}^{'sc}$, $H_{mn}^{'cs}$, $H_{mn}^{'ss}$, H_{mn}^{cc}, H_{mn}^{sc}, H_{mn}^{cs}, H_{mn}^{ss}, $\overline{W}_{mk}^{s}$ and $\overline{W}_{mk}^{c}$ are given as follows ;

$$F_{nk}(z,t) = \frac{2\rho_F I_n(\mu_k\overline{R})}{\pi H \mu_k^2 \; I_n'(\mu_k\overline{R})} \cos(\mu_k z)\, e^{-j\omega t}$$

$$I_{mk}^{<1>} = \frac{I_m(\mu_k\overline{R})}{\mu_k\overline{R} \; I_m'(\mu_k\overline{R})}$$

$$I_{mk}^{<2>} = \frac{\mu_k\overline{R} \; I_m''(\mu_k\overline{R})}{I_m'(\mu_k\overline{R})}$$

$$J_i^{<1>} = \frac{\varepsilon_i^2\overline{R}^2 \; J_1''(\varepsilon_i\overline{R})}{J_1(\varepsilon_i\overline{R})}$$

$$J_i^{<2>} = \frac{2J_2(\varepsilon_i\overline{R})}{\varepsilon_i\overline{R}\left(1 - \frac{1}{(\varepsilon_i\overline{R})^2}\right)J_1(\varepsilon_i\overline{R})}$$

$$H_{mn}^{'cc} = \int_0^{2\pi} \eta'(\theta) \; \cos m\theta \; \cos n\theta \; d\theta$$

$$H_{mn}^{'sc} = \int_0^{2\pi} \eta'(\theta) \; \sin m\theta \; \cos n\theta \; d\theta$$

$$H_{mn}^{'cs} = \int_0^{2\pi} \eta'(\theta) \; \cos m\theta \; \sin n\theta \; d\theta$$

$$H_{mn}^{'ss} = \int_0^{2\pi} \eta'(\theta) \; \sin m\theta \; \sin n\theta \; d\theta$$

$$H_{mn}^{cc} = \int_0^{2\pi} \eta(\theta) \cos m\theta \cos n\theta \, d\theta$$

$$H_{mn}^{sc} = \int_0^{2\pi} \eta(\theta) \sin m\theta \cos n\theta \, d\theta$$

$$H_{mn}^{cs} = \int_0^{2\pi} \eta(\theta) \cos m\theta \sin n\theta \, d\theta$$

$$H_{mn}^{ss} = \int_0^{2\pi} \eta(\theta) \sin m\theta \sin n\theta \, d\theta$$

$$\bar{W}_{mk}^{s} = \int_0^{H} \bar{w}_{m}^{s} \cos(\mu_k z) \, dz$$

$$\bar{W}_{mk}^{c} = \int_0^{H} \bar{w}_{m}^{c} \cos(\mu_k z) \, dz$$

3. Variational Formulation

The expressions of virtual work principle for a structure subjected to hydrodynamic loads due to an inviscid and incompressible fluid are given as [2] ;

Expression by total displacements (Initial state)

$$\int_{\Omega} \rho \delta u_i \, \ddot{u}_i \, d\Omega - \int_{\Gamma} \delta u_i \, I_0^{ad}(x,t) \, n_i \, d\Gamma + \int_{\Omega} \delta u_{i,j} \, \tau_{ij} \, d\Omega$$

$$= \int_{\Gamma} \delta u_i \, P_F(x,t) \, n_i \, d\Gamma + \int_{\Gamma} \delta u_i \, P_L(x) \, n_i \, d\Gamma + \delta W_o^{ext} \tag{3.1}$$

Expression by incremental displacements (Purturbed state)

$$\int_{\Omega} \rho \delta u_i \, \Delta\ddot{u}_i \, d\Omega - \int_{\Gamma} \delta u_i \, I_1^{ad}(x,t) \, n_i \, d\Gamma + \int_{\Omega} \delta u_{i,j} \, C^{t}_{ijkm} \, \Delta u_{k,m} \, d\Omega$$

$$+ \int_{\Omega} \delta u_{i,j} \, \overset{0}{\tau}_{jl} \, \Delta u_{i,l} \, d\Omega - \int_{\Gamma} \delta u_i \, \{ P_F(x,t) + P_L(x) \} \, \Delta n_i \, d\Gamma = 0 \tag{3.2}$$

where Ω, Γ, n_i, x_i and u_i denote the domain of the structure, the fluid-structure interface boundary, the outward normal to the structure surface, the spatial coordinates and the structural displacement components, respectively. A superposed dot and "Δ" designates the temporal derivative and the incremental value, respectively. ρ,I_0^{ad} and I_1^{ad} are the structural mass dencity and added fluid inertia,respectively,and

$$I_0^{ad}(x,t) \equiv \frac{\partial P_d(\ddot{u},x,t)}{\partial \ddot{u}_j} \, \ddot{u}_j \tag{3.3a}$$

$$I_1^{ad}(x,t) \equiv \frac{\partial P_d(\ddot{u},x,t)}{\partial \ddot{u}_j} \, \Delta\ddot{u}_j \tag{3.3b}$$

τ_{ij} is the Cauchy stress tensor, and

$$\Delta\tau_{ij} = \Delta\overset{m}{\tau}_{ij} + \Delta\overset{0}{\tau}_{ij} \tag{3.4a}$$

and

$$\Delta\overset{m}{\tau}_{ij} = \overset{t}{C}_{ijkm}\,\Delta u_{k,m} \tag{3.4b}$$

$$\Delta\overset{0}{\tau}_{ij} = \delta_{ik}\,\overset{0}{\tau}_{jm}\,\Delta u_{k,m} \tag{3.4c}$$

where $\Delta\tau_{ij}$, $\Delta\overset{m}{\tau}_{ij}$,$\Delta\overset{0}{\tau}_{ij}$,$\overset{t}{C}_{ijkm}$ and δ_{ik} are the linearized Cauchy stress tensor, the material response part and the part by the initial stress effect of $\Delta\tau_{ij}$, the material response tensor and Kronecker Delta, respectively.

The pressures acting on the tank wall are tabulated in tables 2.3(1)-(3) in the previous chapter, and

$$\begin{aligned} P_F &= P_h + P_{rm} + P_v \\ &= (P_{h0}|_{\bar{R}} + P_{rm0}|_{\bar{R}} + P_v) + \nu \left\{ \left(\frac{\partial P_{h0}}{\partial r}|_{\bar{R}} + \frac{\partial P_{rm0}}{\partial r}|_{\bar{R}}\right)\bar{R}\eta + P_{h1}|_{\bar{R}} + P_{rm1}|_{\bar{R}} \right\} + O(\nu^2) \end{aligned} \tag{3.5}$$

and

$$\Delta n_i = \Delta u_{k,k}\, n_i - \Delta u_{m,i}\, n_m \tag{3.6}$$

Before applying eqn.(3.2) to the thin structure, some preparations are needed to integrate in the domain and on the boundary of the structure. Geometrical relations for imperfect cylindrical shells are given as [6] (see Fig. 2.1 in the previous chapter)

$$R_s = \frac{(R'^2+R^2)^{3/2}}{2R'^2+R^2-RR''} \tag{3.7}$$

$$dl = \sqrt{R'^2+R^2}\,d\theta \tag{3.8}$$

and

$$\frac{\partial\psi}{\partial\theta} = \frac{2R'^2+R^2-RR''}{R'^2+R^2} \tag{3.9}$$

where R_s, dl are the radius of the curvature of the tank and the differential line element of circumferential direction, respectively.

By applying eqn. (2.1) to eqns. (3.7) to (3.9) and assuming that the parameter ν in eqn. (2.1) is small enough compared to unit, the following approximations can be obtained.

$$R_s \approx \{ 1 + \nu (\eta + \eta'') \} \bar{R} \tag{3.10}$$

$$dl \approx (1 + \nu \eta) \bar{R} \, d\theta \tag{3.11}$$

and

$$\frac{\partial \psi}{\partial \theta} \approx 1 - \nu \eta'' \tag{3.12}$$

The components of the displacements at a distance x from the midsurface of the shell are defined as (refer to Fig.3.1);

$$u_x = u - x \frac{\partial w}{\partial z} \qquad \text{(axial component)} \tag{3.13a}$$

$$v_x = v - \frac{x}{R_s} \frac{\partial w}{\partial \psi} \qquad \text{(circumferential tangential component)}$$

$$\approx v - \frac{x}{\bar{R}} \frac{\partial w}{\partial \theta} + \nu \frac{x}{\bar{R}} \eta \frac{\partial w}{\partial \theta} \tag{3.13b}$$

$$w_x = w \qquad \text{(normal direction component)} \tag{3.13c}$$

where u, v and w are displacements in the axial, circumferential tangential and normal directions at the midsurface, respectively.

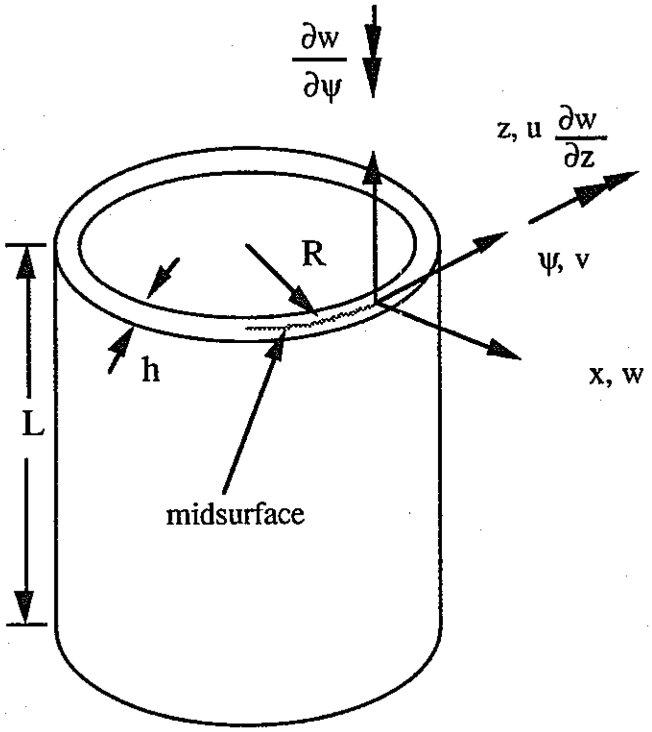


Fig. 3.1 Shell geometry and coordinate system

In the case of thin shells the plane stress assumption is introduced. The stress and the strain tensors are reduced as:

$$\tau = [\ \sigma_z, \sigma_\psi, \sigma_{z\psi}, \sigma_n{=}0, \sigma_{n\psi}, \sigma_{nz}]^T \qquad (3.14)$$

and

$$\varepsilon = [\ \varepsilon_z, \varepsilon_\psi, 2\varepsilon_{z\psi}, \varepsilon_n, 2\varepsilon_{n\psi}{=}0, 2\varepsilon_{nz}{=}0]^T \qquad (3.15)$$

In the cylindrical coordinates the domain and the boundary integrals become

$$\int_\Omega d\Omega = \int_0^L \int \int (R_s+x)\ dx\ d\psi\ dz$$

$$\approx \int_0^L \int_0^{2\pi} \int_{-h/2}^{h/2} (1+\frac{x}{\bar{R}}+\nu\eta)\,\bar{R}\ dx\ d\theta\ dz \qquad (3.16)$$

and

$$\int_{\Gamma} d\Gamma = \int_{0}^{H} \int dl\,dz$$

$$\approx \int_{0}^{H} \int_{0}^{2\pi} (1+\nu\eta)\,\bar{R}\,d\theta\,dz \qquad (3.17)$$

After eqns. (3.13) are substituted into eqn. (3.2) and integrated in the domain or on the boundary, the individual terms for the linearized variational equation of the shell are obtained.

(i) The structural inertia term : the first term of eqn.(3.2)

$$\int_{\Omega} \rho\,(\,\delta u_x\,\Delta\ddot{u}_x + \delta v_x\,\Delta\ddot{v}_x + \delta w_x\,\ddot{w}_x\,)\,d\Omega = I_t + I_{tr} + I_r \qquad (3.18)$$

where

$$I_t \approx \int_{0}^{L} \int_{0}^{2\pi} \rho h\,\bar{R}\,(\,\delta u\Delta\ddot{u} + \delta v\Delta\ddot{v} + \delta w\Delta\ddot{w}\,)\,(1+\nu\eta)\,d\theta\,dz \qquad (3.19a)$$

$$I_{tr} \approx -\int_{0}^{L} \int_{0}^{2\pi} \frac{\rho h^3}{12}\,\{\,\delta u\Delta\ddot{w}_{,z} + \delta w_{,z}\Delta\ddot{u} + \frac{1}{\bar{R}}(\delta v\Delta\ddot{w}_{,\theta} + \delta w_{,\theta}\Delta\ddot{v})\}\,d\theta\,dz \qquad (3.19b)$$

$$I_r \approx \int_{0}^{L} \int_{0}^{2\pi} \frac{\rho h^3\bar{R}}{12}\,\{\,\delta w_{,z}\Delta\ddot{w}_{,z} + \frac{1}{\bar{R}^2}\,\delta w_{,\theta}\Delta\ddot{w}_{,\theta} + \nu\eta\,(\delta w_{,z}\Delta\ddot{w}_{,z} - \frac{1}{\bar{R}^2}\,\delta w_{,\theta}\Delta\ddot{w}_{,\theta})\,\}\,d\theta\,dz \qquad (3.19c)$$

I_t and I_r are the translational and rotational parts, respectively. In the subsequent analysis, the higher order inertias I_{tr} and I_r are neglected.

(ii) The dynamic pressure term: the second term of eqn. (3.2)

Since the surface normal vector direction to the structure at the interface coincides with that of the displacement component w;

$$\mathbf{n} = [\,0\;0\;1\,]^T \tag{3.20}$$

From the equations in Tables 2.3(1)~(3), the dynamic pressures are functions of the normal acceleration. Therefore, eqn. (3.3b) yields to

$$I_1^{ad}(\mathbf{x},t) = \left\{\Delta\,(P_{d0}|_{\bar{R}}) + \nu\cdot\Delta\,(\frac{\partial P_{d0}}{\partial r}|_{\bar{R}}\,\overline{R}\eta + P_{d1}|_{\bar{R}})\right\} \tag{3.21}$$

and hence

$$\int_{\Gamma} \delta u_i\; I_1^{ad}(\mathbf{x},t)\, n_i\; d\Gamma$$
$$\approx \int_0^H \int_0^{2\pi} \delta w\; \left\{\Delta\;(P_{d0}|_{\bar{R}}) + \nu\cdot\Delta(P_{d0}|_{\bar{R}}\cdot\eta + \frac{\partial P_{d0}}{\partial r}|_{\bar{R}}\cdot\overline{R}\eta + P_{d1}|_{\bar{R}})\right\}\overline{R}d\theta\; dz \tag{3.22}$$

(iii) The material response term : the third term in eqn. (3.2)

With the plane stress assumption, the material response term for a linear elastic material can be given in terms of the total strains:

$$I_m \equiv \int_{\Omega} \delta\varepsilon_{ij}^{x}\; C_{ijkm}^{t}\; \Delta\varepsilon_{km}^{x}\; d\Omega \tag{3.23}$$

where

$$\varepsilon_z^x = \varepsilon_z - x\;K_z \tag{3.24a}$$

$$\varepsilon_\psi^x = \varepsilon_\psi - x\;K_\psi \tag{3.24b}$$

$$2\varepsilon_{z\psi}^x = 2\varepsilon_{z\psi} - 2x\;K_{z\psi} \tag{3.24c}$$

ε^x_{ij}, ε_{ij} denote the total strains and the midsurface strains, respectively. K_z and K_ψ are the changes of curvature of the midsurface and $K_{z\psi}$ is the twist of the midsurface.

The strain-displacement and the curvature-displacement relationships are given as

$$\varepsilon_z = u_{,z} \tag{3.25a}$$

$$\varepsilon_\psi = \frac{1}{R_s}(v_{,\psi} + w) = \varepsilon_{\psi 0} + \nu\, \varepsilon_{\psi 1} \tag{3.25b}$$

$$2\varepsilon_{z\psi} = \frac{1}{R_s} u_{,\psi} + v_{,z} = 2\varepsilon_{z\psi 0} + 2\nu\, \varepsilon_{z\psi 1} \tag{3.25c}$$

and

$$K_z = w_{,zz} \tag{3.26a}$$

$$K_\psi = \frac{1}{R_s} \left\{ \left(\frac{1}{R_s} w_{,\psi}\right)_{,\psi} - \frac{1}{R_s} v_{,\psi} \right\} = K_{\psi 0} + \nu\, K_{\psi 1} \tag{3.26b}$$

$$2K_{z\psi} = \frac{2}{R_s} (w_{,z\psi} - v_{,z}) = 2K_{z\psi 0} + 2\nu\, K_{z\psi 1} \tag{3.26c}$$

where

$$\varepsilon_{\psi 0} = \frac{1}{\bar{R}}(v_{,\theta} + w) \tag{3.27a}$$

$$\varepsilon_{\psi 1} = -\frac{1}{\bar{R}}(\eta v_{,\theta} + (\eta+\eta'')w) \tag{3.27b}$$

$$2\varepsilon_{z\psi 0} = \frac{1}{\bar{R}} u_{,\theta} + v_{,z} \tag{3.27c}$$

$$2\varepsilon_{z\psi 1} = -\frac{1}{\bar{R}} \eta u_{,\theta} \tag{3.27d}$$

and

$$K_{\psi 0} = \frac{1}{\bar{R}^2} (w_{,\theta\theta} - v_{,\theta}) \tag{3.28a}$$

$$K_{\psi 1} = -\frac{1}{\bar{R}^2} \{2\eta w_{,\theta\theta} + \eta' w_{,\theta} - (2\eta+\eta'')v_{,\theta}\} \tag{3.28b}$$

$$2K_{z\psi 0} = \frac{2}{\bar{R}} (w_{,z\theta} - v_{,z}) \tag{3.28c}$$

$$2K_{z\psi 1} = -\frac{2}{\bar{R}} \{\eta w_{,z\theta} - (\eta+\eta'')v_{,z}\} \tag{3.28d}$$

By substituting eqns. (3.25) and (3.26) into eqn. (3.23), the material response term can be decomposed to three parts ;

$$I_m \approx I_{m1} + I_{m2} + I_{m3} \tag{3.29}$$

where

$$I_{m1} = \frac{Eh\bar{R}}{1-\bar{\nu}^2} \int_0^L \int_0^{2\pi} \{ \delta\varepsilon_0^T C \, \Delta\varepsilon_0 + \nu \, (\eta\delta\varepsilon_0^T C \, \Delta\varepsilon_0 + \delta\varepsilon_0^T C \, \Delta\varepsilon_1 + \delta\varepsilon_1^T C \, \Delta\varepsilon_0)\} \, d\theta \, dz \tag{3.30}$$

$$I_{m2} = -\frac{Eh^3}{12(1-\bar{\nu}^2)} \int_0^L \int_0^{2\pi} (\delta\varepsilon_0^T C \, \Delta K_0 + \delta K_0^T C \, \Delta\varepsilon_0) \, d\theta \, dz \tag{3.31a}$$

$$I_{m3} = \frac{Eh^3\bar{R}}{12(1-\bar{\nu}^2)} \int_0^L \int_0^{2\pi} \{ \delta K_0^T C \, \Delta K_0 + \nu \, (\eta\delta K_0^T C \, \Delta K_0, + \delta K_0^T C \, \Delta K_1 + \delta K_1^T C \, \Delta K_0)\} \, d\theta \, dz \tag{3.31b}$$

E and $\bar{\nu}$ are the Young's modulus and the Poisson's ratio, respectively, and

$$C = \begin{bmatrix} 1 & \bar{\nu} & 0 \\ \bar{\nu} & 1 & 0 \\ 0 & 0 & \frac{1-\bar{\nu}}{2} \end{bmatrix} \tag{3.32}$$

$$\varepsilon_0 = \begin{Bmatrix} u,_z \\ \frac{1}{\bar{R}}(v,_\theta + w) \\ \frac{1}{\bar{R}} u,_\theta + v,_z \end{Bmatrix}, \quad \varepsilon_1 = \begin{Bmatrix} 0 \\ -\frac{1}{\bar{R}}(\eta v,_\theta + (\eta+\eta'')w) \\ -\frac{1}{\bar{R}} \eta u,_\theta \end{Bmatrix} \tag{3.33a,b}$$

$$\mathbf{K}_0=\left\{\begin{matrix} w_{,zz} \\ \frac{1}{\bar{R}^2}(w_{,\theta\theta}-v_{,\theta}) \\ \frac{2}{\bar{R}}(w_{,z\theta}-v_{,z}) \end{matrix}\right\},\ \mathbf{K}_1=\left\{\begin{matrix} 0 \\ -\frac{1}{\bar{R}^2}(2\eta w_{,\theta\theta}+\eta' w_{,\theta}-(2\eta+\eta'')v_{,\theta}) \\ -\frac{2}{\bar{R}}(\eta w_{,z\theta}-(\eta+\eta'')v_{,z}) \end{matrix}\right\} \tag{3.34a,b}$$

(iv) The initial stress term: the fourth term in eqn. (3.2)

Assuming that the initial state of stress is governed by the membrane stresses and neglecting the higher order curvature terms, the initial stress term can be written as

$$\int_\Omega \delta u_{i,j}\,\overset{0}{\tau}_{jm}\,\Delta u_{i,m}\,d\Omega=\frac{1}{h}\int_0^L\int_0^{2\pi}\int_{-2/h}^{2/h}\delta\bar{\varepsilon}^{\,T}\mathbf{N}^0\Delta\bar{\varepsilon}\,(1+\frac{x}{\bar{R}}+\nu\eta)\,\bar{R}dx\,d\theta\,dz$$

$$=\int_0^L\int_0^{2\pi}\delta\bar{\varepsilon}^{\,T}\mathbf{N}^0\Delta\bar{\varepsilon}\,(1+\nu\eta)\,\bar{R}\,d\theta\,dz \tag{3.35}$$

where

$$\bar{\varepsilon}=[\,u_{,z}\ \ v_{,z}\ \ w_{,z},\ \frac{1}{R_s}u_{,\psi},\ \frac{1}{R_s}(v_{,\psi}+w),\ \frac{1}{R_s}(w_{,\psi}-v)\,]^T \tag{3.36}$$

and

$$\mathbf{N}=\begin{bmatrix} N_z^0\,\mathbf{I} & N_{\psi z}^0\,\mathbf{I} \\ N_{\psi z}^0\,\mathbf{I} & N_\psi^0\,\mathbf{I} \end{bmatrix} \tag{3.37}$$

I is a 3x3 identity matrix and the initial membrane forces are defined as

$$N_a^0(z,\theta,t)\equiv h\sigma_a\,(z,\theta,t) \qquad\qquad a=z,\ \psi\text{ or }(z\psi) \tag{3.38}$$

By applying eqns.(3.10) and (3.12) to eqn. (3.36), eqn. (3.35) can be rewritten as

$$\int_\Omega \delta u_{i,j}\,\overset{0}{\tau}_{jm}\,\Delta u_{i,m}\,d\Omega\approx\int_0^L\int_0^{2\pi}\bar{R}\,\{\delta\bar{\varepsilon}_0^T\,N_0^0\,\Delta\bar{\varepsilon}_0$$

$$+\nu\,(\delta\bar{\varepsilon}_0^T\,N_1^0\ \Delta\bar{\varepsilon}_0+\eta\delta\bar{\varepsilon}_0^T\,N_0^0\ \Delta\bar{\varepsilon}_0+\delta\bar{\varepsilon}_0^T\,N_0^0\ \Delta\bar{\varepsilon}_1+\delta\bar{\varepsilon}_1^T\,N_0^0\,\Delta\bar{\varepsilon}_0)\}\,d\theta\,dz \tag{3.39}$$

where

$$\bar{\varepsilon}=\bar{\varepsilon}_0+\nu\bar{\varepsilon}_1 \tag{3.40a}$$

$$\mathbf{N}^0=\mathbf{N}_0^0+\nu\,\mathbf{N}_1^0 \tag{3.40b}$$

and

$$\bar{\varepsilon}_0 = \left\{ \begin{array}{c} u_{,z} \\ v_{,z} \\ w_{,z} \\ \frac{1}{\bar{R}} u_{,\theta} \\ \frac{1}{\bar{R}} (v_{,\theta}+w) \\ \frac{1}{\bar{R}} (w_{,\theta}-v) \end{array} \right\}, \quad \bar{\varepsilon}_1 = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ -\frac{1}{\bar{R}} \eta u_{,\theta} \\ -\frac{1}{\bar{R}} (\eta v_{,\theta}+(\eta+\eta'')w) \\ -\frac{1}{\bar{R}} (\eta w_{,\theta}-(\eta+\eta'')v) \end{array} \right\} \qquad (3.41a,b)$$

(v.) The load correction term :the fifth term of eqn. (3.2)

$$- I_n = - \int_{\Gamma} \delta u_i \ \{P_F(\mathbf{x},t) + P_L(\mathbf{x})\} \ \Delta n_i \, d\Gamma \qquad (3.42)$$

The change of the normal vector during incremental deformation is given as

$$\Delta n_1 = - \Delta w_{,z} \qquad (3.43a)$$

$$\Delta n_2 = - \frac{1}{R_s} \Delta w_{,\psi} \approx - \frac{1}{\bar{R}} (1-\nu\eta) \ \Delta w_{,\theta} \qquad (3.43b)$$

$$\begin{aligned} \Delta n_3 &= \Delta u_{,z} + \frac{1}{R_s} (\Delta v_{,\psi}+\Delta w) \\ &\approx \Delta u_{,z} + \frac{1}{\bar{R}} (\Delta v_{,\theta}+\Delta w) - \nu \frac{1}{\bar{R}} \{\eta \Delta v_{,\theta}+ (\eta+\eta'')\Delta w\} \end{aligned} \qquad (3.43c)$$

The total pressure acting upon the shell wall can be redefined as follows from Tables 2.3(1)~(3).

$$\begin{aligned} P_F + P_L &= P_L(z) + P_v(z,t) + P_{l0}(z,t) \cos\theta + \nu \ P_{l1}(z,\theta,t) \\ &\equiv P_0^W + \nu \ P_1^W \end{aligned} \qquad (3.44)$$

By substituting eqns. (3.43) and (3.44) into eqn. (3.42), eqn.(3.42) can be rewritten as

$$I_n \approx \int_0^L \int_0^{2\pi} \Big[- \delta u \; P_0^w \, \Delta w_{,z} - \frac{1}{\bar{R}} \delta v P_0^w \, \Delta w_{,\theta} + \delta w \; P_0^w \, (\Delta u_{,z} + \frac{1}{\bar{R}} \Delta v_{,\theta} + \frac{1}{\bar{R}} \Delta w)$$

$$+ \nu \, \{ -\delta u \, (\eta P_0^w + P_1^w) \, \Delta w_{,z} - \frac{1}{\bar{R}} \delta v P_1^w \, \Delta w_{,\theta} + \frac{1}{\bar{R}} \delta w P_1^w \, \Delta v_{,\theta}$$

$$+ \delta w \, (\eta P_0^w + P_1^w) \, \Delta u_{,z} - \frac{1}{\bar{R}} \delta w (\eta'' P_0^w - P_1^w) \, \Delta w \} \Big] \, \bar{R} \, d\theta \, dz \qquad (3.45)$$

Further, eqn. (3.45) can be decomposed into symmetric and skew-symmetric parts by introducing integration by parts.

$$I_n = \frac{1}{2} \int_0^L \int_0^{2\pi} \{ (\delta u_{,z} \; P_0^w \, \Delta w + \delta w \; P_0^w \, \Delta u_{,z}) - (\delta u \; P_0^w \, \Delta w_{,z} + \delta w_{,z} \; P_0^w \, \Delta u)$$

$$+ \frac{1}{\bar{R}} (\delta v_{,\theta} \; P_0^w \, \Delta w + \delta w \; P_0^w \, \Delta v_{,\theta}) - \frac{1}{\bar{R}} (\delta v \; P_0^w \, \Delta w_{,\theta} + \delta w_{,\theta} \; P_0^w \, \Delta v) + \frac{2}{\bar{R}} \delta w \; P_0^w \, \Delta w$$

$$+ \underbrace{(\delta u \; P_{0,z}^w \, \Delta w - \delta w \; P_{0,z}^w \, \Delta u)}_{\text{skew-symmetric}} + \frac{1}{\bar{R}} \underbrace{(\delta v \; P_{0,\theta}^w \, \Delta w - \delta w \; P_{0,\theta}^w \, \Delta v)}_{\text{skew-symmetric}} \} \bar{R} \, d\theta \, dz$$

$$+ \frac{\nu}{2} \int_0^H \int_0^{2\pi} \Big[\{ \delta u_{,z} \, (\eta P_0^w + P_1^w) \, \Delta w + \delta w \, (\eta P_0^w + P_1^w) \, \Delta u_{,z} \}$$

$$- \{ \delta u \, (\eta P_0^w + P_1^w) \, \Delta w_{,z} + \delta w_{,z} \, (\eta P_0^w + P_1^w) \, \Delta u \}$$

$$+ \frac{1}{\bar{R}} \{ (\delta v_{,\theta} \; P_1^w \, \Delta w + \delta w \; P_1^w \, \Delta v_{,\theta}) - (\delta v \; P_1^w \, \Delta w_{,\theta} + \delta w_{,\theta} \; P_1^w \, \Delta v) \}$$

$$- \frac{2}{\bar{R}} \delta w \, (\eta'' P_0^w - P_1^w) \, \Delta w$$

$$+ \underbrace{\{ \delta u \, (\eta P_0^w + P_1^w)_{,z} \, \Delta w - \delta w \, (\eta P_0^w + P_1^w)_{,z} \, \Delta u \}}_{\text{skew-symmetric}}$$

$$+ \frac{1}{\bar{R}} \underbrace{(\delta v \; P_{1,\theta}^w \, \Delta w - \delta w \; P_{1,\theta}^w \, \Delta v)}_{\text{skew-symmetric}} \Big] \bar{R} \, d\theta \, dz + I_\theta + I_z \qquad (3.46)$$

where

$$I_\theta = \frac{1}{2} \int_0^{2\pi} \left[(\delta w P_0^w \Delta u - \delta u P_0^w \Delta w) + \nu \{ \delta w (\eta P_0^w + P_1^w) \Delta u - \delta u (\eta P_0^w + P_1^w) \Delta w \} \right] \bar{R} \Big|_0^H d\theta \qquad (3.47a)$$

$$I_z = \frac{1}{2} \int_0^H \{ (\delta w P_0^w \Delta v - \delta v P_0^w \Delta w) + \nu (\delta w P_1^w \Delta v - \delta v P_1^w \Delta w) \} \Big|_0^{2\pi} dz \qquad (3.47b)$$

The following boundary conditions should be satisfied :

$P_0^w(z{=}H, \theta, t) = P_1^w(z{=}H, \theta, t) = 0$ ~ (free surface) (3.48a)

$\Delta w(z{=}0, \theta, t) = \Delta u(z{=}0, \theta, t) = 0$ ~ (cantilevered shell at the bottom)

or

$\delta w(z{=}0, \theta, t) = \delta u(z{=}0, \theta, t) = 0$ (3.49b)

and

$\delta w P_0^w \Delta v \,\big|_{\theta=0} = \delta w P_0^w \Delta v \,\big|_{\theta=2\pi}$ and so on ~(continuity condition) (3.49c)

Consequently,

$$I_\theta = I_z = 0 \qquad (3.50)$$

Furthermore,eqn. (3.42) can be simplified by neglecting the skew-symmetric parts in eqn. (3.46).

$$I_n \approx \frac{1}{2} \int_0^H \int_0^{2\pi} \delta \varepsilon^{*T} \{ P_0^w \mathbf{P}^{(0)} + \nu (P_1^w \mathbf{P}^{(0)} + P_0^w \mathbf{P}^{(1)}) \} \Delta \varepsilon^* \bar{R} \, d\theta \, dz \qquad (3.51)$$

where

$$\varepsilon^* = [\, u \,,\ v \,,\ w \,,\ u_{,z} \,,\ w_{,z} \,,\ \frac{1}{\bar{R}} v_{,\theta} \,,\ \frac{1}{\bar{R}} w_{,\theta} \,]^T \qquad (3.52)$$

$$\mathbf{P}^{(0)}=\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & -1 \\ & & \frac{2}{\bar{\bar{R}}} & 1 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & \text{sym.} & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix} \tag{3.53a}$$

$$\mathbf{P}^{(1)}=\begin{bmatrix} 0 & 0 & 0 & 0 & -\eta & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \frac{-2\eta''}{\bar{\bar{R}}} & \eta & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & \text{sym.} & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix} \tag{3.53b}$$

(vi) The expression by total displacements ; eqn. (3.1)

The structural inertia and the fluid added inertia are the same as those given in eqns. (3.19) and (3.22), respectively; however, with the "Δ" sign removed.

The internal work term is

$$\delta W^{int} = \int_{\Omega} \delta u_{i,j} \, \tau_{ij} \, d\Omega \tag{3.54}$$

The external work term is

$$\begin{aligned} \delta W^{ext} &= \int_{\Gamma} \delta u_i \; (P_0^w + \nu \, P_1^w) \, n_i \, d\Gamma + \delta W_o{}^{ext} \\ &= \int_{\Gamma} \delta w \; (P_0^w + \nu \, P_1^w) \, d\Gamma + \delta W_o{}^{ext} \end{aligned} \tag{3.55}$$

4. The Membrane Forces Induced by Static and Dynamic Loadings

Using eqns. (3.10) and (3.12), the equilibrium of membrane forces can be expressed as

$$N_{z,z} + (1-\nu\eta)\frac{1}{\bar{R}} N_{z\psi,\theta} = 0 \tag{4.1a}$$

$$(1-\nu\eta)\frac{1}{\bar{R}} N_{\psi,\theta} + N_{z\psi,z} = 0 \tag{4.1b}$$

$$N_{\psi} = P^{w} \{1+\nu(\eta+\eta'')\}\bar{R} \tag{4.1c}$$

where P^{w} represents the pressure induced by rigid body motions (eqn. (3.44)).

We introduce the circumferential geometrical imperfection pattern "$\eta(\theta)$" by the expression of Fourier series.

$$\eta(\theta) = \sum_{k=1}^{K} (c_k^s \sin k\theta + c_k^c \cos k\theta) \tag{4.2}$$

where c_k^s and c_k^c are given coefficients of the functions, sin(k θ) and cos(k θ),respectively.

The membrane force components are derived by solving the coupled eqns. (4.1) with using eqn. (4.2) partially.

$$N_{\psi} = \bar{R}P_0^{w} + \nu\bar{R}\,\{P_1^{w}+(\eta+\eta'')P_0^{w}\} \tag{4.3a}$$

$$N_{z\psi} = -\int P_{0}^{w}{}_{,\theta}\, dz - \nu\left\{ \int P_{1}^{w}{}_{,\theta}\, dz + \eta'' \int P_{0}^{w}{}_{,\theta}\, dz + (\eta'+\eta''') \int P_0^{w}\, dz \right\} + N_{z\psi}^{(M)} \tag{4.3b}$$

$$N_z = \frac{1}{\bar{R}} \int \int P_0^w{}_{,\theta\theta}\, dz\, dz + \frac{\nu}{\bar{R}} \{ \int \int P_1^w{}_{,\theta\theta}\, dz\, dz$$

$$+ (-\eta+\eta'') \int \int P_0^w{}_{,\theta\theta}\, dz\, dz + (\eta'+2\eta''') \int \int P_0^w{}_{,\theta}\, dz\, dz$$

$$+ (\eta''+\eta'''') \int \int P_0^w\, dz\, dz \} + N_z^{(M)} \qquad (4.3c)$$

where $N_{z\psi}^{(M)}$ and $N_z^{(M)}$ are the membrane forces induced by the horizontal ground motion, vertical ground motion, gravitational acceleration and/or rocking motion to a lumped mass "M" located at the height "L_1", and

$$N_{z\psi}^{(M)} = \frac{M}{\pi\bar{R}} (G_h(t)+L_1\ddot{q}(t)) \{\sin\theta - \nu (\eta'\cos\theta + \frac{3}{2} c_2^c \sin\theta)\} \qquad (4.4a)$$

$$N_z^{(M)} = \frac{M}{\pi\bar{R}^2} (G_h(t)+L_1\ddot{q}(t)) \{\cos\theta - \nu ((\eta+\eta'')\cos\theta - \eta'\sin\theta + \frac{3}{2} c_2^c \cos\theta)\} (L_1-z)$$

$$+ \frac{M}{2\pi\bar{R}} (G_v(t)-g) \qquad (4.4b)$$

where c_2^c is the coefficient in eqn. (4.2) for "cos2θ". The details of the derivation of eqns. (4.4) is given in Appendix C.

The membrane forces induced by the hydrostatic pressure are obtained by substituting P_L in Table 2.3(1) into eqns, (4.3).

$$N_\psi^s = \bar{R}\rho_F g(H-z) + \nu\bar{R}(\eta+\eta'')\rho_F g(H-z) \qquad (4.5a)$$

$$N_{z\psi}^s = \frac{1}{2} \nu (\eta'+\eta''')\rho_F g(H-z)^2 \qquad (4.5b)$$

$$N_z^s = \frac{\nu}{6\bar{R}} (\eta''+\eta'''')\rho_F g(H-z)^3 \qquad (4.5c)$$

In deriving these equations the following boundary conditions are employed,

$$N^{s}_{\psi} = N^{s}_{z\psi} = N^{s}_{z} = 0 \qquad \text{at } z=H \qquad (4.6)$$

The membrane forces induced by the pressure due to vertical ground motion are obtained by substituting P_v in Table 2.3(1) into eqns. (4.3) similarly.

$$N^{v}_{\psi} = -\bar{R}\rho_F G_v(t)\,(H-z) - v\bar{R}(\eta+\eta'')\rho_F G_v(t)\,(H-z) \qquad (4.7a)$$

$$N^{v}_{z\psi} = -\frac{v}{2}(\eta'+\eta''')\rho_F G_v(t)\,(H-z)^2 \qquad (4.7b)$$

$$N^{v}_{z} = -\frac{v}{6\bar{R}}(\eta''+\eta'''')\rho_F G_v(t)\,(H-z)^3 \qquad (4.7c)$$

The membrane forces induced by the pressure due to horizontal ground motion are obtained by substituting $P_{h0}|_{\bar{R}}$, $\frac{\partial P_{h0}}{\partial r}|_{\bar{R}}\cdot\bar{R}\eta$ and $P_{h1}|_{\bar{R}}$ in Tables 2.3(1)~(3) into eqns. (4.3).

0-th order

$$N^{H}_{\psi 0} = \bar{R}\,P_{h0}|_{\bar{R}} = -\sum_{i=1}^{\infty} 2\,\bar{G}_h(\omega)\frac{(-1)^{i+1}\bar{R}}{\mu_i}F_{i1}(\bar{R},z,t)\cos\theta \qquad (4.8a)$$

$$N^{H}_{z\psi 0} = -\int P_{h0}|_{\bar{R},\theta}\,dz = -\sum_{i=1}^{\infty} 2\,\bar{G}_h(\omega)\frac{(-1)^{i+1}}{\mu_i}\hat{F}_{i1}(\vec{R},z,t)\sin\theta \qquad (4.8b)$$

$$N^{H}_{z0} = \frac{1}{\bar{R}}\int\int P_{h0}|_{\bar{R},\theta\theta}\,dz\,dz = \sum_{i=1}^{\infty} 2\,\bar{G}_h(\omega)\frac{(-1)^{i+1}}{\mu_i\bar{R}}\hat{\hat{F}}_{i1}(\bar{R},z,t)\cos\theta \qquad (4.8c)$$

1st order

$$N^{H}_{\psi 1} = \bar{R}\left\{P_{h1}|_{\bar{R}} + \frac{\partial P_{h0}}{\partial r}|_{\bar{R}}\cdot\bar{R}\eta + (\eta+\eta'')P_{h0}|_{\bar{R}}\right\}$$

$$= \sum_{i=1}^{\infty}\frac{(-1)^{i}\cdot 2\,\bar{G}_h(\omega)\bar{R}}{\mu_i}F_{i1}(\bar{R},z,t)\left\{\left(\frac{1}{I^{<1>}_{1i}}+1\right)\eta + \eta''\right\}\cos\theta$$

$$+\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\overline{G}_h(\omega)F_{0k}(z,t)\overline{R}\left\{(I^{<1>}_{1k}-1)\,H'^{sc}_{10}+I^{<2>}_{1k}H^{cc}_{10}\right\}$$

$$+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+1}\overline{G}_h(\omega)F_{nk}(z,t)\overline{R}\left\{(I^{<1>}_{1k}-1)\,(H'^{sc}_{1n}\cos n\theta+H'^{ss}_{1n}\sin n\theta)\right.$$

$$\left.+I^{<2>}_{1k}(H^{cc}_{1n}\cos n\theta+H^{cs}_{1n}\sin n\theta)\right\} \tag{4.9a}$$

$$N^{H}_{z\psi 1}=-\left\{\int(P_{h1}|_{\bar{R}}+\frac{\partial P_{h0}}{\partial r}|_{\bar{R}}\overline{R}\eta)_{,\theta}\,dz+\eta''\int P_{h0}|_{\bar{R},\theta}\,dz+(\eta'+\eta''')\int P_{h0}|_{\bar{R}}\,dz\right\}$$

$$=\sum_{i=1}^{\infty}\frac{(-1)^{i}2\,\overline{G}_h(\omega)}{\mu_i}\hat{F}_{i1}(\overline{R},z,t)\left\{(\eta''+\frac{1}{I^{<1>}_{1i}}\eta)\sin\theta-\left((1+\frac{1}{I^{<1>}_{1i}})\eta'+\eta'''\right)\cos\theta\right\}$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+1}\overline{G}_h(\omega)\hat{F}_{nk}(z,t)\cdot n\left\{(I^{<1>}_{1k}-1)\,(-H'^{sc}_{1n}\sin n\theta+H'^{ss}_{1n}\cos n\theta)\right.$$

$$\left.+I^{<2>}_{1k}(-H^{cc}_{1n}\sin n\theta+H^{cs}_{1n}\cos n\theta)\right\} \tag{4.9b}$$

$$N^{H}_{z1}=\frac{1}{\overline{R}}\left\{\int\int(P_{h1}|_{\bar{R}}+\frac{\partial P_{h0}}{\partial r}|_{\bar{R}}\cdot\overline{R}\eta)_{,\theta\theta}\,dz\,dz+(-\eta+\eta'')\int\int P_{h0}|_{\bar{R},\theta\theta}\,dz\,dz\right.$$

$$\left.+(\eta'+2\eta''')\int\int P_{h0}|_{\bar{R},\theta}\,dz\,dz+(\eta''+\eta'''')\int\int P_{h0}|_{\bar{R}}\,dz\,dz\right.$$

$$=-\frac{1}{\overline{R}}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}2\,\overline{G}_h(\omega)}{\mu_i}\hat{F}_{i1}(\overline{R},z,t)\left[\left\{(1-\frac{1}{I^{<1>}_{1i}})\eta+\frac{1}{I^{<1>}_{1i}}\eta''+\eta''''\right\}\cos\theta\right.$$

$$\left.-\left\{(1+\frac{2}{I^{<1>}_{1i}})\eta'+2\eta'''\right\}\sin\theta\right]$$

$$-\frac{1}{\overline{R}}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+1}\overline{G}_h(\omega)\hat{F}_{nk}(z,t)\cdot n^2\left\{(I^{<1>}_{1k}-1)\,(H'^{sc}_{1n}\cos n\theta+H'^{ss}_{1n}\sin n\theta)\right.$$

$$\left.+I^{<2>}_{1k}(H^{cc}_{1n}\cos n\theta+H^{cs}_{1n}\sin n\theta)\right\} \tag{4.9c}$$

where

$$F_{in}(\bar{R},z,t) = \frac{\rho_F I_n(\mu_i \bar{R})}{\mu_i H \; I_n'(\mu_i \bar{R})} \cos(\mu_i z)\, e^{-j\omega t} \tag{4.10a}$$

$$\hat{F}_{in}(\bar{R},z,t) = \frac{\rho_F I_n(\mu_i \bar{R})}{\mu_i^2 H \; I_n'(\mu_i \bar{R})} \{\sin(\mu_i z)+(-1)^i\}\, e^{-j\omega t} \tag{4.10b}$$

$$\hat{\hat{F}}_{in}(\bar{R},z,t) = \frac{\rho_F I_n(\mu_i \bar{R})}{\mu_i^3 H \; I_n'(\mu_i \bar{R})} \{-\cos(\mu_i z)+(-1)^i \mu_i (z-H)\}\, e^{-j\omega t} \tag{4.10c}$$

$$\mathcal{I}_{mk}^{<1>} = \frac{I_m(\mu_k \bar{R})}{\mu_k \bar{R} \; I_m'(\mu_k \bar{R})} \quad , \quad \mathcal{I}_{mk}^{<2>} = \frac{\mu_k \bar{R} \; I_m''(\mu_k \bar{R})}{I_m'(\mu_k \bar{R})} \tag{4.11a,b}$$

$$\mathcal{F}_{nk}(z,t) = \frac{2\rho_F I_n(\mu_k \bar{R})}{\pi H \mu_k^2 \; I_n'(\mu_k \bar{R})} \cos(\mu_k z)\, e^{-j\omega t} \tag{4.12a}$$

$$\hat{\mathcal{F}}_{nk}(z,t) = \frac{2\rho_F I_n(\mu_k \bar{R})}{\pi H \mu_k^3 \; I_n'(\mu_k \bar{R})} \{\sin(\mu_k z)+(-1)^k\}\, e^{-j\omega t} \tag{4.12b}$$

$$\hat{\hat{\mathcal{F}}}_{nk}(z,t) = \frac{2\rho_F I_n(\mu_k \bar{R})}{\pi H \mu_k^4 \; I_n'(\mu_k \bar{R})} \{-\cos(\mu_k z)+(-1)^k \mu_k (z-H)\}\, e^{-j\omega t} \tag{4.12c}$$

In deriving eqns. (4.8) and (4.9), the same conditions as eqn. (4.6) are employed.

The membrane forces induced by the pressure due to rocking motion are obtained by substituting $P_{rm0}|_{\bar{R}}$, $\frac{\partial P_{rm0}}{\partial r}|_{\bar{R}} \cdot \bar{R}\eta$ and $P_{rm1}|_{\bar{R}}$ in Tables 3.2(1)~(3) into eqns. (4.3).

<u>0-th order</u>

$$N_{\psi 0}^{Rm} = \bar{R} \cdot P_{rm0}|_{\bar{R}}$$

$$= \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{R} \bar{q}(\omega) \left\{ \frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} F_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} Q_{i1}(\bar{R},z,t) \right\} \cos\theta \tag{4.13a}$$

$$N_{z\psi 0}^{Rm} = - \int P_{rm0}|_{\bar{R},\theta}\, dz$$

$$= \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \left\{ \frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t) \right\} \sin\theta \qquad (4.13b)$$

$$N_{z0}^{Rm} = \frac{1}{\bar{R}} \int \int P_{rm0}|_{\bar{R},\theta\theta}\, dz\, dz$$

$$= - \sum_{i=1}^{\infty} \frac{2\omega^2 H^2 \bar{q}(\omega)}{\bar{R}} \left\{ \frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t) \right\} \cos\theta \qquad (4.13c)$$

<u>1st order</u>

$$N_{\psi 1}^{Rm} = \bar{R} \left\{ P_{rm1}|_{\bar{R}} + \frac{\partial P_{rm0}}{\partial r}|_{\bar{R}} \cdot \bar{R}\eta + (\eta+\eta'')P_{rm0}|_{\bar{R}} \right\}$$

$$= \bar{R} \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \left\{ \frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} F_{i1}(\bar{R},z,t) \left\{ (1+\frac{1}{I_{1i}^{<1>}})\eta+\eta'' \right\} \right.$$

$$\left. - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} Q_{i1}(\bar{R},z,t)\, (\eta+\eta'') \right\} \cos\theta$$

$$- \frac{\bar{R}}{2} \sum_{k=1}^{\infty} \omega^2 \bar{q}(\omega)\, \mu_k F_{0k}(z,t) \cdot \left[\frac{(-1)^{k+1} \cdot \mu_k H-1}{\mu_k^2} \left\{ (I_{1i}^{<1>}-1) H_{10}^{'sc} + I_{1k}^{<2>} H_{10}^{cc} \right\} \right.$$

$$\left. + \sum_{i=1}^{\infty} \frac{J_i^{<2>}}{\varepsilon_i^2 + \mu_k^2} (H_{10}^{'sc} + J_i^{<1>} H_{10}^{cc}) \right]$$

$$- \bar{R} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \omega^2 \bar{q}(\omega)\, \mu_k F_{nk}(z,t) \cdot \left[\frac{(-1)^{k+1} \cdot \mu_k H-1}{\mu_k^2} \left\{ (I_{1k}^{<1>}-1)(H_{1n}^{'sc} \cos n\theta + H_{1n}^{'ss} \sin n\theta) \right. \right.$$

$$\left. + I_{1k}^{<2>} (H_{1n}^{cc} \cos n\theta + H_{1n}^{cs} \sin n\theta) \right\}$$

$$\left. + \sum_{i=1}^{\infty} \frac{J_i^{<2>}}{\varepsilon_i^2 + \mu_k^2} \left\{ (H_{1n}^{'sc} \cos n\theta + H_{1n}^{'ss} \sin n\theta) + J_i^{<1>} (H_{1n}^{cc} \cos n\theta + H_{1n}^{cs} \sin n\theta) \right\} \right] \qquad (4.14a)$$

$$N_{z\psi 1}^{Rm} = - \left\{ \int (P_{rm1}|_{\bar{R}} + \frac{\partial P_{rm0}}{\partial r}|_{\bar{R}} \cdot \bar{R}\eta)_{,\theta}\, dz + \eta'' \int P_{rm0}|_{\bar{R},\theta}\, dz + (\eta'+\eta''') \int P_{rm0}|_{\bar{R}}\, dz \right\}$$

$$= - \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \Big[\frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) \Big\{ ((1+\frac{1}{I^{<1>}_{1i}})\eta'+\eta''')\cos\theta - (\frac{1}{I^{<1>}_{1i}}\eta+\eta'')\sin\theta \Big\}$$

$$- \frac{\bar{R}J_2(\varepsilon_i\bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t)\{(\eta'+\eta''')\cos\theta - \eta''\sin\theta\}\Big]$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \omega^2 \bar{q}(\omega)\, \mu_k \hat{\mathcal{F}}_{nk}(z,t) \cdot n \Big[\frac{(-1)^{k+1}\cdot\mu_k H-1}{\mu_k^2} \{(I^{<1>}_{1k}-1)(-H'^{sc}_{1n} \sin n\theta + H'^{ss}_{1n} \cos n\theta)$$

$$+ I^{<2>}_{1k}(-H^{cc}_{1n} \sin n\theta + H^{cs}_{1n}\cos n\theta)\}$$

$$+ \sum_{i=1}^{\infty} \frac{J^{<2>}_{i}}{\varepsilon_i^2 + \mu_k^2}\{(- H'^{sc}_{1n} \sin n\theta + H'^{ss}_{1n}\cos n\theta) + J^{<1>}_{i}(-H^{cc}_{1n} \sin n\theta + H^{cs}_{1n}\cos n\theta) \}\Big] \qquad (4.14b)$$

$$N^{Rm}_{z1} = \frac{1}{\bar{R}} \int \int \{ (P_{rm1}|_{\bar{R}} + \frac{\partial P_{rm0}}{\partial r}|_{\bar{R}}\cdot\bar{R}\eta)_{,\theta\theta} + (-\eta+\eta'')P_{rm0}|_{\bar{R},\theta\theta}$$

$$+ (\eta'+2\eta''')P_{rm0}|_{\bar{R},\theta} + (\eta''+\eta'''')P_{rm0}|_{R}\} \, dz \, dz$$

$$= \frac{1}{\bar{R}} \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \Big[\frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} \hat{\hat{F}}_{i1}(\bar{R},z,t) \Big\{ \Big((1 - \frac{1}{I^{<1>}_{1i}})\eta + \frac{1}{I^{<1>}_{1i}}\eta''+\eta''''\Big) \cos\theta$$

$$- \Big((1+\frac{2}{I^{<1>}_{1i}})\eta'+2\eta'''\Big) \sin\theta \Big\}$$

$$- \frac{\bar{R}J_2(\varepsilon_i\bar{R})}{\varepsilon_i H^2} \hat{\hat{Q}}_{i1}(\bar{R},z,t)\{(\eta+\eta'''')\cos\theta - (\eta'+2\eta''')\sin\theta\}\Big]$$

$$+ \frac{1}{\bar{R}} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \omega^2 \bar{q}(\omega)\, \mu_k \hat{\hat{\mathcal{F}}}_{nk}(z,t)\cdot n^2\cdot\Big[\frac{(-1)^{k+1}\cdot\mu_k H-1}{\mu_k^2} \{(I^{<1>}_{1k}-1)(H'^{sc}_{1n} \text{ con } n\theta + H'^{ss}_{1n} \sin n\theta)$$

$$+ I^{<2>}_{1k}(H^{cc}_{1n} \cos n\theta + H^{cs}_{1n}\sin n\theta)\}$$

$$+ \sum_{i=1}^{\infty} \frac{J^{<2>}_{i}}{\varepsilon_i^2 + \mu_k^2}\{(H'^{sc}_{1n} \cos n\theta + H'^{ss}_{1n}\sin n\theta) + J^{<1>}_{i}(H^{cc}_{1n} \cos n\theta + H^{cs}_{1n}\sin n\theta) \}\Big] \qquad (4.14c)$$

where

$$Q_{in}(\bar{R},z,t) = \frac{\rho_F}{\varepsilon_i\bar{R}\left(1-\frac{n^2}{(\varepsilon_i\bar{R})^2}\right)J_n(\varepsilon_i\bar{R})}\{\sinh(\varepsilon_i z) - \tanh(\varepsilon_i H)\cosh(\varepsilon_i z)\}\ e^{-j\omega t} \qquad (4.15a)$$

$$\hat{Q}_{in}(\bar{R},z,t) = \frac{\rho_F}{\varepsilon_i^2\bar{R}\left(1-\frac{n^2}{(\varepsilon_i\bar{R})^2}\right)J_n(\varepsilon_i\bar{R})}\{\cosh(\varepsilon_i z) - \tanh(\varepsilon_i H)\sinh(\varepsilon_i z) - \frac{1}{\cosh(\varepsilon_i H)}\}\ e^{-j\omega t} \qquad (4.15b)$$

$$\hat{\hat{Q}}_{in}(\bar{R},z,t) = \frac{\rho_F}{\varepsilon_i^3\bar{R}\left(1-\frac{n^2}{(\varepsilon_i\bar{R})^2}\right)J_n(\varepsilon_i\bar{R})}\{\sinh(\varepsilon_i z) - \tanh(\varepsilon_i H)\cosh(\varepsilon_i z) - \frac{\varepsilon_i(z-H)}{\cosh(\varepsilon_i H)}\}\ e^{-j\omega t} \qquad (4.15c)$$

$$J_i^{<2>} = \frac{2J_2(\varepsilon_i\bar{R})}{\varepsilon_i\bar{R}\left(1-\frac{1}{(\varepsilon_i\bar{R})^2}\right)J_1(\varepsilon_i\bar{R})} \qquad (4.16a)$$

$$J_i^{<1>} = \frac{\varepsilon_i^2\bar{R}^2 J_1''(\varepsilon_i\bar{R})}{J_1(\varepsilon_i\bar{R})} \qquad (4.16b)$$

$$H_{mn}^{'sc} = \int_0^{2\pi} \eta'(\theta)\ \sin m\theta\ \cos n\theta\ d\theta = \frac{\pi}{2}\{(n-m)\ c_{|n-m|}^{c} - (n+m)\ c_{n+m}^{c}\} \qquad (4.17a)$$

$$H_{mn}^{'ss} = \int_0^{2\pi} \eta'(\theta)\ \sin m\theta\ \sin n\theta\ d\theta = \frac{\pi}{2}\{|n-m|\ c_{|n-m|}^{s} - (n+m)\ c_{n+m}^{s}\} \qquad (4.17b)$$

$$H_{mn}^{cc} = \int_0^{2\pi} \eta(\theta)\ \cos m\theta\ \cos n\theta\ d\theta = \frac{\pi}{2}(c_{|n-m|}^{c} + c_{n+m}^{c}) \qquad (4.17c)$$

$$H_{mn}^{cs} = \int_0^{2\pi} \eta(\theta)\ \cos m\theta\ \sin n\theta\ d\theta = \frac{\pi}{2}\{\mathrm{sign}(n-m)\ c_{|n-m|}^{s} + c_{n+m}^{s}\} \qquad (4.17d)$$

In deriving eqns.(4.13) and (4.14), the same conditions as eqn.(4.6) are employed again.

Eqn.(4.2) is used to obtain the final forms of eqns.(4.17).

The components of menbrane forces obtained are summarized in Table 4.1.

the special cases of eqns.(4.17)

$$H'^{sc}_{10} = \int_0^{2\pi} \eta'(\theta) \sin\theta \, d\theta = -\pi c^c_1$$

$$H'^{ss}_{10} = 0$$

$$H^{cc}_{10} = \int_0^{2\pi} \eta(\theta) \cos\theta \, d\theta = \pi c^c_1$$

$$H^{cs}_{10} = 0$$

$$H'^{sc}_{1n} = \frac{\pi}{2} \{(n-1) c^c_{n-1} - (n+1) c^c_{n+1}\}$$

$$H'^{ss}_{1n} = \frac{\pi}{2} \{(n-1) c^s_{n-1} - (n+1) c^s_{n+1}\}$$

$$H^{cc}_{1n} = \frac{\pi}{2} (c^c_{n-1} + c^c_{n+1})$$

$$H^{cs}_{1n} = \frac{\pi}{2} (c^s_{n-1} + c^s_{n+1})$$

Table 4.1 The components of membrane forces

	induced by a lumped mass	
	0-th order	1st order
$N_{Z}^{(M)}$	$\frac{M}{\pi\bar{R}^2}\{(G_h(t)+L_1\ddot{q}(t))\cdot(L_1-z)\cos\theta$ $+\frac{\bar{R}}{2}(G_V(t)-g)\}$	$-\frac{M}{\pi\bar{R}^2}(G_h(t)+L_1\ddot{q}(t))(L_1-z)$ $\times\{(\eta+\eta'')\cos\theta-\eta'\sin\theta+\frac{3}{2}c_2^c\cos\theta\}$
$N_{Z\Psi}^{(M)}$	$\frac{M}{\pi\bar{R}}(G_h(t)+L_1\ddot{q}(t))\sin\theta$	$-\frac{M}{\pi\bar{R}}(G_h(t)+L_1\ddot{q}(t))(\eta'\cos\theta+\frac{3}{2}c_2^c\sin\theta)$
$N_{\Psi}^{(M)}$	0	0
	induced by pressures	
	due to hydrostatic pressure	
	0-th order	1st order
N_{Z}^{S}	0	$\frac{1}{6\bar{R}}\rho_{F}g(H-z)^3(\eta''+\eta'''')$
$N_{Z\Psi}^{S}$	0	$\frac{1}{2}\rho_{F}g(H-z)^2(\eta'+\eta''')$
N_{Ψ}^{S}	$\bar{R}\rho_{F}g(H-z)$	$\bar{R}\rho_{F}g(H-z)(\eta+\eta'')$

Table 4.1 The components of membrane forces

continue (2)

	due to vertical ground motion	
	0th order	1st order
N_Z^V	0	$-\frac{1}{6\bar{R}}\rho_F G_V(t)\,(H-z)^3\,(\eta''+\eta'''')$
$N_{Z\Psi}^V$	0	$-\frac{1}{2}\rho_F G_V(t)\,(H-z)^2\,(\eta'+\eta''')$
N_Ψ^V	$-\bar{R}\,\rho_F G_V(t)\,(H-z)$	$-\bar{R}\,\rho_F G_V(t)\,(H-z)\,(\eta+\eta'')$
	due to horizontal ground motion	
	0-th order	1st order
N_Z^H	$\sum_{i=1}^{\infty} 2\,\bar{G}_h(\omega)\frac{(-1)^{i+1}}{\mu_i\bar{R}}\hat{F}_{i1}(\bar{R},z,t)\cos\theta$	see eqn. (4.9c)
$N_{Z\Psi}^H$	$-\sum_{i=1}^{\infty} 2\,\bar{G}_h(\omega)\frac{(-1)^{i+1}}{\mu_i}\hat{F}_{i1}(\bar{R},z,t)\sin\theta$	see eqn. (4.9b)
N_Ψ^H	$-\sum_{i=1}^{\infty} 2\,\bar{G}_h(\omega)\frac{(-1)^{i+1}\bar{R}}{\mu_i}F_{i1}(\bar{R},z,t)\cos\theta$	see eqn. (4.9a)

Table 4.1 The components of membrane forces

continue(3)

	due to rocking motion	
	0-th order	1st order
N_{Z}^{Rm}	$-\sum_{i=1}^{\infty} \frac{2\omega^2 H^2 \bar{q}(\omega)}{\bar{R}} \left\{ \frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} \hat{\hat{F}}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{\hat{Q}}_{i1}(\bar{R},z,t) \right\} \cos\theta$	see eqn. (4.14c)
$N_{Z\Psi}^{Rm}$	$\sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \left\{ \frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t) \right\} \sin\theta$	see eqn. (4.14b)
N_{Ψ}^{Rm}	$\sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{R} \bar{q}(\omega) \left\{ \frac{\mu_i H(-1)^{i+1}-1}{(\mu_i H)^2} F_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} Q_{i1}(\bar{R},z,t) \right\} \cos\theta$	see eqn. (4.14a)

5. Galerkin / Finite Element Discretization

We assume the displacement components which are expanded in terms of the mode shapes in the circumferential direction and the nodal displacements in the axial direction.If Hermite interpolation functions are adopted.

$$u(z,\theta,t) = \sum_{i=1}^{I} \sum_{n=0}^{N} H_{i0}^{I0}(z)(\sin n\theta \cdot u_{in}^{s} + \cos n\theta \cdot u_{in}^{c}) \tag{5.1a}$$

$$\begin{aligned} v(z,\theta,t) = \sum_{i=1}^{I} \sum_{n=0}^{N} \{ & (H_{i0}^{I1}(z)\, v_{in}^{s} + H_{i1}^{I1}(z)\, v_{in}^{s'}) \sin n\theta \\ & + (H_{i0}^{I1}(z)\, v_{in}^{c} + H_{i1}^{I1}(z)\, v_{in}^{c'}) \cos n\theta \} \end{aligned} \tag{5.1b}$$

$$\begin{aligned} w(z,\theta,t) = \sum_{i=1}^{I} \sum_{n=0}^{N} \{ & (H_{i0}^{I1}(z)\, w_{in}^{s} + H_{i1}^{I1}(z)\, w_{in}^{s'}) \sin n\theta \\ & + (H_{i0}^{I1}(z)\, w_{in}^{c} + H_{i1}^{I1}(z)\, w_{in}^{c'}) \cos n\theta \} \end{aligned} \tag{5.1c}$$

where $H_{i0}^{I0}(z)$, $H_{i0}^{I1}(z)$ and $H_{i1}^{I1}(z)$ are Hermite interpolation function I and N are the total numbers of nodes in the axial direction and the total numbers of modes in the circumferential direction excluding the breathing type mode, respectively. u, v and w are the axial, circumferential tangential and normal direction displacement components to the tank, respectively.u_{in}^{s}, u_{in}^{c},v_{in}^{s},v_{in}^{c} w_{in}^{s} and w_{in}^{c} denote the displacement coefficients in each direction which belong to node number "i" and mode number "n", and the superscript "s"and "c" represent "sine"and "cosine"components,respectively. and

$$v_{in}^{\alpha'} = \left(\frac{dv^{\alpha}}{dz}\right)_{in} \qquad \alpha = \text{s or c} \tag{5.2a}$$

$$w_{in}^{\alpha'} = \left(\frac{dw^{\alpha}}{dz}\right)_{in} \qquad \alpha = \text{s or c} \tag{5.2b}$$

Equs. (5.1) can be expressed as the matrix form,

$$\mathbf{d} = \sum_{i=1}^{I} \sum_{n=0}^{N} \mathbf{N}_{\theta n} \mathbf{N}_{zi} \mathbf{d}_{in} \tag{5.3}$$

where

$$\mathbf{d} = [\, u , v , w]^T \tag{5.4a}$$

$$\mathbf{d}_{in} = [\, u_{in}^{s}\ u_{in}^{c} : v_{in}^{s}\ v_{in}^{s'}\ v_{in}^{c}\ v_{in}^{c'} : w_{in}^{s}\ w_{in}^{s'}\ w_{in}^{c}\ w_{in}^{c'}]^T \tag{5.4b}$$

$\mathbf{N}_{\theta n}$ and $\mathbf{N}_{zi}$ are tabulated in Table 5.1.

(i) The structural mass matrices ; refer to eqn. (3.19a)

The structural inertia term (eqn. (3.19a)) can be expressed as follows by using eqn. (5.3)

$$I_t = \int_0^L \int_0^{2\pi} \rho h\ \bar{R}\ \delta\mathbf{d}^T \Delta\ddot{\mathbf{d}}(1+\nu\eta)\, d\theta\, dz$$

$$= \delta\mathbf{d}_{in}^T (\mathbf{M}_{ijnm}^{s0} + \nu\, \mathbf{M}_{ijnm}^{s1}) \Delta\ddot{\mathbf{d}}_{jm} \tag{5.5}$$

where $\mathbf{M}_{ij\,nm}^{s0}$ and $\mathbf{M}_{ij\,nm}^{s1}$ are the 0-th order and 1st order mass matrices given as follows,respectively.

$$\mathbf{M}_{ijnm}^{s0} = \delta_{nm}\pi\rho h\bar{R} \int_0^L \mathbf{N}_{zi}^T \mathbf{N}_{zj}\, dz \tag{5.6a}$$

$$\mathbf{M}_{ijnm}^{s1} = \rho h\bar{R} \int_0^L \mathbf{N}_{zi}^T \mathbf{m}_{nm}^{s1} \mathbf{N}_{zj}\, dz \tag{5.6b}$$

and

$$\mathbf{m}_{nm}^{s1} = \begin{bmatrix} (\eta_1)_{nm} & 0 & 0 \\ 0 & (\eta_1)_{nm} & 0 \\ 0 & 0 & (\eta_1)_{nm} \end{bmatrix} \tag{5.7}$$

where

$$(\eta_1)_{nm} = \begin{bmatrix} \eta_{nm}^{ss} & \eta_{nm}^{sc} \\ \eta_{nm}^{cs} & \eta_{nm}^{cc} \end{bmatrix}$$

$$= \frac{\pi}{2} \left[\begin{array}{ccc} c^{c}_{|n-m|} - c^{c}_{n+m} & : & \mathrm{sign}(n-m)\, c^{s}_{|n-m|} + c^{s}_{n+m} \\ & : & \\ \mathrm{sign}(m-n)\, c^{s}_{|n-m|} + c^{s}_{n+m} & : & c^{c}_{|n-m|} + c^{c}_{n+m} \end{array} \right] \tag{5.8}$$

$$(\eta^{cs}_{nm} = \eta^{sc}_{mn})$$

In deriving eqn. (5.8), eqn. (4.2) is applied. The details of the components of $(\eta_1)_{nm}$ are given in Appendix D.

Furthermore,eqns.(5.6) can be expressed by the forms after integrating in the axial direcion.

$$\mathbf{M}^{s0}_{ijnm} = \delta_{nm} \cdot \pi \rho h \bar{R}\, \mathbf{M}^{*s0}_{ij} \tag{5.9a}$$

(in the case q m=n=o,see Appendix E)

$$\mathbf{M}^{s1}_{ijnm} = \rho h \bar{R} \mathbf{M}^{*s1}_{ij\ nm} \tag{5.9b}$$

where $\mathbf{M}^{*s0}_{ij}$ and $\mathbf{M}^{*s1}_{ijnm}$ are given in Table 5.2.

(ii) The added mass matrices : refer to eqn.(3.22)

The normal displacement "w" is written as follows from eqn.(5.1c)

$$w = \sum_{i=1}^{I} \sum_{n=0}^{N} (\mathbf{SC})_n\, \mathbf{N}^{w}_{zi}\, \mathbf{d}_{in} \tag{5.10}$$

where

$$(\mathbf{SC})_n = [\sin n\theta \ \ \cos n\theta] \tag{5.11a}$$

$$\mathbf{N}^{w}_{zi} = [\mathbf{0} \ \ \mathbf{H}^{1}_{i}(z)] \tag{5.11b}$$

Refering to Tables 2.3(1)~(3), each term in eqn.(3.21) can be written as

$$\Delta(P_{d0}|_{\bar{R}}) = -\sum_{q=1}^{\infty} \sum_{m=0}^{N} \sum_{j=1}^{I} \frac{2\rho_F I_m(\mu_q \bar{R})}{\mu_q H\, I'_m(\mu_q \bar{R})} \cos(\mu_q z)\, (\mathbf{SC})_m \left(\int_0^H \cos(\mu_q \zeta)\, \mathbf{N}^{w}_{zj}\, d\zeta \right) \Delta \ddot{\mathbf{d}}_{jm} \tag{5.12a}$$

$$\Delta\left(\frac{\partial P_{d0}}{\partial r}\Big|_{\bar{R}}\bar{R}\eta\right)=-\sum_{q=1}^{\infty}\sum_{m=0}^{N}\sum_{j=1}^{I}\frac{2\rho_F\bar{R}}{H}\cos(\mu_q z)\,\eta(\theta)\,(\mathbf{SC})_m\left(\int_0^H\cos(\mu_q\zeta)\,\mathbf{N}_{zj}^{w}\,d\zeta\right)\Delta\ddot{\mathbf{d}}_{jm}\tag{5.12b}$$

$$\begin{aligned}\Delta(P_{d1}|_{\bar{R}})=&-\sum_{q=1}^{\infty}\sum_{m=0}^{N}\sum_{j=1}^{I}\frac{\rho_F I_o(\mu_q\bar{R})}{\pi H\mu_q I_o'(\mu_q\bar{R})}\cos(\mu_q z)\\&\times\left\{I_{mq}^{<1>}\left(\int_0^{2\pi}\eta'(\theta)\,(\mathbf{SC})_m'\,d\theta\right)-I_{mq}^{<2>}\left(\int_0^{2\pi}\eta(\theta)\,(\mathbf{SC})_m\,d\theta\right)\right\}\left(\int_0^H\cos(\mu_q\zeta)\,\mathbf{N}_{zj}^{w}\,d\zeta\right)\Delta\ddot{\mathbf{d}}_{jm}\\&-\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{N}\sum_{j=1}^{I}\frac{2\rho_F I_p(\mu_q\bar{R})}{\pi H\mu_q I_p'(\mu_q\bar{R})}\cos(\mu_q z)\\&\times\Big[I_{mq}^{<1>}\Big\{\left(\int_0^{2\pi}\eta'(\theta)\cos p\theta\,(\mathbf{SC})_m'\,d\theta\right)\cos p\theta+\left(\int_0^{2\pi}\eta'(\theta)\sin p\theta\,(\mathbf{SC})_m'\,d\theta\right)\sin p\theta\Big\}\\&-I_{mq}^{<2>}\Big\{\left(\int_0^{2\pi}\eta(\theta)\cos p\theta\,(\mathbf{SC})_m\,d\theta\right)\cos p\theta\\&+\left(\int_0^{2\pi}\eta(\theta)\sin p\theta\,(\mathbf{SC})_m\,d\theta\right)\sin p\theta\Big\}\Big]\left(\int_0^H\cos(\mu_q\zeta)\mathbf{N}_{zj}^{w}\,d\zeta\right)\Delta\ddot{\mathbf{d}}_{jm}\end{aligned}\tag{5.12c}$$

In eqn.(5.12c)

$$(\mathbf{SC})_m'=\mathrm{m}[\cos m\theta\ \ -\sin m\theta]\tag{5.13}$$

Using eqns.(5.12), eqn.(3.22) can be expressed as

$$-\int_{\Gamma}\delta u_i\ I_1^{ad}(\mathbf{x},t)\ n_i\ d\Gamma=\delta\mathbf{d}_{in}^{T}\left\{\mathbf{M}_{ijnm}^{ad0}+\nu\left(\mathbf{M}_{ijnm}^{ad11}+\mathbf{M}_{ijnm}^{ad12}+\mathbf{M}_{ijnm}^{ad13}\right)\right\}\Delta\ddot{\mathbf{d}}_{jm}\tag{5.14}$$

where

$$\mathbf{M}_{ijnm}^{ad0} = \pi\delta_{nm} \sum_{q=1}^{\infty} \frac{2\rho_F \, I_m(\mu_q \bar{R})}{\mu_q H \; I_m'(\mu_q \bar{R})} \gamma_{iq}^T \gamma_{jq} \tag{5.15a}$$

(in case of m=n=0, Appendix E)

$$\mathbf{M}_{ijnm}^{ad11} = \sum_{q=1}^{\infty} \frac{2\rho_F \bar{R} \; I_m(\mu_q \bar{R})}{\mu_q H \; I_m'(\mu_q \bar{R})} \gamma_{iq}^T (\eta_1)_{nm} \gamma_{jq} \tag{5.15b}$$

$$\mathbf{M}_{ijnm}^{ad12} = \sum_{q=1}^{\infty} \frac{2\rho_F \bar{R}^2}{H} \gamma_{iq}^T (\eta_1)_{nm} \gamma_{jq} \tag{5.15c}$$

$$\mathbf{M}_{ijnm}^{ad13} = \sum_{q=1}^{\infty} \frac{2\rho_F \bar{R} \; I_n(\mu_q \bar{R})}{H\mu_q \; I_n'(\mu_q \bar{R})} \gamma_{iq}^T \{ m \, I_{mq}^{<1>} (\eta_3')_{nm} - I_{mq}^{<2>} (\eta_1)_{nm} \} \gamma_{jq} \tag{5.15d}$$

In above equations, $(\eta_1)_{nm}$ is given by eqn. (5.8) and γ_{iq} and $(\eta_3')_{nm}$ are given as

$$\gamma_{iq} = \int_0^H \cos(\mu_q z) \, \mathbf{N}_{zi}^{w} \, dz \tag{5.16}$$

$$(\eta_3')_{nm} = \begin{bmatrix} \eta_{nm}^{'sc} & -\eta_{nm}^{'ss} \\ \eta_{nm}^{'cc} & -\eta_{nm}^{'cs} \end{bmatrix} = \frac{\pi}{2} \begin{bmatrix} -(n-m)c_{|n-m|}^{c} - (n+m)c_{n+m}^{c} & : & -|n-m| \, c_{|n-m|}^{s} + (n+m)c_{n+m}^{s} \\ & : & \\ |n-m| \, c_{|n-m|}^{s} + (n+m)c_{n+m}^{s} & : & -(n-m)c_{|n-m|}^{c} + (n+m)c_{n+m}^{c} \end{bmatrix}$$

In deriving eqn.(5.17), eqn.(4.2) is applied. The details of the components of $(\eta_3')_{nm}$ are given in Appendix D.

(iii) The material stiffness matrices : refer to eqns.(3.30) and (3.31)

By using eqn. (5.3), eqns.(3.33) and (3.34) can be written as

$$\varepsilon_0 = \mathbf{B}_{\theta n}^{10} \, \mathbf{B}_{zi}^{10} \, \mathbf{d}_{in} \quad , \quad \varepsilon_1 = \mathbf{B}_{\theta n}^{11} \, \mathbf{B}_{zi}^{11} \, \mathbf{d}_{in} \tag{5.18a,b}$$

$$\kappa_0 = \mathbf{B}_{\theta n}^{20} \, \mathbf{B}_{zi}^{20} \, \mathbf{d}_{in} \quad , \quad \kappa_1 = \mathbf{B}_{\theta n}^{21} \, \mathbf{B}_{zi}^{21} \, \mathbf{d}_{in} \tag{5.19a,b}$$

where the matrices which relate strains and curvatures to modal and nodal displacements are given in Table 5.1.

Applying eqns. (5.18) and (5.19), eqns. (3.30) and (3.31) can be expressed as follows.

$$I_{m1} = \delta \mathbf{d}_{in}^{T} [\mathbf{K}_{ijnm}^{D01} + \nu \{\mathbf{K}_{ijnm}^{D11\text{-}0} + \mathbf{K}_{ijnm}^{D11\text{-}1} + (\mathbf{K}_{jimn}^{D11\text{-}1})^{T} \}] \Delta \mathbf{d}_{jm} \tag{5.20a}$$

$$I_{m2} = \delta \mathbf{d}_{in}^{T} [\mathbf{K}_{ijnm}^{D3} + (\mathbf{K}_{jimn}^{D3})^{T}] \Delta \mathbf{d}_{jm} \tag{5.20b}$$

$$I_{m3} = \delta \mathbf{d}_{in}^{T} [\mathbf{K}_{ijnm}^{D02} + \nu \{\mathbf{K}_{ijnm}^{D12\text{-}0} + \mathbf{K}_{ijnm}^{D12\text{-}1} + (\mathbf{K}_{jimn}^{D12\text{-}1})^{T} \}] \Delta \mathbf{d}_{jm} \tag{5.20c}$$

where the stiffness matrices are as follows ;

$$\mathbf{K}_{ijnm}^{D01} = \frac{Eh\bar{R}}{1-\bar{\nu}^{2}} \int_{0}^{L} \mathbf{B}_{zi}^{10^{T}} \left(\int_{0}^{2\pi} \mathbf{B}_{\theta n}^{10^{T}} \mathbf{C} \, \mathbf{B}_{\theta m}^{10} \, d\theta \right) \mathbf{B}_{zj}^{10} \, dz \tag{5.21a}$$

$$\mathbf{K}_{ijnm}^{D11\text{-}0} = \frac{Eh\bar{R}}{1-\bar{\nu}^{2}} \int_{0}^{L} \mathbf{B}_{zi}^{10^{T}} \left(\int_{0}^{2\pi} \eta(\theta) \, \mathbf{B}_{\theta n}^{10^{T}} \mathbf{C} \, \mathbf{B}_{\theta m}^{10} \, d\theta \right) \mathbf{B}_{zj}^{10} \, dz \tag{5.21b}$$

$$\mathbf{K}_{ijnm}^{D11\text{-}1} = \frac{Eh\bar{R}}{1-\bar{\nu}^{2}} \int_{0}^{L} \mathbf{B}_{zi}^{10^{T}} \left(\int_{0}^{2\pi} \mathbf{B}_{\theta n}^{10^{T}} \mathbf{C} \, \mathbf{B}_{\theta m}^{11} \, d\theta \right) \mathbf{B}_{zj}^{11} \, dz \tag{5.21c}$$

$$\mathbf{K}_{ijnm}^{D02} = \frac{Eh^{3}\bar{R}}{12(1-\bar{\nu}^{2})} \int_{0}^{L} \mathbf{B}_{zi}^{20^{T}} \left(\int_{0}^{2\pi} \mathbf{B}_{\theta n}^{20^{T}} \mathbf{C} \, \mathbf{B}_{\theta m}^{20} \, d\theta \right) \mathbf{B}_{zj}^{20} \, dz \tag{5.22a}$$

$$\mathbf{K}_{ijnm}^{D12\text{-}0} = \frac{Eh^{3}\bar{R}}{12(1-\bar{\nu}^{2})} \int_{0}^{L} \mathbf{B}_{zi}^{20^{T}} \left(\int_{0}^{2\pi} \eta(\theta) \mathbf{B}_{\theta n}^{20^{T}} \mathbf{C} \, \mathbf{B}_{\theta m}^{20} \, d\theta \right) \mathbf{B}_{zj}^{20} \, dz \tag{5.22b}$$

$$\mathbf{K}_{ijnm}^{D12\text{-}1} = \frac{Eh^{3}\bar{R}}{12(1-\bar{\nu}^{2})} \int_{0}^{L} \mathbf{B}_{zi}^{20^{T}} \left(\int_{0}^{2\pi} \mathbf{B}_{\theta n}^{20^{T}} \mathbf{C} \, \mathbf{B}_{\theta m}^{21} \, d\theta \right) \mathbf{B}_{zj}^{21} \, dz \tag{5.22c}$$

$$\mathbf{K}_{ijnm}^{D3} = - \frac{Eh^{3}}{12(1-\bar{\nu}^{2})} \int_{0}^{L} \mathbf{B}_{zi}^{10^{T}} \left(\int_{0}^{2\pi} \mathbf{B}_{\theta n}^{10^{T}} \mathbf{C} \, \mathbf{B}_{\theta m}^{20} \, d\theta \right) \mathbf{B}_{zj}^{20} \, dz \tag{5.23}$$

Furthermore, by integrating in the direction of θ (0 to 2π), eqns.(5.21) to (5.23) can be rewritten as explicit forms on θ.

$$\mathbf{K}_{ijnm}^{D01} = \pi \delta_{nm} \int_{0}^{L} \mathbf{B}_{zi}^{10^{T}} \mathbf{C}^{10} \mathbf{B}_{zj}^{10} \, dz \tag{5.24a}$$

(in the case of m=n=0. see Appendix E)

$$\mathbf{K}^{D11\text{-}0}_{ijnm} = \int_0^L \mathbf{B}^{10^T}_{zi}\, \mathbf{C}^{110}_{nm}\, \mathbf{B}^{10}_{zj}\, dz \tag{5.24b}$$

$$\mathbf{K}^{D11\text{-}1}_{ijnm} = \int_0^L \mathbf{B}^{10^T}_{zi}\, \mathbf{C}^{111}_{nm}\, \mathbf{B}^{11}_{zj}\, dz = -\frac{1}{\bar{R}} \int_0^L \mathbf{B}^{10^T}_{zi}\, \mathbf{C}^{111}_{nm}\, \mathbf{N}_{zj}\, dz \tag{5.24c}$$

$$\mathbf{K}^{D02}_{ijnm} = \pi\delta_{nm} \int_0^L \mathbf{B}^{20^T}_{zi}\, \mathbf{C}^{20}\, \mathbf{B}^{20}_{zj}\, dz \tag{5.25a}$$

(in the case of m=n=0. see Appendix E)

$$\mathbf{K}^{D12\text{-}0}_{ijnm} = \int_0^L \mathbf{B}^{20^T}_{zi}\, \mathbf{C}^{210}_{nm}\, \mathbf{B}^{20}_{zj}\, dz \tag{5.25b}$$

$$\mathbf{K}^{D12\text{-}1}_{ijnm} = \int_0^L \mathbf{B}^{20^T}_{zi}\, \mathbf{C}^{211}_{nm}\, \mathbf{B}^{21}_{zj}\, dz \tag{5.25c}$$

$$\mathbf{K}^{D3}_{ijnm} = -\pi\delta_{nm} \int_0^L \mathbf{B}^{10^T}_{zi}\, \mathbf{C}^{3}\, \mathbf{B}^{20}_{zj}\, dz \tag{5.26}$$

(in the case of m=n=0. see Appendix E)

In above equations, the matrices, $\mathbf{C}^{10}$, $\mathbf{C}^{110}_{nm}$, $\mathbf{C}^{111}_{mn}$, $\mathbf{C}^{20}$, $\mathbf{C}^{210}_{nm}$, $\mathbf{C}^{211}_{nm}$ and $\mathbf{C}^{3}$ are given in Appendix F.

(iv) The geometric stiffness matrices ; refer to eqn.(3.39)

By using eqn. (5.3), eqns. (3.41) can be written as

$$\bar{\varepsilon}_0 = \mathbf{B}^{G0}_{\theta n}\, \mathbf{B}^{G0}_{zi}\, \mathbf{d}_{in} \quad , \quad \bar{\varepsilon}_1 = \mathbf{B}^{G1}_{\theta n}\, \mathbf{B}^{G0}_{zi}\, \mathbf{d}_{in} \tag{5.27a,b}$$

where the matrices, $\mathbf{B}^{G0}_{\theta n}$, $\mathbf{B}^{G1}_{\theta n}$, $\mathbf{B}^{G0}_{zi}$ and $\mathbf{B}^{G1}_{zi}$, are tabulated in Table 5.1.

Applying eqns. (5.27) to eqn. (3.39), the following equation is obtained.

$$\int_{\Omega} \delta u_{i,j} \overset{0}{\tau}_{jm} \Delta u_{i,m} d\Omega$$

$$\approx \delta \mathbf{d}^{T}_{in} \{ \mathbf{K}^{G0}_{ijnm} + \nu \, (\mathbf{K}^{G11}_{ijnm} + \mathbf{K}^{G12}_{ijnm} + \mathbf{K}^{G13}_{ijnm} + (\mathbf{K}^{G13}_{jimn})^{T}) \} \Delta \mathbf{d}_{jm} \tag{5.28}$$

where the geometric stiffness matrices are as follows:

$$\mathbf{K}^{G0}_{ijnm} = \bar{R} \int_{0}^{L} \mathbf{B}^{G0^T}_{zi} \Big(\int_{0}^{2\pi} \mathbf{B}^{G0^T}_{\theta n} \mathbf{N}^{0}_{0} \mathbf{B}^{G0}_{\theta m} d\theta \Big) \mathbf{B}^{G0}_{zj} dz \tag{5.29}$$

$$\mathbf{K}^{G11}_{ijnm} = \bar{R} \int_{0}^{L} \mathbf{B}^{G0^T}_{zi} \Big(\int_{0}^{2\pi} \mathbf{B}^{G0^T}_{\theta n} \mathbf{N}^{0}_{1} \mathbf{B}^{G0}_{\theta m} d\theta \Big) \mathbf{B}^{G0}_{zj} dz \tag{5.30a}$$

$$\mathbf{K}^{G12}_{ijnm} = \bar{R} \int_{0}^{L} \mathbf{B}^{G0^T}_{zi} \Big(\int_{0}^{2\pi} \eta(\theta) \, \mathbf{B}^{G0^T}_{\theta n} \mathbf{N}^{0}_{0} \mathbf{B}^{G0}_{\theta m} d\theta \Big) \mathbf{B}^{G0}_{zj} dz \tag{5.30b}$$

$$\mathbf{K}^{G13}_{ijnm} = \bar{R} \int_{0}^{L} \mathbf{B}^{G0^T}_{zi} \Big(\int_{0}^{2\pi} \mathbf{B}^{G0^T}_{\theta n} \mathbf{N}^{0}_{0} \mathbf{B}^{G1}_{\theta m} d\theta \Big) \mathbf{B}^{G1}_{zj} dz \tag{5.30c}$$

As can be seen in Table 4.1 (chapter 4), the membrane force are decomposed to five components for 0-th and 1st order terms, respectively.

$$\mathbf{N}^{0}_{0} = \mathbf{N}^{(M)}_{0} + \mathbf{N}^{S}_{0} + \mathbf{N}^{V}_{0} + \mathbf{N}^{H}_{0} + \mathbf{N}^{Rm}_{0} \tag{5.31a}$$

$$\mathbf{N}^{0}_{1} = \mathbf{N}^{(M)}_{1} + \mathbf{N}^{S}_{1} + \mathbf{N}^{V}_{1} + \mathbf{N}^{H}_{1} + \mathbf{N}^{Rm}_{1} \tag{5.31b}$$

where the superscripts (M), S, V, H and Rm, represent the lumped mass, hydrostatic pressure, vertical ground motion, horizontal ground motion and rocking motion, respectively, and

$$\mathbf{N}^{\beta}_{\alpha} = \begin{bmatrix} N^{\beta}_{z\alpha}\mathbf{I} & N^{\beta}_{z\psi\alpha}\mathbf{I} \\ N^{\beta}_{z\psi\alpha}\mathbf{I} & N^{\beta}_{\psi\alpha}\mathbf{I} \end{bmatrix} \qquad \begin{pmatrix} \alpha = 0, \ 1 \\ \beta = (M), \ S, \ V, \ H, \ Rm \end{pmatrix} \tag{5.32}$$

Using eqns.(5.31), eqns.(5.29) and (5.30) are rewritten as,

$$\mathbf{K}^{G\beta 0}_{ijnm} = \bar{R} \int_{0}^{L} \mathbf{B}^{G0^T}_{zi} \Big(\int_{0}^{2\pi} \mathbf{B}^{G0^T}_{\theta n} \mathbf{N}^{\beta}_{0} \mathbf{B}^{G0}_{\theta m} d\theta \Big) \mathbf{B}^{G0}_{zj} dz \tag{5.33}$$

$$\mathbf{K}^{G\beta 11}_{ijnm} = \bar{R} \int_{0}^{L} \mathbf{B}^{G0^T}_{zi} \Big(\int_{0}^{2\pi} \mathbf{B}^{G0^T}_{\theta n} \mathbf{N}^{\beta}_{1} \mathbf{B}^{G0}_{\theta m} d\theta \Big) \mathbf{B}^{G0}_{zj} dz \tag{5.34a}$$

$$\mathbf{K}^{G\beta 12}_{ijnm} = \bar{R} \int_0^L \mathbf{B}^{G0^T}_{zi} \left(\int_0^{2\pi} \eta(\theta)\, \mathbf{B}^{G0^T}_{\theta n}\, \mathbf{N}^{\beta}_{0}\, \mathbf{B}^{G0}_{\theta m}\, d\theta \right) \mathbf{B}^{G0}_{zj}\, dz \tag{5.34b}$$

$$\mathbf{K}^{G\beta 13}_{ijnm} = \bar{R} \int_0^L \mathbf{B}^{G0^T}_{zi} \left(\int_0^{2\pi} \mathbf{B}^{G0^T}_{\theta n}\, \mathbf{N}^{\beta}_{0}\, \mathbf{B}^{G1}_{\theta m}\, d\theta \right) \mathbf{B}^{G1}_{zj}\, dz \tag{5.34c}$$

$$(\beta = (M),\ S,\ V,\ H,\ Rm)$$

Furthermore, by integrating in the direction of θ (0~2π) using the equations in Table 4.1 and Table 5.1, eqns.(5.33) and (5.34) can be expressed as explicit forms on θ.

$$\mathbf{K}^{G\beta\alpha}_{ijnm} = \int_0^L \mathbf{B}^{G0^T}_{zi}\, \mathbf{C}^{\beta\alpha}_{nm}(z,t)\, \mathbf{B}^{G0}_{zj}\, dz \tag{5.35}$$

and

$$\mathbf{K}^{G\beta 13}_{ijnm} = \int_0^L \mathbf{B}^{G0^T}_{zi}\, \mathbf{C}^{\beta 13}_{nm}(z,t)\, \mathbf{B}^{G1}_{zj}\, dz \tag{5.36}$$

$$\begin{pmatrix} \alpha = 0,\ 11,\ 12 \\ \beta = (M),\ S,\ V,\ H,\ Rm \end{pmatrix}$$

where $\mathbf{C}^{\beta\alpha}_{nm}(z,t)$ and $\mathbf{C}^{\beta 13}_{nm}(z,t)$ are composed from four blocks of submatrices,

$$\mathbf{C}^{\beta\alpha}_{nm}(z,t) = \begin{bmatrix} \mathbf{C}^{\beta\alpha}_{Anm} & \mathbf{C}^{\beta\alpha}_{Cnm} \\ \mathbf{C}^{\beta\alpha}_{Dnm} & \mathbf{C}^{\beta\alpha}_{Bnm} \end{bmatrix}, \quad \mathbf{C}^{\beta\alpha}_{Dnm} = (\mathbf{C}^{\beta\alpha}_{Cmn})^T \tag{5.37}$$

and

$$\mathbf{C}^{\beta 13}_{nm}(z,t) = \begin{bmatrix} \mathbf{C}^{\beta 13}_{Anm} & \mathbf{C}^{\beta 13}_{Cnm} \\ \mathbf{C}^{\beta 13}_{Dnm} & \mathbf{C}^{\beta 13}_{Bnm} \end{bmatrix} \tag{5.38}$$

The submatrices in eqns.(5.37) and (5.38) are given as:

$$\mathbf{C}^{\beta 0}_{Anm}(z,t) = \bar{R} \int_0^{2\pi} \mathbf{N}^{\beta}_{z0} \cdot \mathbf{B}^{G0}_{1nm}(\theta)\, d\theta \tag{5.39a}$$

$$\mathbf{C}^{\beta 0}_{Bnm}(z,t) = \bar{R} \int_0^{2\pi} \mathbf{N}^{\beta}_{\Psi 0} \cdot \mathbf{B}^{G0}_{2nm}(\theta)\, d\theta \tag{5.39b}$$

$$\mathbf{C}^{\beta 0}_{\mathrm{Cnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{z\Psi_0}^{\beta} \cdot \mathbf{B}^{G0}_{3nm}\,(\theta)\; d\theta \tag{5.39c}$$

$$\mathbf{C}^{\beta 11}_{\mathrm{Anm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{z1}^{\beta} \cdot \mathbf{B}^{G0}_{1nm}\,(\theta)\; d\theta \tag{5.40a}$$

$$\mathbf{C}^{\beta 11}_{\mathrm{Bnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{\psi_1}^{\beta} \cdot \mathbf{B}^{G0}_{2nm}\,(\theta)\; d\theta \tag{5.40b}$$

$$\mathbf{C}^{\beta 11}_{\mathrm{Cnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{z\Psi_1}^{\beta} \cdot \mathbf{B}^{G0}_{3nm}\,(\theta)\; d\theta \tag{5.40c}$$

$$\mathbf{C}^{\beta 12}_{\mathrm{Anm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \eta(\theta)\; \mathbf{N}_{z0}^{\beta} \cdot \mathbf{B}^{G0}_{1nm}\,(\theta)\; d\theta \tag{5.41a}$$

$$\mathbf{C}^{\beta 12}_{\mathrm{Bnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \eta(\theta)\; \mathbf{N}_{\psi_0}^{\beta} \cdot \mathbf{B}^{G0}_{2nm}\,(\theta)\; d\theta \tag{5.41b}$$

$$\mathbf{C}^{\beta 12}_{\mathrm{Cnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \eta(\theta)\; \mathbf{N}_{z\Psi_0}^{\beta} \cdot \mathbf{B}^{G0}_{3nm}\,(\theta)\; d\theta \tag{5.41c}$$

and

$$\mathbf{C}^{\beta 13}_{\mathrm{Anm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{z\Psi_0}^{\beta} \cdot \mathbf{B}^{G1}_{1nm}\,(\theta)\; d\theta \tag{5.42a}$$

$$\mathbf{C}^{\beta 13}_{\mathrm{Bnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{\psi_0}^{\beta} \cdot \mathbf{B}^{G1}_{2nm}\,(\theta)\; d\theta \tag{5.42b}$$

$$\mathbf{C}^{\beta 13}_{\mathrm{Cnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{z\Psi_0}^{\beta} \cdot \mathbf{B}^{G1}_{3nm}\,(\theta)\; d\theta \tag{5.42c}$$

$$\mathbf{C}^{\beta 13}_{\mathrm{Dnm}}\,(z,t) = \bar{R} \int_{0}^{2\pi} \mathbf{N}_{\psi_0}^{\beta} \cdot \mathbf{B}^{G1}_{4nm}\,(\theta)\; d\theta \tag{5.42d}$$

where the matries, $\mathbf{B}^{G0}_{1nm}$, $\mathbf{B}^{G0}_{2nm}$, $\mathbf{B}^{G0}_{3nm}$, $\mathbf{B}^{G1}_{1nm}$, $\mathbf{B}^{G1}_{2nm}$, $\mathbf{B}^{G1}_{3nm}$ and $\mathbf{B}^{G1}_{4nm}$ are given in Appendix H.

By applying the membrane forces in Table 4.1 (chapter 4), eqns.(5.39) through (5.42) can be obtained explicitly as follows.

(a) Effect of a lumped mass.

$$\mathbf{C}_{\mathrm{Anm}}^{(\mathrm{M})0}(z,t) = \frac{\mathrm{M}}{\pi\bar{\mathrm{R}}^2}\left\{(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))(\mathrm{L_1} - z)\ \mathbf{A}_{1\mathrm{nm}}^{0\mathrm{C}} + \frac{\bar{\mathrm{R}}}{2}(\mathrm{G_V}(t) - g)\ \mathbf{A}_{1\mathrm{nm}}^{01}\right\} \tag{5.43a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{(\mathrm{M})0}(z,t) = \mathbf{0} \tag{5.43b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{(\mathrm{M})0}(z,t) = \frac{\mathrm{M}}{\pi\bar{\mathrm{R}}}(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))\ \mathbf{A}_{3\mathrm{nm}}^{0\mathrm{S}} \tag{5.43c}$$

$$\mathbf{C}_{\mathrm{Anm}}^{(\mathrm{M})11}(z,t) = -\frac{\mathrm{M}}{\pi\bar{\mathrm{R}}^2}(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))(\mathrm{L_1} - z)\left\{\mathbf{A}_{1\mathrm{nm}}^{0\eta\mathrm{C}} + \mathbf{A}_{1\mathrm{nm}}^{0\eta2\mathrm{C}} - \mathbf{A}_{1\mathrm{nm}}^{0\eta1\mathrm{S}} + \frac{3}{2}\mathbf{C}_2^{\mathrm{C}}\ \mathbf{A}_{1\mathrm{nm}}^{0\mathrm{C}}\right\} \tag{5.44a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{(\mathrm{M})11}(z,t) = \mathbf{0} \tag{5.44b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{(\mathrm{M})11}(z,t) = -\frac{\mathrm{M}}{\pi\bar{\mathrm{R}}}(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))\left\{\mathbf{A}_{3\mathrm{nm}}^{0\eta1\mathrm{C}} + \frac{3}{2}\mathbf{C}_2^{\mathrm{C}}\ \mathbf{A}_{3\mathrm{nm}}^{0\mathrm{S}}\right\} \tag{5.44c}$$

$$\mathbf{C}_{\mathrm{Anm}}^{(\mathrm{M})12}(z,t) = \frac{\mathrm{M}}{\pi\bar{\mathrm{R}}^2}\left\{(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))(\mathrm{L_1} - z)\ \mathbf{A}_{1\mathrm{nm}}^{0\eta\mathrm{C}} + \frac{\bar{\mathrm{R}}}{2}(\mathrm{G_V}(t) - g)\ \mathbf{A}_{1\mathrm{nm}}^{0\eta}\right\} \tag{5.45a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{(\mathrm{M})12}(z,t) = \mathbf{0} \tag{5.45b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{(\mathrm{M})12}(z,t) = \frac{\mathrm{M}}{\pi\bar{\mathrm{R}}}(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))\ \mathbf{A}_{3\mathrm{nm}}^{0\eta\mathrm{S}} \tag{5.45c}$$

$$\mathbf{C}_{\mathrm{Anm}}^{(\mathrm{M})13}(z,t) = \frac{\mathrm{M}}{\pi\bar{\mathrm{R}}}(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))\ \mathbf{A}_{1\mathrm{nm}}^{1\mathrm{S}} \tag{5.46a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{(\mathrm{M})13}(z,t) = \mathbf{0} \tag{5.46b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{(\mathrm{M})13}(z,t) = \frac{\mathrm{M}}{\pi\bar{\mathrm{R}}}(\mathrm{G_h}(t) + \mathrm{L_1}\ddot{q}(t))\ \mathbf{A}_{3\mathrm{nm}}^{1\mathrm{S}} \tag{5.46c}$$

$$\mathbf{C}_{\mathrm{Dnm}}^{(\mathrm{M})13}(z,t) = \mathbf{0} \tag{5.46d}$$

(b) Effect of the hydrostatic pressure.

$$C^{S0}_{Anm}(z) = 0 \tag{5.47a}$$

$$C^{S0}_{Bnm}(z) = \bar{R}\rho_F g\,(H - z)\,A^{01}_{2nm} \tag{5.47b}$$

$$C^{S0}_{Cnm}(z) = 0 \tag{5.47c}$$

$$C^{S11}_{Anm}(z) = \frac{1}{6\bar{R}}\rho_F g\,(H - z)^3\left\{A^{0\eta 2}_{1nm} + A^{0\eta 4}_{1nm}\right\} \tag{5.48a}$$

$$C^{S11}_{Bnm}(z) = \bar{R}\,\rho_F g\,(H - z)\left\{A^{0\eta}_{2nm} + A^{0\eta 2}_{2nm}\right\} \tag{5.48b}$$

$$C^{S11}_{Cnm}(z) = \frac{1}{2}\rho_F g\,(H - z)^2\left\{A^{0\eta 1}_{3nm} + A^{0\eta 3}_{3nm}\right\} \tag{5.48c}$$

$$C^{S12}_{Anm}(z) = 0 \tag{5.49a}$$

$$C^{S12}_{Bnm}(z) = \bar{R}\rho_F g\,(H - z)\,A^{0\eta}_{2nm} \tag{5.49b}$$

$$C^{S12}_{Cnm}(z) = 0 \tag{5.49c}$$

$$C^{S13}_{Anm}(z) = 0 \tag{5.50a}$$

$$C^{S13}_{Bnm}(z) = \bar{R}\rho_F g\,(H - z)\,A^{11}_{2nm} \tag{5.50b}$$

$$C^{S13}_{Cnm}(z) = 0 \tag{5.50c}$$

$$C^{S13}_{Dnm}(z) = \bar{R}\rho_F g\,(H - z)\,A^{11}_{4nm} \tag{5.50d}$$

(c) Effect of the pressure due to vertical ground motion.

$$C^{V0}_{Anm}(z,t) = 0 \tag{5.51a}$$

$$C^{V0}_{Bnm}(z,t) = -\,\bar{R}\rho_F G_V(t)\,(H - z)\,A^{01}_{2nm} \tag{5.51b}$$

$$C^{V0}_{Cnm}(z,t) = 0 \tag{5.51c}$$

$$C^{V11}_{Anm}(z,t) = -\,\frac{1}{6\bar{R}}\rho_F G_V(t)\,(H - z)^3\left\{A^{0\eta 2}_{1nm} + A^{0\eta 4}_{1nm}\right\} \tag{5.52a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{\mathrm{V11}}(z,t) = -\bar{R}\,\rho_F G_V(t)\,(H-z)\left\{\mathbf{A}_{2nm}^{0\eta} + \mathbf{A}_{2nm}^{0\eta 2}\right\} \tag{5.52b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{\mathrm{V11}}(z,t) = -\frac{1}{2}\rho_F G_V(t)\,(H-z)^2\left\{\mathbf{A}_{3nm}^{0\eta 1} + \mathbf{A}_{3nm}^{0\eta 3}\right\} \tag{5.52c}$$

$$\mathbf{C}_{\mathrm{Anm}}^{\mathrm{V12}}(z,t) = \mathbf{0} \tag{5.53a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{\mathrm{V12}}(z,t) = -\bar{R}\rho_F G_V(t)\,(H-z)\,\mathbf{A}_{2nm}^{0\eta} \tag{5.53b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{\mathrm{V12}}(z,t) = \mathbf{0} \tag{5.53c}$$

$$\mathbf{C}_{\mathrm{Anm}}^{\mathrm{V13}}(z,t) = \mathbf{0} \tag{5.54a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{\mathrm{V13}}(z,t) = -\bar{R}\rho_F G_V(t)\,(H-z)\,\mathbf{A}_{2nm}^{11} \tag{5.54b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{\mathrm{V13}}(z,t) = \mathbf{0} \tag{5.54c}$$

$$\mathbf{C}_{\mathrm{Dnm}}^{\mathrm{V13}}(z,t) = -\bar{R}\rho_F G_V(t)\,(H-z)\,\mathbf{A}_{4nm}^{11} \tag{5.54d}$$

(d) Effect of the pressure due to horizontal ground motion

$$\mathbf{C}_{\mathrm{Anm}}^{\mathrm{H0}}(z,t) = \sum_{i=1}^{\infty} 2G^{-}{}_{h}(\omega)\,\frac{(-1)^{i+1}}{\mu_i\bar{R}}\hat{F}_{i1}(\bar{R},z,t)\,\mathbf{A}_{1nm}^{0C} \tag{5.55a}$$

$$\mathbf{C}_{\mathrm{Bnm}}^{\mathrm{H0}}(z,t) = -\sum_{i=1}^{\infty} 2\bar{G}_h(\omega)\,\frac{(-1)^{i+1}\bar{R}}{\mu_i}F_{i1}(\bar{R},z,t)\,\mathbf{A}_{2nm}^{0C} \tag{5.55b}$$

$$\mathbf{C}_{\mathrm{Cnm}}^{\mathrm{H0}}(z,t) = -\sum_{i=1}^{\infty} 2\bar{G}_h(\omega)\,\frac{(-1)^{i+1}}{\mu_i}\hat{F}_{i1}(\bar{R},z,t)\,\mathbf{A}_{3nm}^{0S} \tag{5.55c}$$

$$\mathbf{C}_{\mathrm{Anm}}^{\mathrm{H11}}(z,t) = -\frac{2\bar{G}_h(\omega)}{\bar{R}}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{\mu_i}\hat{F}_{i1}(\bar{R},z,t)$$

$$\left\{\left(1-\frac{1}{I_{1i}^{<1>}}\right)\mathbf{A}_{1nm}^{0\eta C} + \frac{1}{I_{1i}^{<1>}}\mathbf{A}_{1nm}^{0\eta 2C} + \mathbf{A}_{1nm}^{0\eta 4C} - \left(1+\frac{2}{I_{1i}^{<1>}}\right)\mathbf{A}_{1nm}^{0\eta 1S} - 2\,\mathbf{A}_{1nm}^{0\eta 3S}\right\}$$

$$-\frac{\pi^2}{4}\bar{G}_h(\omega)\sum_{k=1}^{\infty}(-1)^{k+1}\left\{(n-m)^2\,\hat{\mathcal{F}}_{|n-m|,k}(z,t)\left(K_{knm}^{C<1>}\,\mathbf{I} + K_{knm}^{S<1>}\,\mathbf{L}_{11}^{S}\right)\right.$$

$$+ (n+m)^2 \hat{F}_{n+m,k}(z,t) \left(K^{C<2>}_{knm} \mathbf{L}^{C}_{12} + K^{S<2>}_{knm} \mathbf{L}^{S}_{12} \right) \Big\} \tag{5.56a}$$

$$\mathbf{C}^{H11}_{Bnm}(z,t) = 2\bar{G}_h(\omega)\bar{R} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{\mu_i} F_{i1}(\bar{R},z,t) \left\{ \left(\frac{1}{I^{<1>}_{1i}} + 1 \right) \mathbf{A}^{0\eta C}_{2nm} + \mathbf{A}^{0\eta 2C}_{2nm} \right\}$$

$$+ \frac{\pi}{2} \bar{G}_h(\omega) \bar{R} c^{c}_{i} \sum_{k=1}^{\infty} (-1)^{k+1} F_{0k}(z,t) \left(1 - I^{<1>}_{1k} + I^{<2>}_{1k} \right) \mathbf{A}^{01}_{2nm}$$

$$+ \frac{\pi^2}{4} \bar{G}_h(\omega)\bar{R}^2 \sum_{k=1}^{\infty} (-1)^{k+1} \Big\{ F_{|n-m|,k}(z,t) \left(K^{C<1>}_{knm} \mathbf{L}^{C}_{21} + K^{S<1>}_{knm} \mathbf{L}^{S}_{21} \right) (1 - \delta_{nm})$$

$$+ F_{n+m,k}(z,t) \left(K^{C<2>}_{knm} \mathbf{L}^{C}_{22} + K^{S<2>}_{knm} \mathbf{L}^{S}_{22} \right) \Big\} \tag{5.56b}$$

$$\mathbf{C}^{H11}_{Cnm}(z,t) = 2\bar{G}_h(\omega) \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{\mu_i} \hat{F}_{i1}(\bar{R},z,t)$$

$$* \left\{ \mathbf{A}^{0\eta 2S}_{3nm} + \frac{1}{I^{<1>}_{1i}} \mathbf{A}^{0\eta S}_{3nm} - \left(1 + \frac{1}{I^{<1>}_{1i}} \right) \mathbf{A}^{0\eta 1C}_{3nm} - \mathbf{A}^{0\eta 3C}_{3nm} \right\}$$

$$+ \frac{\pi^2}{4} \bar{G}_h(\omega)\bar{R} \sum_{k=1}^{\infty} (-1)^{k+1} \Big\{ |n-m| \hat{F}_{|n-m|,k}(z,t) \left(K^{C<3>}_{knm} \mathbf{L}^{S}_{31} - K^{S<3>}_{knm} \mathbf{L}^{C}_{31} \right)$$

$$+ (n+m) \hat{F}_{n+m,k}(z,t) \left(K^{C<2>}_{knm} \mathbf{L}^{S}_{32} - K^{S<2>}_{knm} \mathbf{L}^{C}_{32} \right) \Big\} \tag{5.56c}$$

where $K^{C<i>}_{knm}$ and $K^{S<i>}_{knm}$ (i = 1,3) are given in Appendix L.

$\mathbf{L}^{C}_{ij}$ and $\mathbf{L}^{S}_{ij}$ (i,j = 1,2) are given in Appendix I (eqns.(I - 12) and (I - 13)).

$$\mathbf{C}^{H12}_{Anm}(z,t) = \sum_{i=1}^{\infty} 2\bar{G}_h(\omega) \frac{(-1)^{i+1}}{\mu_i \bar{R}} \hat{F}_{i1}(\bar{R},z,t) \mathbf{A}^{0\eta C}_{1nm} \tag{5.57a}$$

$$\mathbf{C}^{H12}_{Bnm}(z,t) = - \sum_{i=1}^{\infty} 2\bar{G}_h(\omega) \frac{(-1)^{i+1}\bar{R}}{\mu_i} F_{i1}(\bar{R},z,t) \mathbf{A}^{0\eta C}_{2nm} \tag{5.57b}$$

$$\mathbf{C}^{H12}_{Cnm}(z,t) = - \sum_{i=1}^{\infty} 2\bar{G}_h(\omega) \frac{(-1)^{i+1}}{\mu_i} \hat{F}_{i1}(\bar{R},z,t) \mathbf{A}^{0\eta S}_{3nm} \tag{5.57c}$$

$$C_{Anm}^{H13}(z,t) = -\sum_{i=1}^{\infty} 2\overline{G}_h(\omega)\frac{(-1)^{i+1}}{\mu_i}\hat{F}_{i1}(\overline{R},z,t)\ A_{1nm}^{1S} \qquad (5.58a)$$

$$C_{Bnm}^{H13}(z,t) = -\sum_{i=1}^{\infty} 2\overline{G}_h(\omega)\frac{(-1)^{i+1}\overline{R}}{\mu_i}F_{i1}(\overline{R},z,t)\ A_{2nm}^{1C} \qquad (5.58b)$$

$$C_{Cnm}^{H13}(z,t) = -\sum_{i=1}^{\infty} 2\overline{G}_h(\omega)\frac{(-1)^{i+1}}{\mu_i}\hat{F}_{i1}(\overline{R},z,t)\ A_{3nm}^{1S} \qquad (5.58c)$$

$$C_{Dnm}^{H13}(z,t) = -\sum_{i=1}^{\infty} 2\overline{G}_h(\omega)\frac{(-1)^{i+1}\overline{R}}{\mu_i}F_{i1}(\overline{R},z,t)\ A_{4nm}^{1C} \qquad (5.58d)$$

(e) Effect of the pressure due to rocking motion

$$C_{Anm}^{Rm0}(z,t) = -\sum_{i=1}^{\infty}\frac{2\omega^2H^2\overline{q}(\omega)}{\overline{R}}\left\{\frac{\mu_iH\cdot(-1)^{i+1}-1}{(\mu_iH)^2}\hat{F}_{i1}(\overline{R},z,t) - \frac{\overline{R}J_2(\varepsilon_i\overline{R})}{\varepsilon_iH^2}\hat{Q}_{i1}(\overline{R},z,t)\right\}\ A_{1nm}^{0C} \qquad (5.59a)$$

$$C_{Bnm}^{Rm0}(z,t) = \sum_{i=1}^{\infty} 2\omega^2H^2\overline{R}\ \overline{q}(\omega)\left\{\frac{\mu_iH\cdot(-1)^{i+1}-1}{(\mu_iH)^2}F_{i1}(\overline{R},z,t) - \frac{\overline{R}J_2(\varepsilon_i\overline{R})}{\varepsilon_iH^2}Q_{i1}(\overline{R},z,t)\right\}\ A_{2nm}^{0C} \qquad (5.59b)$$

$$C_{Cnm}^{Rm0}(z,t) = \sum_{i=1}^{\infty} 2\omega^2H^2\overline{q}(\omega)\left\{\frac{\mu_iH\cdot(-1)^{i+1}-1}{(\mu_iH)^2}\hat{F}_{i1}(\overline{R},z,t) - \frac{\overline{R}J_2(\varepsilon_i\overline{R})}{\varepsilon_iH^2}\hat{Q}_{i1}(\overline{R},z,t)\right\}\ A_{3nm}^{0S} \qquad (5.59c)$$

$$\begin{aligned}C_{Anm}^{Rm11}(z,t) = \frac{2\omega^2\overline{q}(\omega)}{\overline{R}}\sum_{i=1}^{\infty}\Bigg[&\frac{\mu_iH\cdot(-1)^{i+1}-1}{\mu_i^2}\hat{F}_{i1}(\overline{R},z,t)\\ &*\left\{\left(1-\frac{1}{I_{1i}^{<1>}}\right)A_{1nm}^{0\eta C}+\frac{1}{I_{1i}^{<1>}}A_{1nm}^{0\eta 2C}+A_{1nm}^{0\eta 4C}-\left(1+\frac{2}{I_{1i}^{<1>}}\right)A_{1nm}^{0\eta 1S}-2A_{1nm}^{0\eta 3S}\right\}\\ &-\frac{\overline{R}J_2(\varepsilon_i\overline{R})}{\varepsilon_i}\hat{Q}_{i1}(\overline{R},z,t)\left\{A_{1nm}^{0\eta C}+A_{1nm}^{0\eta 4C}-A_{1nm}^{0\eta 1S}-2A_{1nm}^{0\eta 3S}\right\}\Bigg]\end{aligned}$$

$$+\frac{\pi^2\omega^2\overline{q}(\omega)}{4}\sum_{k=1}^{\infty}\mu_k\left\{(n-m)^2\,\hat{F}_{|n-m|,k}(z,t)\,(\overline{K}\,{}_{knm}^{C<1>}\,\mathbf{I}+\overline{K}\,{}_{knm}^{S<1>}\,\mathbf{L}_{11}^{S})\right.$$

$$\left.+(n+m)^2\,\hat{F}_{n+m,k}(z,t)\,(\overline{K}\,{}_{knm}^{C<2>}\,\mathbf{L}_{12}^{C}+\overline{K}\,{}_{knm}^{S<2>}\,\mathbf{L}_{12}^{S})\right\} \tag{5.60a}$$

$$\mathbf{C}_{Bnm}^{Rm11}(z,t)=2\omega^2\overline{q}(\omega)\overline{R}\sum_{i=1}^{\infty}\left[\frac{\mu_i H\cdot(-1)^{i+1}-1}{\mu_i^2}F_{i1}(\overline{R},z,t)\right.$$

$$\left.*\left\{(1+\frac{1}{I\,{}_{1i}^{<1>}})\,\mathbf{A}_{2nm}^{0\eta C}+\mathbf{A}_{2nm}^{0\eta 2C}\right\}-\frac{\overline{R}J_2(\varepsilon_i\overline{R})}{\varepsilon_i}Q_{i1}(\overline{R},z,t)\,\left(\mathbf{A}_{2nm}^{0\eta C}+\mathbf{A}_{2nm}^{0\eta 2C}\right)\right]$$

$$-\frac{\pi\omega^2\overline{q}(\omega)\overline{R}\cdot c_1^c}{2}\sum_{k=1}^{\infty}\mu_k F_{0k}(z,t)\left\{\frac{\mu_k H\cdot(-1)^{k+1}-1}{\mu_k{}^2}(1-I\,{}_{1k}^{<1>}+I\,{}_{1k}^{<2>})\right.$$

$$\left.+\sum_{i=1}^{\infty}\frac{J\,{}_{i}^{<2>}}{\varepsilon_i^2+\mu_k^2}(J\,{}_{i}^{<1>}-1)\right\}\mathbf{A}_{2nm}^{01}$$

$$-\frac{\pi^2\omega^2\overline{q}(\omega)\overline{R}^2}{4}\sum_{k=1}^{\infty}\mu_k\left\{F_{|n-m|,k}(z,t)\,(\overline{K}\,{}_{knm}^{C<1>}\,\mathbf{L}_{21}^{C}+\overline{K}\,{}_{knm}^{S<1>}\,\mathbf{L}_{21}^{S})\,(1-\delta_{nm})\right.$$

$$\left.+F_{n+m,k}(z,t)\,(\overline{K}\,{}_{knm}^{C<2>}\,\mathbf{L}_{22}^{C}+\overline{K}\,{}_{knm}^{S<2>}\,\mathbf{L}_{22}^{S})\right\} \tag{5.60b}$$

$$\mathbf{C}_{Cnm}^{Rm11}(z,t)=+2\omega^2\overline{q}(\omega)\sum_{i=1}^{\infty}\left[\frac{\mu_i H\cdot(-1)^{i+1}-1}{\mu_i^2}\hat{F}_{i1}(\overline{R},z,t)\right.$$

$$*\left\{\mathbf{A}_{3nm}^{0\eta 2S}+\frac{1}{I\,{}_{1i}^{<1>}}\mathbf{A}_{3nm}^{0\eta S}-(1+\frac{1}{I\,{}_{1i}^{<1>}})\,\mathbf{A}_{3nm}^{0\eta 1C}-\mathbf{A}_{3nm}^{0\eta 3C}\right\}$$

$$\left.+\frac{\overline{R}J_2(\varepsilon_i\overline{R})}{\varepsilon_i}\hat{Q}_{i1}(\overline{R},z,t)\,\left(\mathbf{A}_{3nm}^{0\eta 1C}+\mathbf{A}_{3nm}^{0\eta 3C}-\mathbf{A}_{3nm}^{0\eta 2S}\right)\right]$$

$$-\frac{\pi^2\omega^2\overline{q}(\omega)\overline{R}}{4}\sum_{k=1}^{\infty}\mu_k\left\{|n-m|\,\hat{F}_{|n-m|,k}(z,t)\,(\overline{K}\,{}_{knm}^{C<3>}\,\mathbf{L}_{31}^{C}-\overline{K}\,{}_{knm}^{S<3>}\,\mathbf{L}_{31}^{S})\right.$$

$$\left.+(n+m)\,\hat{F}_{n+m,k}(z,t)\,(\overline{K}\,{}_{knm}^{C<2>}\,\mathbf{L}_{32}^{C}-\overline{K}\,{}_{knm}^{S<2>}\,\mathbf{L}_{32}^{S})\right\} \tag{5.60c}$$

where $\overline{K}\,{}_{knm}^{C<i>}$ and $\overline{K}\,{}_{knm}^{S<i>}$ (i = 1,3) are given in Appendix L.

$\mathbf{L}^{S}_{ij}$ and $\mathbf{L}^{c}_{ij}$ (i,j = 1,2) are given in Appendix I (eqns. (I - 12) and (I - 13)).

$$\mathbf{C}^{Rm12}_{Anm}(z,t) = -\sum_{i=1}^{\infty} \frac{2\omega^2 H^2 \bar{q}(\omega)}{\bar{R}} \left\{ \frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t) \right\} \mathbf{A}^{0\eta C}_{1nm} \tag{5.61a}$$

$$\mathbf{C}^{Rm12}_{Bnm}(z,t) = \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{R}\, \bar{q}(\omega) \left\{ \frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} F_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} Q_{i1}(\bar{R},z,t) \right\} \mathbf{A}^{0\eta C}_{2nm} \tag{5.61b}$$

$$\mathbf{C}^{Rm12}_{Cnm}(z,t) = \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \left\{ \frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t) \right\} \mathbf{A}^{0\eta S}_{3nm} \tag{5.61c}$$

$$\mathbf{C}^{Rm13}_{Anm}(z,t) = \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \left\{ \frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t) \right\} \mathbf{A}^{1S}_{1nm} \tag{5.62a}$$

$$\mathbf{C}^{Rm13}_{Bnm}(z,t) = \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{R}\, \bar{q}(\omega) \left\{ \frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} F_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} Q_{i1}(\bar{R},z,t) \right\} \mathbf{A}^{1C}_{2nm} \tag{5.62b}$$

$$\mathbf{C}^{Rm13}_{Cnm}(z,t) = \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{q}(\omega) \left\{ \frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} \hat{F}_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} \hat{Q}_{i1}(\bar{R},z,t) \right\} \mathbf{A}^{1S}_{3nm} \tag{5.62c}$$

$$\mathbf{C}^{Rm13}_{Dnm}(z,t) = \sum_{i=1}^{\infty} 2\omega^2 H^2 \bar{R}\, \bar{q}(\omega) \left\{ \frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} F_{i1}(\bar{R},z,t) - \frac{\bar{R} J_2(\varepsilon_i \bar{R})}{\varepsilon_i H^2} Q_{i1}(\bar{R},z,t) \right\} \mathbf{A}^{1C}_{4nm} \tag{5.62d}$$

The matrices in eqns.(5.43) through (5.62), $\mathbf{A}^{01}_{\alpha nm}$, $\mathbf{A}^{0S}_{\alpha nm}$, $\mathbf{A}^{0C}_{\alpha nm}$, $\mathbf{A}^{0\eta}_{\alpha nm}$, $\mathbf{A}^{0\eta\beta}_{\alpha nm}$, $\mathbf{A}^{0\eta\beta S}_{\alpha nm}$, $\mathbf{A}^{0\eta\beta C}_{\alpha nm}$ (β = 1,4), $\mathbf{A}^{0S}_{\alpha nmp}$, $\mathbf{A}^{0C}_{\alpha nmp}$ (α = 1,3), $\mathbf{A}^{11}_{\alpha nm}$, $\mathbf{A}^{1S}_{\alpha nm}$ and $\mathbf{A}^{1C}_{\alpha nm}$ (α = 1,4), are given in Appendix I.

(v) The load correction matrices : refer to eqn.(3.51)

By using eqn.(5.3), eqn.(3.52) can be written as

$$\varepsilon^* = \mathbf{B}^*_{\theta n} \mathbf{B}^*_{zi} \mathbf{d}_{in} \tag{5.63}$$

where the matrices, $\mathbf{B}^*_{\theta n}$ and $\mathbf{B}^*_{zi}$, are tabulated in Table 5.1. Applying eqn.(5.63) to eqn.(3.51), the following equation is obtained.

$$I_n \approx \delta \mathbf{d}^T_{in} \left\{ \mathbf{K}^{P0}_{ijnm} + \nu \left(\mathbf{K}^{P11}_{ijnm} + \mathbf{K}^{P12}_{ijnm} \right) \right\} \Delta \mathbf{d}_{jm} \qquad (5.64)$$

where the load correction matrices are as follows.

$$\mathbf{K}^{P0}_{ijnm} = \frac{\overline{R}}{2} \int_0^H \mathbf{B}^{*T}_{zi} \left(\int_0^{2\pi} P^W_0 \, \mathbf{B}^*_{0nm} \, d\theta \right) \mathbf{B}^*_{zj} \, dz \qquad (5.65a)$$

$$\mathbf{K}^{P11}_{ijnm} = \frac{\overline{R}}{2} \int_0^H \mathbf{B}^{*T}_{zi} \left(\int_0^{2\pi} P^W_1 \, \mathbf{B}^*_{0nm} \, d\theta \right) \mathbf{B}^*_{zj} \, dz \qquad (5.65b)$$

$$\mathbf{K}^{P12}_{ijnm} = \frac{\overline{R}}{2} \int_0^H \mathbf{B}^{*T}_{zi} \left(\int_0^{2\pi} P^W_0 \, \mathbf{B}^*_{1nm} \, d\theta \right) \mathbf{B}^*_{zj} \, dz \qquad (5.65c)$$

and

$$\mathbf{B}^*_{0nm} = \mathbf{B}^{*T}_{\theta n} \, \mathbf{P}^{(0)} \, \mathbf{B}^*_{\theta m} \qquad (5.66a)$$

$$\mathbf{B}^*_{1nm} = \mathbf{B}^{*T}_{\theta n} \, \mathbf{P}^{(1)} \, \mathbf{B}^*_{\theta m} \qquad (5.66b)$$

Eqns.(5.66) are given in Appendix H (eqns.(H - 3)).

Considering eqn.(3.44) with Tables 2.3(1)~(3), eqns.(5.65) can be written as.

$$\mathbf{K}^{P\alpha\beta}_{ijnm} = \int_0^H \mathbf{B}^{*T}_{zi} \, \mathbf{C}^{P\alpha\beta}_{nm} (z,t) \, \mathbf{B}^*_{zj} \, dz \qquad (5.67a)$$

$$(\alpha = S, V, H, Rm \;\; ; \;\; \beta = 0, 12)$$

$$\mathbf{K}^{P\alpha 11}_{ijnm} = \int_0^H \mathbf{B}^{*T}_{zi} \, \mathbf{C}^{P\alpha 11}_{nm} (z,t) \, \mathbf{B}^*_{zj} \, dz \qquad (5.67b)$$

$$(\alpha = H, Rm)$$

where

$$\mathbf{C}^{PS0}_{nm} (z) = \frac{\overline{R}}{2} \rho_F g (H - z) \, \mathbf{C}^{01}_{nm} \qquad (5.68a)$$

$$\mathbf{C}_{nm}^{PV0}(z,t) = -\frac{\overline{R}}{2}\rho_F G_V(t)(H-z)\,\mathbf{C}_{nm}^{01} \tag{5.68b}$$

$$\mathbf{C}_{nm}^{PH0}(z,t) = -\sum_{i=1}^{\infty}\overline{G}_h(\omega)\overline{R}\cdot\frac{(-1)^{i+1}}{\mu_i}F_{i1}(\overline{R},z,t)\,\mathbf{C}_{nm}^{0C} \tag{5.68c}$$

$$\mathbf{C}_{nm}^{PRm0}(z,t) = \sum_{i=1}^{\infty}\omega^2H^2\overline{R}\overline{q}(\omega)\left\{\frac{\mu_iH\cdot(-1)^{i+1}-1}{(\mu_iH)^2}F_{i1}(\overline{R},z,t) - \frac{\overline{R}J_2(\varepsilon_i\overline{R})}{\varepsilon_iH^2}Q_{i1}(\overline{R},z,t)\right\}\mathbf{C}_{nm}^{0C} \tag{5.68d}$$

$$\begin{aligned}
\mathbf{C}_{nm}^{PH11}(z,t) = &-\overline{G}_h(\omega)\overline{R}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{\mu_i\,I_{1i}^{<1>}}F_{i1}(\overline{R},z,t)\,\mathbf{C}_{nm}^{0\eta C}\\
&+\frac{\pi}{4}\overline{G}_h(\omega)\overline{R}\cdot c_1^c\sum_{k=1}^{\infty}(-1)^{k+1}F_{0k}(z,t)\left(1-I_{1k}^{<1>}+I_{1k}^{<2>}\right)\mathbf{C}_{nm}^{01}\\
&+\frac{\pi^2}{8}\overline{G}_h(\omega)\overline{R}\sum_{k=1}^{\infty}(-1)^{k+1}\left\{F_{|n-m|,k}(z,t)\left(K_{knm}^{C<1>}\mathbf{L}_{p1}^{C}+K_{knm}^{S<1>}\mathbf{L}_{p1}^{S}\right)(1-\delta_{nm})\right.\\
&\left.+F_{n+m,k}(z,t)\left(K_{knm}^{C<2>}\mathbf{L}_{p2}^{C}+K_{knm}^{S<2>}\mathbf{L}_{p2}^{S}\right)\right\}
\end{aligned} \tag{5.69a}$$

$$\begin{aligned}
\mathbf{C}_{nm}^{PRm11}(z,t) = &\;\omega^2\overline{q}(\omega)\overline{R}\sum_{i=1}^{\infty}\frac{\mu_iH(-1)^{i+1}-1}{\mu_i^2\,I_{1i}^{<1>}}F_{i1}(\overline{R},z,t)\,\mathbf{C}_{nm}^{0\eta C}\\
&-\frac{\pi}{4}\omega^2\overline{q}(\omega)\overline{R}\cdot c_1^c\sum_{k=1}^{\infty}\mu_k F_{0k}(z,t)\\
&*\left\{\frac{\mu_kH(-1)^{k+1}-1}{\mu_k^2}\left(1-I_{1k}^{<1>}+I_{1k}^{<2>}\right)+\sum_{i=1}^{\infty}\frac{J_i^{<2>}}{\varepsilon_i^2+\mu_k^2}\left(J_i^{<1>}-1\right)\right\}\mathbf{C}_{nm}^{01}\\
&-\frac{\pi^2}{8}\omega^2\overline{q}(\omega)\overline{R}\sum_{k=1}^{\infty}\mu_k\left\{F_{|n-m|,k}(z,t)\left(\overline{K}_{knm}^{C<1>}\mathbf{L}_{p1}^{C}+\overline{K}_{knm}^{S<1>}\mathbf{L}_{p1}^{S}\right)(1-\delta_{nm})\right.\\
&\left.+F_{n+m,k}(z,t)\left(\overline{K}_{knm}^{C<2>}\mathbf{L}_{p2}^{C}+\overline{K}_{knm}^{S<2>}\mathbf{L}_{p2}^{S}\right)\right\}
\end{aligned} \tag{5.69b}$$

where $K_{knm}^{C<i>}$, $K_{knm}^{S<i>}$, $\overline{K}_{knm}^{C<i>}$ and $\overline{K}_{knm}^{S<i>}$ (i = 1,3) are given in Appendix L.

$\mathbf{L}_{pi}^{C}$ and $\mathbf{L}_{pi}^{S}$ (i = 1,2) are given in Appendix J (eqns. (J - 13) and (J - 14)).

$$C^{PS12}_{nm}(z) = \frac{\overline{R}}{2}\rho_F g(H - z)\, C^{11}_{nm} \tag{5.70a}$$

$$C^{PV12}_{nm}(z,t) = -\frac{\overline{R}}{2}\rho_F G_V(t)(H - z)\, C^{11}_{nm} \tag{5.70b}$$

$$C^{PH12}_{nm}(z,t) = -\sum_{i=1}^{\infty} \overline{G}_h(\omega)\overline{R} \cdot \frac{(-1)^{i+1}}{\mu_i} F_{i1}(\overline{R},z,t)\, C^{1C}_{nm} \tag{5.70c}$$

$$C^{PRm12}_{nm}(z,t) = \sum_{i=1}^{\infty} \omega^2 H^2 \overline{R}\overline{q}(\omega) \left\{\frac{\mu_i H \cdot (-1)^{i+1} - 1}{(\mu_i H)^2} F_{i1}(\overline{R},z,t) - \frac{\overline{R} J_2(\varepsilon_i \overline{R})}{\varepsilon_i H^2} Q_{i1}(\overline{R},z,t)\right\} C^{1C}_{nm} \tag{5.70d}$$

The matrices in eqns.(5.68) through (5.70), $C^{0\alpha}_{nm}$ ($\alpha = 1, C, S, \eta C$) and $C^{1\alpha}_{nm}$ ($\alpha = 1, C$), are given in Appendix J.

The obtained matrices which compose the governing equatins of motion are summerized in Table 5.3.

Table 5.1 Explicit forms of matrices employed in the discretization

	$\alpha=\theta\ ,\ \beta=n$	$\alpha=z\ ,\ \beta=i$
$\mathbf{N}_{\alpha\beta}$	$\begin{bmatrix} \sin n\theta & \cos n\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin n\theta & \cos n\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin n\theta & \cos n\theta \end{bmatrix}$ (3×6)	$\begin{bmatrix} \mathbf{H}_i^0(z) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_i^1(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_i^1(z) \end{bmatrix}$ (6×10) where $\mathbf{H}_i^0(z)=\begin{bmatrix} H_{i0}^{I0} & 0 \\ 0 & H_{i0}^{I0} \end{bmatrix},$ $\mathbf{H}_i^1(z)=\begin{bmatrix} H_{i0}^{I1} & H_{i1}^{I1} & 0 & 0 \\ 0 & 0 & H_{i0}^{I1} & H_{i0}^{I1} \end{bmatrix}$
$\mathbf{B}_{\alpha\beta}^{10}$	$\left[\begin{matrix} \sin n\theta & \cos n\theta & 0 & 0 & 0 \\ 0 & 0 & n\cos n\theta & -n\sin n\theta & \sin n\theta \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right.$ $\left.\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ \cos n\theta & 0 & 0 & 0 & 0 \\ 0 & n\cos n\theta & -n\sin n\theta & \sin n\theta & \cos n\theta \end{matrix}\right]$ (3×10)	$\begin{bmatrix} \mathbf{H}_i^{0'}(z) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\bar{R}}\mathbf{H}_i^1(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\bar{R}}\mathbf{H}_i^1(z) \\ \frac{1}{\bar{R}}\mathbf{H}_i^0(z) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_i^{0'}(z) & \mathbf{0} \end{bmatrix}$ (10×10) where $\mathbf{H}_i^{0'}(z)=\begin{bmatrix} H_{i0,z}^{I0} & 0 \\ 0 & H_{i0,z}^{I0} \end{bmatrix}$ $\mathbf{H}_i^{1'}(z)=\begin{bmatrix} H_{i0,z}^{I1} & H_{i1,z}^{I1} & 0 & 0 \\ 0 & 0 & H_{i0,z}^{I1} & H_{i1,z}^{I1} \end{bmatrix}$
$\mathbf{B}_{\alpha\beta}^{11}$	$\left[\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \eta n\cos n\theta & -\eta n\sin n\theta \\ \eta n\cos n\theta & -\eta n\sin n\theta & 0 & 0 \end{matrix}\right.$ $\left.\begin{matrix} 0 & 0 \\ (\eta+\eta'')\sin n\theta & (\eta+\eta'')\cos n\theta \\ 0 & 0 \end{matrix}\right]$ (3×6)	$-\frac{1}{\bar{R}}\ \mathbf{N}_{zi}$ (6×10)

Table 5.1 Explicit forms of matrices employed in the discretization (2)
(continue)

$\mathbf{B}_{\alpha\beta}^{20}$	$\left[\begin{matrix} \sin n\theta & \cos n\theta & 0 & 0 & 0 \\ 0 & 0 & -n^2\sin n\theta & -n^2\cos n\theta & n\cos n\theta \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right.$ $\left.\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ -n\sin n\theta & 0 & 0 & 0 & 0 \\ 0 & n\cos n\theta & -n\sin n\theta & \sin n\theta & \cos n\theta \end{matrix}\right]$ (3×10)	$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{H}_i^{1''}(z) \\ \mathbf{0} & \mathbf{0} & \frac{1}{\bar{R}^2}\mathbf{H}_i^{1}(z) \\ \mathbf{0} & -\frac{1}{\bar{R}^2}\mathbf{H}_i^{1}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2}{\bar{R}}\mathbf{H}_i^{1'}(z) \\ \mathbf{0} & -\frac{2}{\bar{R}}\mathbf{H}_i^{1'}(z) & \mathbf{0} \end{bmatrix}$ (10×10) where $\mathbf{H}_i^{1''}(z) =$ $\begin{bmatrix} H_{i0}^{I1},zz & H_{i1}^{I1},zz & 0 & 0 \\ 0 & 0 & H_{i0}^{I1},zz & H_{i1}^{I1},zz \end{bmatrix}$
$\mathbf{B}_{\alpha\beta}^{21}$	$\left[\begin{matrix} 0 & 0 & 0 & 0 \\ (2\eta+\eta'')n\cos n\theta & -(2\eta+\eta'')\sin n\theta & \begin{pmatrix} -2\eta n^2\sin n\theta \\ +\eta' n\cos n\theta \end{pmatrix} & \begin{pmatrix} -2\eta n^2\cos n\theta \\ -\eta' n\sin n\theta \end{pmatrix} \\ 0 & 0 & 0 & 0 \end{matrix}\right.$ $\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (\eta+\eta'')\sin n\theta & (\eta+\eta'')\cos n\theta & \eta n\cos n\theta & -\eta n\sin n\theta \end{matrix}\right]$ (3×8)	$\begin{bmatrix} \mathbf{0} & \frac{1}{\bar{R}^2}\mathbf{H}_i^{1'}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{\bar{R}^2}\mathbf{H}_i^{1}(z) \\ \mathbf{0} & \frac{2}{\bar{R}}\mathbf{H}_i^{1'}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{2}{\bar{R}}\mathbf{H}_i^{1'}(z) \end{bmatrix}$ (8×10)

Table 5.1 Explicit forms of matrices employed in the discretization (3) (continue)

$\mathbf{B}^{G0}_{\alpha\beta}$	$\left[\begin{array}{cccccc\|cccccc} \sin n\theta & \cos n\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin n\theta & \cos n\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin n\theta & \cos n\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n\cos n\theta & -n\sin n\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n\cos n\theta & -\sin n\theta & \sin n\theta & \cos n\theta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin n\theta & -\cos n\theta & n\cos n\theta & -n\sin n\theta \end{array}\right]$ (6×12)	$\begin{bmatrix} \mathbf{H}^{0'}_{i}(z) & 0 & 0 \\ 0 & \mathbf{H}^{1'}_{i}(z) & 0 \\ 0 & 0 & \mathbf{H}^{1'}_{i}(z) \\ \frac{1}{\bar{R}}\mathbf{H}^{0}_{i}(z) & 0 & 0 \\ 0 & \frac{1}{\bar{R}}\mathbf{H}^{1}_{i}(z) & 0 \\ 0 & 0 & \frac{1}{\bar{R}}\mathbf{H}^{1}_{i}(z) \end{bmatrix}$ (12×10)
$\mathbf{B}^{G1}_{\alpha\beta}$	$\left[\begin{array}{ccc\|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \eta n\cos n\theta & -\eta n\sin n\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta n\cos n\theta & -\eta n\sin n\theta & (\eta+\eta'')\sin n\theta & (\eta+\eta'')\cos n\theta \\ 0 & 0 & -(\eta+\eta'')\sin n\theta & -(\eta+\eta'')\cos n\theta & \eta n\cos n\theta & -\eta n\sin n\theta \end{array}\right]$ (6×6)	$-\frac{1}{\bar{R}}\begin{bmatrix} \mathbf{H}^{0}_{i}(z) & 0 & 0 \\ 0 & \mathbf{H}^{1}_{i}(z) & 0 \\ 0 & 0 & \mathbf{H}^{1}_{i}(z) \end{bmatrix}$ $= -\frac{1}{\bar{R}}\mathbf{H}_{zi}$ (6×10)

Table 5.1 Explicit forms of matrices employed in the discretization (4)

(continue)

$\mathbf{B}^{*}_{\alpha\beta}$	$\left[\begin{array}{ccccccc} \sin n\theta & \cos n\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin n\theta & \cos n\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin n\theta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$ $\left.\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos n\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin n\theta & \cos n\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n\cos n\theta & -n\sin n\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -n\cos n\theta & -n\sin n\theta \end{array}\right]$ (7×14)	$\begin{bmatrix} \mathbf{H}_{i}^{0}(z) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{i}^{1}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{i}^{1}(z) \\ \mathbf{H}_{i}^{0'}(z) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{i}^{1'}(z) \\ \mathbf{0} & \frac{1}{\overline{R}}\mathbf{H}_{i}^{1}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\overline{R}}\mathbf{H}_{i}^{1}(z) \end{bmatrix}$ (14×10)

Table 5.2 Parts of mass matrices

matrix	expression
$\mathbf{M}^{*s0}_{ij}$ 10×10	$\begin{bmatrix} \underset{(2\times 2)}{\int_0^L \mathbf{H}_{i}^{0T} \mathbf{H}_{j}^{0}\, dz} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \underset{(4\times 4)}{\int_0^L \mathbf{H}_{i}^{1T} \mathbf{H}_{j}^{1}\, dz} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \underset{(4\times 4)}{\int_0^L \mathbf{H}_{i}^{1T} \mathbf{H}_{j}^{1}\, dz} \end{bmatrix}$
$\mathbf{M}^{*s1}_{ijnm}$ 10×10	$\begin{bmatrix} \underset{(2\times 2)}{\int_0^L \mathbf{H}_{i}^{0T} (\eta_1)_{nm} \mathbf{H}_{j}^{0}\, dz} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \underset{(4\times 4)}{\int_0^L \mathbf{H}_{i}^{1T} (\eta_1)_{nm} \mathbf{H}_{j}^{1}\, dz} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \underset{(4\times 4)}{\int_0^L \mathbf{H}_{i}^{1T} (\eta_1)_{nm} \mathbf{H}_{j}^{1}\, dz} \end{bmatrix}$

Table 5.3 Obtained matrices

(∗ the number in the parenthesis denotes the equation number)

	0-th order		1st order	
structural mass matrices $\mathbf{M}^{s}_{ijnm}$	$\mathbf{M}^{s0}_{ijnm}$	(5.9a)	$\mathbf{M}^{s1}_{ijnm}$	(5.9b)
added mass matrices $\mathbf{M}^{ad}_{ijnm}$	$\mathbf{M}^{ad0}_{ijnm}$	(5.15a)	$\mathbf{M}^{ad11}_{ijnm}$ $\mathbf{M}^{ad12}_{ijnm}$ $\mathbf{M}^{ad13}_{ijnm}$	(5.15b) (5.15c) (5.15d)
material stiffness matrices $\mathbf{K}^{D}_{ijnm}$	$\mathbf{K}^{D01}_{ijnm}$ $\mathbf{K}^{D02}_{ijnm}$ $\mathbf{K}^{D3}_{ijnm} + (\mathbf{K}^{D3}_{jimn})^{T}$	(5.24a) (5.25a) (5.26)	$\mathbf{K}^{D11\text{-}0}_{ijnm}$ $\mathbf{K}^{D11\text{-}1}_{ijnm} + (\mathbf{K}^{D11\text{-}1}_{jimn})^{T}$ $\mathbf{K}^{D12\text{-}0}_{ijnm}$ $\mathbf{K}^{D12\text{-}1}_{ijnm} + (\mathbf{K}^{D12\text{-}1}_{jimn})^{T}$	(5.24b) (5.24c) (5.25b) (5.25c)
geometric stiffness matrices $\mathbf{K}^{G}_{ijnm}$	$\mathbf{K}^{G(M)0}_{ijnm}$ $\mathbf{K}^{GS0}_{ijnm}$	(5.35) (5.35)	$\mathbf{K}^{G(M)11}_{ijnm}$ $\mathbf{K}^{G(M)12}_{ijnm}$ $\mathbf{K}^{G(M)13}_{ijnm} + (\mathbf{K}^{G(M)13}_{jimn})^{T}$ $\mathbf{K}^{GS11}_{ijnm}$ $\mathbf{K}^{GS12}_{ijnm}$ $\mathbf{K}^{GS13}_{ijnm} + (\mathbf{K}^{GS13}_{jimn})^{T}$	(5.35) (5.35) (5.36) (5.35) (5.35) (5.36)

Table 5.3 Obtained matrices (2)

(continue)

(continue) $\mathbf{K}^{G}_{ijnm}$	$\mathbf{K}^{GV0}_{ijnm}$	(5.35)	$\mathbf{K}^{GV11}_{ijnm}$	(5.35)
			$\mathbf{K}^{GV12}_{ijnm}$	(5.35)
			$\mathbf{K}^{GV13}_{ijnm} + (\mathbf{K}^{GV13}_{jimn})^{T}$	(5.36)
	$\mathbf{K}^{GH0}_{ijnm}$	(5.35)	$\mathbf{K}^{GH11}_{ijnm}$	(5.35)
			$\mathbf{K}^{GH12}_{ijnm}$	(5.35)
			$\mathbf{K}^{GH13}_{ijnm} + (\mathbf{K}^{GH13}_{jimn})^{T}$	(5.36)
	$\mathbf{K}^{GRm0}_{ijnm}$	(5.35)	$\mathbf{K}^{GRm11}_{ijnm}$	(5.35)
			$\mathbf{K}^{GRm12}_{ijnm}$	(5.35)
			$\mathbf{K}^{GRm13}_{ijnm} + (\mathbf{K}^{GRm13}_{jimn})^{T}$	(5.36)
load correction matrices $\mathbf{K}^{P}_{ijnm}$ (approximation)	$\mathbf{K}^{PS0}_{ijnm}$	(5.67a)	$\mathbf{K}^{PS12}_{ijnm}$	(5.67a)
	$\mathbf{K}^{PV0}_{ijnm}$	(5.67a)	$\mathbf{K}^{PV12}_{ijnm}$	(5.67a)
	$\mathbf{K}^{PH0}_{ijnm}$	(5.67a)	$\mathbf{K}^{PH11}_{ijnm}$	(5.67b)
			$\mathbf{K}^{PH12}_{ijnm}$	(5.67a)
	$\mathbf{K}^{PRm0}_{ijnm}$	(5.67a)	$\mathbf{K}^{PRm11}_{ijnm}$	(5.67b)
			$\mathbf{K}^{PRm12}_{ijnm}$	(5.67a)

6. Governing Equations for the Dynamic Stability Analysis

The matrix form of equation of motion can be obtained by applying all the matrices in Table 5.3 to eqn. (3.2) as follows,

$$(\mathbf{M}^{s}_{ijnm} + \mathbf{M}^{ad}_{ijnm})\, \Delta\ddot{\mathbf{d}}_{jm} + (\mathbf{K}^{D}_{ijnm},\, + \mathbf{K}^{G}_{ijnm} - \mathbf{K}^{P}_{ijnm})\, \Delta\mathbf{d}_{jm} = \mathbf{0} \tag{6.1}$$

where

$$\mathbf{M}^{s}_{ijnm} = \mathbf{M}^{s0}_{ijnm} + \nu\, \mathbf{M}^{s1}_{ijnm} \tag{6.2a}$$

$$\mathbf{M}^{ad}_{ijnm} = \mathbf{M}^{ad0}_{ijnm} + \nu\, (\mathbf{M}^{ad11}_{ijnm} + \mathbf{M}^{ad12}_{ijnm} + \mathbf{M}^{ad13}_{ijnm}) \tag{6.2b}$$

$$\begin{aligned}\mathbf{K}^{D}_{ijnm} = {} & \mathbf{K}^{D01}_{ijnm} + \mathbf{K}^{D02}_{ijnm} + \mathbf{K}^{D3}_{ijnm} + (\mathbf{K}^{D3}_{jimn})^{T} \\ & + \nu\, \{\mathbf{K}^{D11\text{-}0}_{ijnm} + \mathbf{K}^{D11\text{-}1}_{ijnm} + (\mathbf{K}^{D11\text{-}1}_{jimn})^{T} + \mathbf{K}^{D12\text{-}0}_{ijnm} + \mathbf{K}^{D12\text{-}1}_{ijnm} + (\mathbf{K}^{D12\text{-}1}_{jimn})^{T}\}\end{aligned} \tag{6.2c}$$

$$\begin{aligned}\mathbf{K}^{G}_{ijnm} = {} & \mathbf{K}^{G(M)0}_{ijnm} + \mathbf{K}^{GS0}_{ij\ \ nm} + \mathbf{K}^{GV0}_{ijnm} + \mathbf{K}^{GH0}_{ijnm} + \mathbf{K}^{GRm0}_{ijnm} \\ & + \nu\, \{\mathbf{K}^{G(M)11}_{ijnm} + \mathbf{K}^{G(M)12}_{ijnm} + \mathbf{K}^{G(M)13}_{ijnm} + (\mathbf{K}^{G(M)13}_{jimn})^{T} \\ & + \mathbf{K}^{GS11}_{ijnm} + \mathbf{K}^{GS12}_{ijnm} + \mathbf{K}^{GS13}_{ijnm} + (\mathbf{K}^{GS13}_{jimn})^{T} \\ & + \mathbf{K}^{GV11}_{ijnm} + \mathbf{K}^{GV12}_{ijnm} + \mathbf{K}^{GV13}_{ijnm} + (\mathbf{K}^{GV13}_{jimn})^{T} \\ & + \mathbf{K}^{GH11}_{ijnm} + \mathbf{K}^{GH12}_{ijnm} + \mathbf{K}^{GH13}_{ijnm} + (\mathbf{K}^{GH13}_{jimn})^{T} \\ & + \mathbf{K}^{GRm11}_{ijnm} + \mathbf{K}^{GRm12}_{ijnm} + \mathbf{K}^{GRm13}_{ijnm} + (\mathbf{K}^{GRm13}_{jimn})^{T}\}\end{aligned} \tag{6.2d}$$

$$\begin{aligned}\mathbf{K}^{P}_{ijnm} = {} & \mathbf{K}^{PS0}_{ijnm} + \mathbf{K}^{PV0}_{ijnm} + \mathbf{K}^{PH0}_{ijnm} + \mathbf{K}^{PRm0}_{ijnm} \\ & + \nu\, (\mathbf{K}^{PS12}_{ijnm} + \mathbf{K}^{PV12}_{ijnm} + \mathbf{K}^{PH11}_{ijnm} + \mathbf{K}^{PH12}_{ijnm} + \mathbf{K}^{PRm11}_{ijnm} + \mathbf{K}^{PRm12}_{ijnm})\end{aligned} \tag{6.2e}$$

As can be seen from eqns. (5.43) and (5.45), the matrices $\mathbf{K}^{G(M)0}_{ijnm}$ and $\mathbf{K}^{GM12}_{ijnm}$ can be decomposed to time-dependent matrices and time-independent gravitational matrices.

$$\mathbf{K}^{G(M)0}_{ijnm} = \mathbf{K}^{G(M)0<\tau>}_{ijnm}(t) + \mathbf{K}^{G(M)0<g>}_{ijnm} \tag{6.3a}$$

$$\mathbf{K}^{G(M)12}_{ijnm} = \mathbf{K}^{G(M)12<\tau>}_{ijnm}(t) + \mathbf{K}^{G(M)12<g>}_{ijnm} \tag{6.3b}$$

Eqn. (6.1) is recoganized by separating the time-dependent matrices as follows.

$$\begin{aligned}(\mathbf{M}^{<0>}_{ijnm} + \nu\, \mathbf{M}^{<1>}_{ijnm})\, \Delta\ddot{\mathbf{d}}_{jm} + (\mathbf{K}^{<0>}_{ijnm} + \nu\, \mathbf{K}^{<1>}_{ijnm}\,)\, \Delta\mathbf{d}_{jm} \\ + (\mathbf{K}^{\tau<0>}_{ijnm}(t) + \nu\, \mathbf{K}^{\tau<1>}_{ijnm}(t))\, \Delta\mathbf{d}_{jm} = \mathbf{0}\end{aligned} \tag{6.4}$$

where

$$\mathbf{M}^{<0>}_{ijnm} = \mathbf{M}^{s0}_{ijnm} + \mathbf{M}^{ad0}_{ijnm} \tag{6.5a}$$

$$\mathbf{M}^{<1>}_{ijnm} = \mathbf{M}^{s1}_{ijnm} + \mathbf{M}^{ad11}_{ijnm} + \mathbf{M}^{ad12}_{ijnm} + \mathbf{M}^{ad13}_{ijnm} \tag{6.5b}$$

$$\mathbf{K}^{<0>}_{ijnm} = \mathbf{K}^{D01}_{ijnm} + \mathbf{K}^{D02}_{ijnm} + \mathbf{K}^{D3}_{ijnm} + (\mathbf{K}^{D3}_{jimn})^T + \mathbf{K}^{G(M)0<g>}_{ijnm} + \mathbf{K}^{GS0}_{ijnm} + \mathbf{K}^{PS0}_{ijnm} \tag{6.5c}$$

$$\begin{aligned}\mathbf{K}^{<1>}_{ijnm} = \mathbf{K}^{D11\text{-}0}_{ijnm} + \mathbf{K}^{D11\text{-}1}_{ijnm} + (\mathbf{K}^{D11\text{-}1}_{jinm})^T + \mathbf{K}^{D12\text{-}0}_{ijnm} + \mathbf{K}^{D12\text{-}1}_{ijnm} + (\mathbf{K}^{D12\text{-}1}_{jimn})^T \\ + \mathbf{K}^{G(M)12<g>}_{ijnm} + \mathbf{K}^{GS11}_{ijnm} + \mathbf{K}^{GS12}_{ijnm} + \mathbf{K}^{GS13}_{ijnm} + (\mathbf{K}^{GS13}_{jinm})^T + \mathbf{K}^{PS12}_{ijnm}\end{aligned} \tag{6.5d}$$

$$\begin{aligned}\mathbf{K}^{\tau<0>}_{ijnm}(t) = \mathbf{K}^{G(M)0<\tau>}_{ijnm} + \mathbf{K}^{GV0}_{ijnm} + \mathbf{K}^{GH0}_{ijnm} + \mathbf{K}^{GRm0}_{ijnm} \\ + \mathbf{K}^{PV0}_{ijnm} + \mathbf{K}^{PH0}_{ijnm} + \mathbf{K}^{PRm0}_{ijnm}\end{aligned} \tag{6.5e}$$

$$\begin{aligned}\mathbf{K}^{\tau<1>}_{ijnm}(t) = \mathbf{K}^{G(M)11}_{ijnm} + \mathbf{K}^{G(M)12<\tau>}_{ijnm} + \mathbf{K}^{G(M)13}_{ijnm} + (\mathbf{K}^{G(M)13}_{jimn})^T \\ + \mathbf{K}^{GV11}_{ijnm} + \mathbf{K}^{GV12}_{ijnm} + \mathbf{K}^{GV13}_{ijnm} + (\mathbf{K}^{GV13}_{jimn})^T\end{aligned}$$

$$+ \mathbf{K}^{GH11}_{ijnm} + \mathbf{K}^{GH12}_{ijnm} + \mathbf{K}^{GH13}_{ijnm} + (\mathbf{K}^{GH13}_{jinm})^T$$

$$+ \mathbf{K}^{GRm11}_{ijnm} + \mathbf{K}^{GRm12}_{ijnm} + \mathbf{K}^{GRm13}_{ijnm} + (\mathbf{K}^{GRm13}_{jinm})^T$$

$$+ \mathbf{K}^{PV12}_{ijnm} + \mathbf{K}^{PH11}_{ijnm} + \mathbf{K}^{PH12}_{ijnm} + \mathbf{K}^{PRm11}_{ijnm} + \mathbf{K}^{PRm12}_{ijnm} \qquad (6.5f)$$

The global matrix equation of motion can be expressed considering eqn. (6.4) as follows.

$$\mathbf{M}(\nu)\, \Delta\ddot{\mathbf{d}} + \mathbf{K}(\nu)\, \Delta\mathbf{d} + \mathbf{K}^{\tau}(\nu,t)\, \Delta\mathbf{d} = \mathbf{0} \qquad (6.6)$$

where

$$\underset{(10I \times (N+1)) \times (10I \times (N+1))}{\mathbf{M}(\nu)} = \mathbf{M}^{<0>} + \nu\, \mathbf{M}^{<1>} \qquad (6.7a)$$

$$\underset{(10I \times (N+1)) \times (10I \times (N+1))}{\mathbf{K}(\nu)} = \mathbf{K}^{<0>} + \nu\, \mathbf{K}^{<1>} \qquad (6.7b)$$

$$\underset{(10I \times (N+1)) \times (10I \times (N+1))}{\mathbf{K}^{\tau}(\nu,t)} = \mathbf{K}^{\tau<0>}(t) + \nu\, \mathbf{K}^{\tau<1>}(t) \qquad (6.7c)$$

Each global matrices consists of (10I×10I) block matrices as illustrated in Fig. 6.1.

In order to solve eqn. (6.6), the boundary conditions have to be introduced.

In the case of the cantilever tank with open roof:

$$\Delta\ddot{\mathbf{d}}_{1m} = \Delta\mathbf{d}_{1m} = \mathbf{0} \quad (\text{ m=0,N}) \ \ (\text{at the fixed end}) \qquad (6.8)$$

Therefore, the dimension of the matrices of eqn. (6.6) is reduced to {10(I-1)*(N+1)}×{10(I-1)*(N+1)}.

In the case of the cantilever tank with closed roof which is considered rigid, the following condition is added to eqn. (6.8)

$$
\mathbf{M}^{<0>} = \left[\begin{array}{ccccc}
\underset{|0| \times |0|}{(\mathbf{M}^0)_{00}} & & & & \\
 & (\mathbf{M}^0)_{11} & & \mathbf{0} & \\
 & & \ddots & & \\
 & & (\mathbf{M}^0)_{nn} & & \\
 & \mathbf{0} & & \ddots & \\
 & & & & (\mathbf{M}^0)_{NN}
\end{array} \right]
$$

$$
\mathbf{M}^{<1>} = \sum_{k=1}^{K} \mathbf{M}^{<1>}_{(k)}
$$

$$
\mathbf{M}^{<1>}_{(k)} = \left[\begin{array}{ccccccccc}
 & & & & & (\mathbf{M}^1)_{0k} & & & \\
 & \mathbf{0} & & & (\mathbf{M}^1)_{1,k-1} & & (\mathbf{M}^1)_{1,k+1} & \mathbf{0} & \\
 & & & \iddots & & & \ddots & & \\
 & & & (\mathbf{M}^1)_{n,k-n} & & & & (\mathbf{M}^1)_{n,k+n} & \\
 & & \iddots & & & & & \ddots & \\
 & (\mathbf{M}^1)_{k-1,1} & & & & & & & (\mathbf{M}^1)_{N-k,N} \\
(\mathbf{M}^1)_{k0} & & & & & & & & \\
 & & (\mathbf{M}^1)_{k+1,1} & & & & & & \\
 & & \ddots & & & & & & \\
 & \mathbf{0} & & (\mathbf{M}^1)_{m,m-k} & & & \mathbf{0} & & \\
 & & & \ddots & & & & & \\
 & & & & (\mathbf{M}^1)_{N,N-k} & & & &
\end{array} \right]
$$

(k : the coefficient indeces of eqn. (4.2))

Fig.6.1 Block-wise matrix distribution pattern of global matrices

((1) mass matrices)

$$\mathbf{K}^{<0>} = \begin{bmatrix} (\mathbf{K}^0)_{00} & & & & \\ & (\mathbf{K}^0)_{11} & & \mathbf{0} & \\ & & \ddots & & \\ & & & (\mathbf{K}^0)_{nn} & \\ & \mathbf{0} & & \ddots & \\ & & & & (\mathbf{K}^0)_{NN} \end{bmatrix}$$

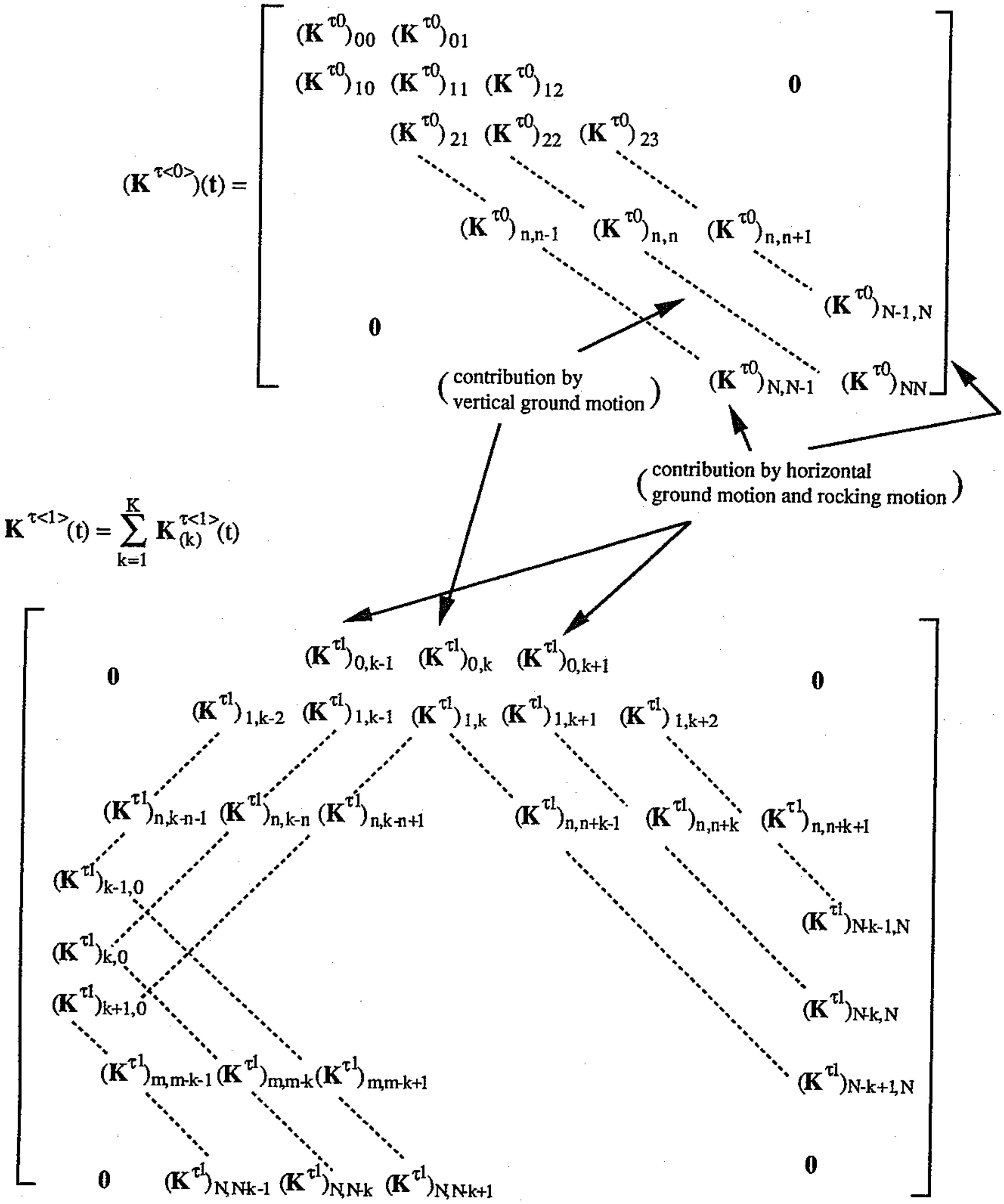
$$\mathbf{K}^{<1>} = \sum_{k=1}^{K} \mathbf{K}^{<1>}_{(k)}$$

$$\mathbf{K}^{<1>}_{(k)} = \begin{bmatrix} \mathbf{0} & & & & (\mathbf{K}^1)_{0k} & & & \\ & & & (\mathbf{K}^1)_{1,k-1} & & (\mathbf{K}^1)_{1,k+1} & \mathbf{0} & \\ & & \iddots & & & \ddots & & \\ & & (\mathbf{K}^1)_{n,k-n} & & & & (\mathbf{K}^1)_{n,n+k} & \\ & \iddots & & & & & \ddots & \\ & (\mathbf{K}^1)_{k-1,1} & & & & & & (\mathbf{K}^1)_{N-k,N} \\ (\mathbf{K}^1)_{k0} & & & & & & & \\ & (\mathbf{K}^1)_{k+1,1} & & & & & & \\ & & \ddots & & & & & \\ & & & (\mathbf{K}^1)_{m,m-k} & & & \mathbf{0} & \\ & \mathbf{0} & & & \ddots & & & \\ & & & & (\mathbf{K}^1)_{N,N-k} & & & \end{bmatrix}$$

(k : the coefficient indeces of eqn.(4.2))

Fig 6.1 Block-wise matrix distribution pattern of global matrices

((2) stiffness matrices)



(k : the cocfficient indeces of eqn.(4.2))

Fig. 6.1 Block-wise matrix distribution pattern of global matrices

((3) time-dependent matrices)

$$\Delta\ddot{\mathbf{d}}_{\mathrm{Im}} = \Delta\mathbf{d}_{\mathrm{Im}} = \mathbf{0} \quad (m=0,N \,;\, m\neq 1) \tag{6.9}$$

Therefore, the dimension of the matrices of eqn. (6.6) is reduced to {10(I-2)*(N+1)+4}x{10(I-2)*(N+1)+4}.

We express the equation of motion after introducing boundary conditions by the same form as eqn. (6.6).

In order to obtain the eigen values and eigen vectors efficiently, firstly we consider to solve the following equation for each matrix block.

$$\mathbf{M}(0)\,\Delta\ddot{\mathbf{d}} + \mathbf{K}(0)\,\Delta\mathbf{d} = \mathbf{0} \tag{6.10}$$

By using the obtained eigen vectors, the following relation is formed.

$$\underbrace{\Delta\,\mathbf{d}}_{10(I-1)\times(N+1)} = \underbrace{\Phi}_{\{10(I-1)\times(N+1)\}\times\{a\times(N+1)\}}\ \underbrace{\Delta\,\mathbf{d}^{(r)}}_{a\times(N+1)} \tag{6.11}$$

* for the case of an open cantilever tank

where a (a<<I) is the number of mode shapes. The matrix form is illustrated schematically in Appendix K.

The reduced matrix system can be obtained by applying eqn. (6.11) to eqn. (6.6).

$$\bar{\mathbf{M}}(\nu)\,\Delta\ddot{\mathbf{d}}^{(r)} + \bar{\mathbf{K}}(\nu)\,\Delta\mathbf{d}^{(r)} + \bar{\mathbf{K}}^{\tau}(\nu,t)\,\Delta\mathbf{d}^{(r)} = \mathbf{0} \tag{6.12}$$

where

$$\underset{\{a\times(N+1)\}\times\{a\times(N+1)\}}{\bar{\mathbf{M}}(\nu)} = \Phi^{T}\,\mathbf{M}(\nu)\,\Phi = \mathbf{I} + \nu\,\bar{\mathbf{M}}^{*} \tag{6.13a}$$

$$\bar{\mathbf{K}}(\nu) = \Phi^{T}\,\mathbf{K}(\nu)\,\Phi = \bar{\Lambda} + \nu\,\bar{\mathbf{K}}^{*} \tag{6.13b}$$

$$\bar{\mathbf{K}}^{\tau}(\nu,t) = \Phi^{T}\,\mathbf{K}^{\tau}(\nu,t)\,\Phi = \bar{\mathbf{G}}(t) + \nu\,\bar{\mathbf{K}}^{\tau*}(t) \tag{6.13c}$$

$\bar{\mathbf{M}}(v)$ and $\bar{\mathbf{K}}(v)$ are semi-diagonal matrices which have first order off-diagonal terms. $\bar{\Lambda}$ and $\bar{\mathbf{G}}(t)$ are illustrated in Appendix K.

By solving eqn. (6.12) with excluding the last term, $\Delta\mathbf{d}^{(r)}$ can be expressed as.

$$\Delta\mathbf{d}^{(r)} = \mathbf{Q}\ \Delta\mathbf{q} \tag{6.14}$$

where $\mathbf{Q}$ and $\mathbf{q}$ are a modal matrix and generalized coordinates, respectively.

$\bar{\mathbf{M}}(v)$ and $\bar{\mathbf{K}}(v)$ are orthogonalized by applying eqn. (6.14) to eqn. (6.12). Adding a modal damping matrix for the completeness of the stability analysis, eqn. (6.12) can be translated to the following form.

$$\Delta\ddot{\mathbf{q}} + \mathbf{C}\ \Delta\dot{\mathbf{q}} + \Lambda\ \Delta\mathbf{q} + \sum_{s=1}^{S} \{\mathbf{G}_1^s \cos(s\omega t) + \mathbf{G}_2^s \sin(s\omega t)\}\ \Delta\mathbf{q} = \mathbf{0} \tag{6.15}$$

where $\mathbf{C}$ and Λ are diagonal matrices of damping and natural frequencies, respectively. The components of the matrix $\mathbf{C}$ are given by

$$c_{in} = 2\zeta_{in}\omega_{in} \tag{6.16}$$

where ζ_{in} is the damping coefficient associated with eigen frequency ω_{in}.

The form of eqn. (6.15) is commonly referred to as the coupled Hill's equation. The dynamic stability can be analized by applying the criteria established by Hsu [7] to eqn. (6.15).

7. Numerical Examples

7.1 Instability criteria

In the previous section, the final equation to evaluate the instability regions is obtained (eqn.(6.15)). If we consider only one dominant excitation frequency, eqn.(6.15) is simplified to

$$\Delta\ddot{\mathbf{q}} + \mathbf{C}\,\Delta\dot{\mathbf{q}} + \Lambda\,\Delta\mathbf{q} + \varepsilon\,\mathbf{G}\,\Delta\mathbf{q}\cos\omega t = \mathbf{0} \tag{7.1}$$

where ω and ε represent a typical dominant frequency and the normalized amplitude of the seismic excitation, respectively.

A superposed dot denotes time differentiation and q is the generalized coordinate. When mass proportional damping is adopted, the i-th and n-th component of the damping matrix **C** is given by

$$c_{in} = 2\zeta_{min}\frac{\omega_{in}^2}{\omega_{min}} \qquad \text{for } i=1, I \text{ and } n=1,N \tag{7.2}$$

where I and N are the total numbur of modes in the axial and the circumferential directions, respectively. ω_{min} and ζ_{min} are the lowest natural frequency and the corresponding damping coefficient, respectively. Λ is a diagonal matrix of the natural frequencies of an imperfect cylindrical shell, ω_{in}^2. **G** is the coefficient matrix of the time-dependent matrix obtained by

$$\varepsilon\,\mathbf{G}\cos\omega t = \mathbf{Q}^T\,(\bar{\mathbf{G}}(t) + \nu\,\bar{\mathbf{K}}^{\tau *}(t))\,\mathbf{Q} \tag{7.3}$$

The dynamic stability analysis can be performed by the same criteria used by Liu and Uras[3].

$$1 < \frac{\varepsilon^2}{\varepsilon_{cr}^2} - \frac{(\omega-\varpi)^2}{\sigma^2} \tag{7.4}$$

In eqn.(7.4) ϖ, ε_{cr} and σ are given by follows in terms of the circumferential wave number n and m and the axial mode number i and j.

$$\varpi = \omega_{in} + \omega_{jm} \tag{7.5a}$$

$$\varepsilon_{cr}^2 = \frac{16\zeta_{min}^2\,\omega_{in}^3\,\omega_{jm}^3}{\omega_{min}^2\,G_{ijnm}\,G_{jimn}} \tag{7.5b}$$

$$\sigma = \frac{\zeta_{min}}{\omega_{min}}(\omega_{in}^2 + \omega_{jm}^2) \tag{7.5c}$$

Considering the interest in seismic analyses, two lowest axial modes are enough for each circumferential wave n to make the stability charts. Since the displacement vector includes not only symmetric part (cos nθ) but also antisymmetric part (sin nθ) with respect to the axis of θ = 0° (eqn.(5.4b) in Section 5), in this case,the axial block matrices become

$$(G)_{nm} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \qquad (7.6a)$$

If only symmetric deformations (cos nθ) are assumed, the matrix of eqn.(7.6) can be reduced as

$$(G)_{nm} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad (7.6b)$$

7.2 Applications

In this section, example analyses are shown. The examples are from the experiments by Chiba et al.[4].

Uras and Liu used it as well, comparing it with their analysis results assuming perfect shells [8].

The specification of the test opecimens is shown in Table 7-1.

In order to identify the effect of imperfections, four simple imperfections are assumed separately, i, e. cosθ, cos2θ, cos3θ and cos4θ.

THe number of circumferential waves and the number of elements in the axial direction of the analysis model are 20 and 10, respectively. 1.0×10^{-4} was chosen as ζ_{min}.

Table 7-1 Shell and fluid material data

Shell and fluid data	Chiba et al.
R Radius	0.1m
h Thickness	0.0025R
L Length : broad	1.607R
tall	2.270R
E Young's modulus	5.56GPa
ν Poisson ratio	0.3
ρ Shell mass density	1.405×10^3 kg/m^3
ρ_F Fluid mass density	1.0×10^3 kg/m^3

(1) 50% full tall shell[4]

perfect shell

Table 7-2 shows the comparison of experimental and theoretical buckling fiequencies due to n th and (n+1) th circumferential coupling. The present results are very close to those by Uras and Liu[8].

Fig.7-1 shows the stability charts according to the analysis by Uras and Liu[8], the present analysis and the experiments by Chiba et al.[4].The regions of the present results without imperfections are between those by Uras and Liu and those by Chiba et al, however, these are closer to the analysis results obtained by Uras and Liu than the experimental results obtained by Chiba et al.

Table 7-2 Comparison of experimental and theoretical buckling frequencies due to n th and (n+1)th circumferential coupling for a 50 percent full tall shell.

n	n+1	Experimental[4] frequencies (Hz)	Theoretical[8] frequencies (Hz)	Theoretical (Present) frequencies (Hz)
6	7	100	103	103
7	8	108	110	110
8	9	117	119	119
9	10	130	133	133
10	11	147	151	151
11	12	168	174	173
12	13	182	202	200
13	14	212	234	231

imperfect shell

The main influences of the imperfection of the shell are;

1) changes of the natural frequencies.

2) appearance of the high order instability regions.

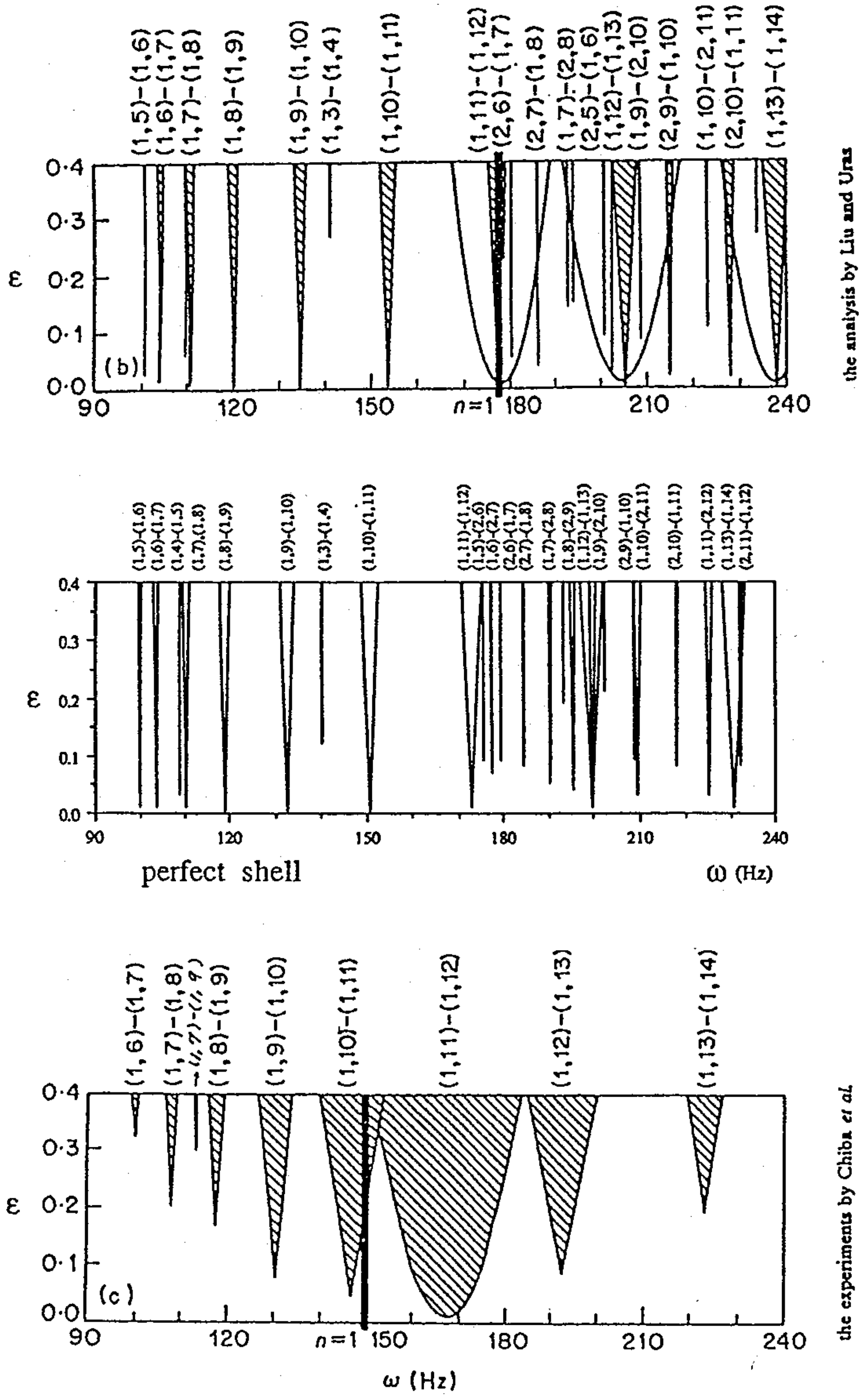


Fig. 7-1 Stability charts for 50 percent full tall shell (perfect shell)

These can be explained from the patterns of matrices. If we pick up the case of the imperfection pattern, cos4θ, as an example, the matrix block patterns of the mass matrix and the time-independent stiffness matrix (eqns.(6-7a) and (6-7b)) can be illustrated as Fig.7-2 (the breathing modes (n=0) are eliminated and the total number of the circumferential waves is 20.). In this figure the black squares and the shaded squares represent the 0th order sub-matrices and the 1st order sub-matrices which appear due to the imperfection, respectively. And the white squares represent 0 matrices. The locations of the shaded squares are determined according to Fig.6.1 (1) and (2) for k=4. The time-dependent stiffness matrix is also illustrated as Fig.7.3. The locations of the shaded squares are determined according to Fig.6.1(3) for k=4.

After trensformation by the eigen mode matrix which is obtained from the eigen-value analyses of eqn.(6.10),the mass matrix changes as shown in Fig.7-4. The 0th order diagonal block matrices become unit matrices, however, the pattern of the 1st order matrix doesn't change (eqn.(6.13a)). The time-independent stiffness matrix also changes as shown in Fig.7-5. The 0th order diagonal block matrices become diagonal matrices which consist of the natural frequencies of a perfect shell. The pattern of the 1st order matrix doesn't change either (eqn.(6.13b)). On the other hand, the pattern of the time-dependent stiffness matrix doesn't change (eqn.(6.13c)).

After solving the eigen-value problem of eqn.(6.12) with excluding the last term, the natural frequencies and eigen modes of the imperfect shell can be obtained. 0th order natural frequencies are slightly changed by the effect of the 1st order block matrices in Fig.7-4 and 7-5. The eigen modes which belong to the wave number (n) are no longer pure, that is, the small components which belong to another wave number (m) appear in them due to the effect of imperfection. Consequently, after the second transformation some new block matrices appear in the time-dependent stiffness matrix as shown in Fig.7-6 with the cross striped squares. The components of these block matrices are usually very small. However, they become bigger with increase of the amplitude of the imperfection. In this case the combinations of circumferential waves of the instability regions are;

1) n and n+1 ; main regions

2) n and n+2m+1 (m=1,9) ; higher order regions

Fig.7-7 through Fig.7-10 show the stability charts with excluding the main instability regions for the imperfection patterns of cosθ, cos2θ, cos3θ and cos4θ, respectively. The patterns of the combination of two different modes, (i,n)-(j,m) (see eqn.(7.5a)), due to the imperfections are ;

1) cosθ ; (i,n)-(j,n) and (i,n)-(j,n+2)

2) cos2θ ; (i,n)-(j,n+3)

* The combinations of (i,n)-(j,n+1) are absorbed in the main regions.

3) $\cos 3\theta$; (i,n)-(j,n) , (i,n)-(j,n+2) and (i,n)-(j,n+4)

4) $\cos 4\theta$; (i,n)-(j,n+3) , (i,n)-(j,n+5) , (i,n)-(j,n+7) and (i,n)-(j,n+9).

In the case of $\cos 4\theta$ imperfection, the last two combinations are obtained by the higher order block matrices which appear after the second step transformation of the time-dependent stiffness matrix.

Generally speaking, the higher order instability regions grow bigger as the imperfection amplitudes increase. However, there is the possibility of the interference among two or more instability regions which have very close combination frequencies to each other. As the result the phenomenon will not be simple. As an example the case of the imperfection $\cos 4\theta$ can be focused. In Fig.7-10 the instability regions of (1,3)-(1,10) and (1,3)-(1,12) do not appear in the chart at the imperfection amplitude $\nu = 2.5\times10^{-3}$, and they become very large at $\nu = 5.0\times10^{-3}$. However, they become much smaller at $\nu = 1.0\times10^{-2}$. It is inferred that a kind of resonance has occurred. As can be seen in Fig. 7-1, the main instability regions of (1,10)-(1,11) and (1-11)-(1-12) are very close to the regions of (1,3)-(1,10) and (1,3)-(1,12), respectively. Table 7-3 shows the comparison of these four combination frequencies corresponding to ν values. The frequencies of the main instability regions and those of the higher order instability regions are very close, but those at $\nu = 5\times10^{-3}$ are much closer to each other than the rest.

Table 7-3 Comparison of combination frequencies

ν	ϖ (1,10)-(1,11) (Hz)	ϖ (1,3)-(1,10) (Hz)	difference (%)
2.5×10^{-3}	150.67	151.05	0.252
5.0×10^{-3}	150.59	150.55	- 0.0266
1.0×10^{-2}	150.40	149.84	- 0.372

(a) ϖ (1,10)-(1,11) and ϖ (1,3)-(1,10)

ν	ϖ (1,11)-(1,12) (Hz)	ϖ (1,3)-(1,12) (Hz)	difference (%)
2.5×10^{-3}	173.14	173.53	0.225
5.0×10^{-3}	173.09	173.05	- 0.0231
1.0×10^{-2}	172.95	172.39	- 0.324

(b) ϖ (1,11)-(1,12) and ϖ (1,3)-(1,12)

Fig.7-11 through Fig.7-14 show the changes of the natural frequencies of first axial modes according to the amplitudes of the imperfertions ν for each imperfection pattern. In the case of cosθ imperfection (Fig.7-11), the natural frequencies almost do not change with respect to the change of the impefection amplitude. In the case of cos 2θ imperfection (Fig.7-12), the natural frequency of wave number 1 (the beam mode) increases with the increase of the imperfection amplitude. This seems to be reasonable, since the shape of the shell cross section is slightly elliptical due to the imperfection and the direction of the longer radius corresponds to that of the excitation. In the case of cos 3θ imperfection (Fig.7-13), the frequencies of wave numbers 1, 2, 4 and 5 decreases slightly with the increase of the imperfection amplitude. In the case of cos 4θ imperfection (Fig.7-14), the frequencies of wave number 1, 2 and 4 increase with the increase of the imperfection amplitude. However, those of wave number 3, 8, 7, 6 and 5 decrease.

It should be noted that the high order instability region,the combination of (1,7) and (1,9), which was observed in the experiments by Chiba et al, can be found in the present analyses of cosθ and cos3θ imperfections as can be seen in Fig. 7-7 and Fig.7-9.

cos 4θ imperfection

M,K

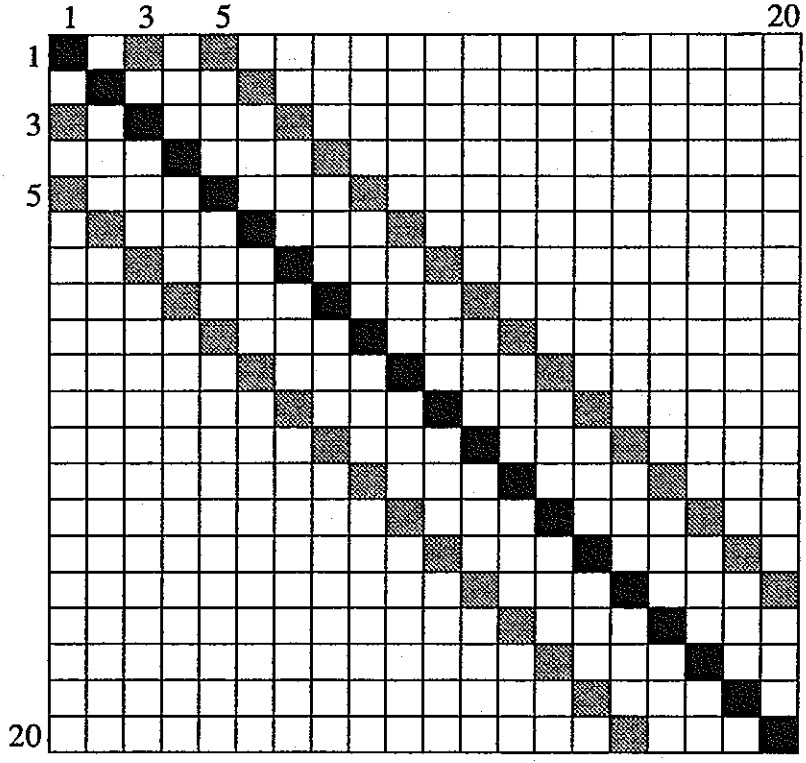


Fig. 7-2

K_G

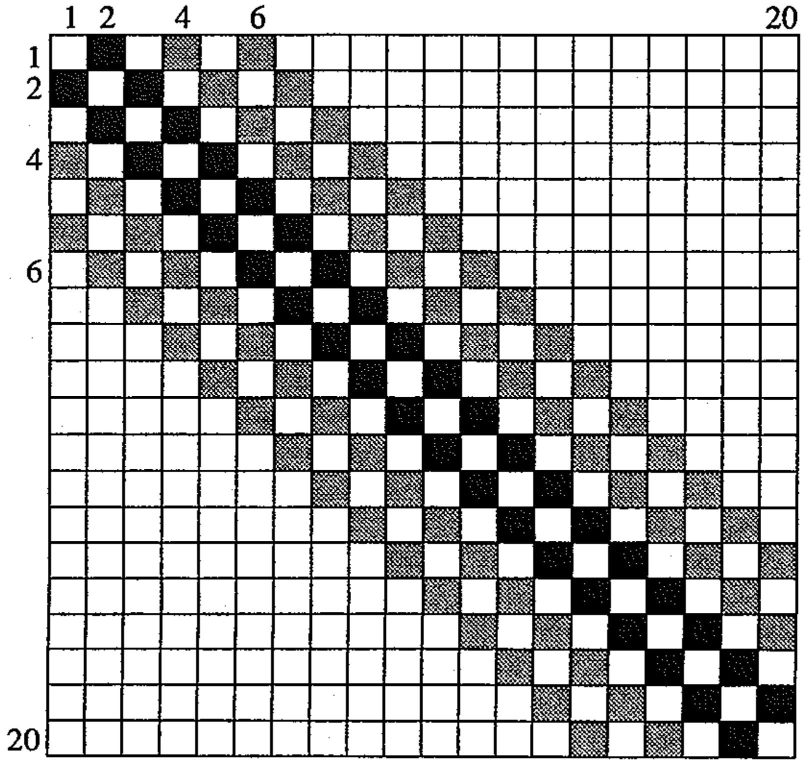


Fig. 7-3

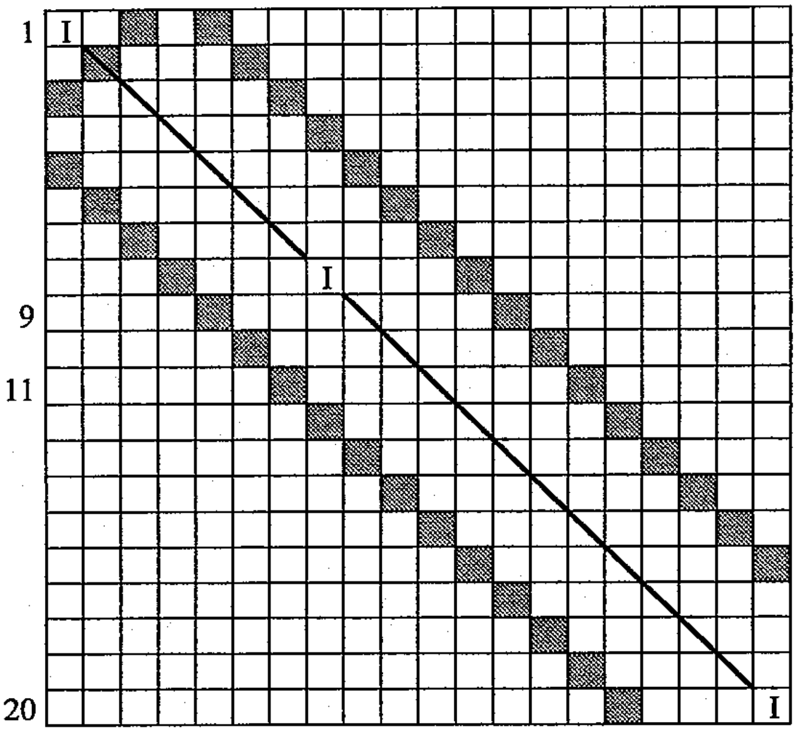


Fig.7-4

Φ TK (ν) Φ

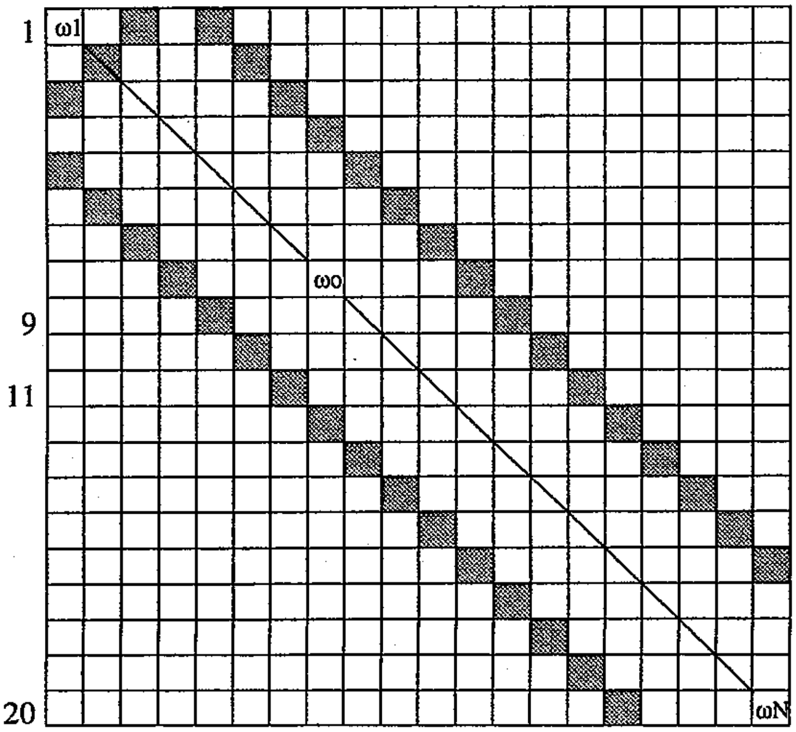


Fig.7-5

cos 4θ imperfection

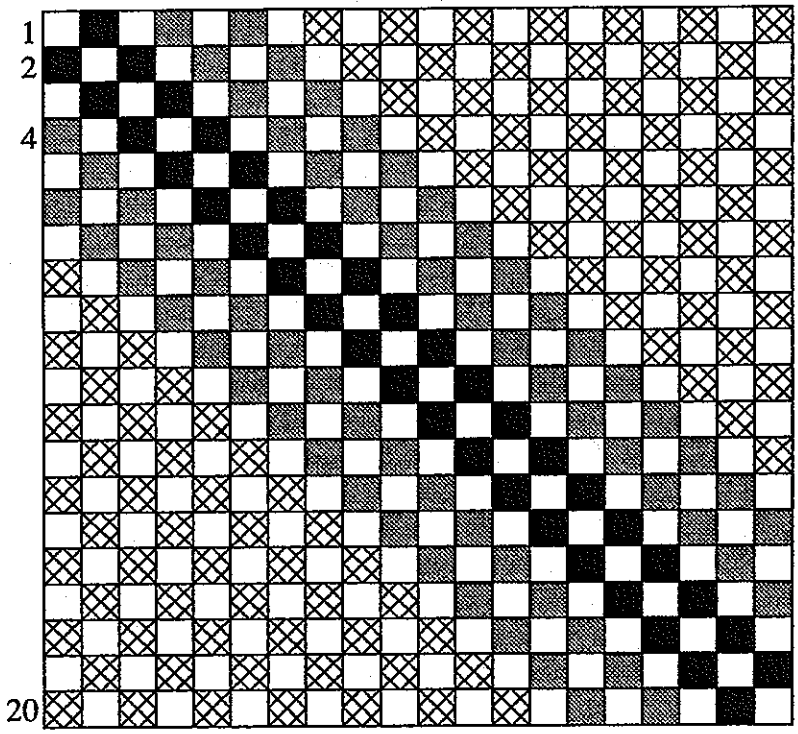


Fig. 7-6

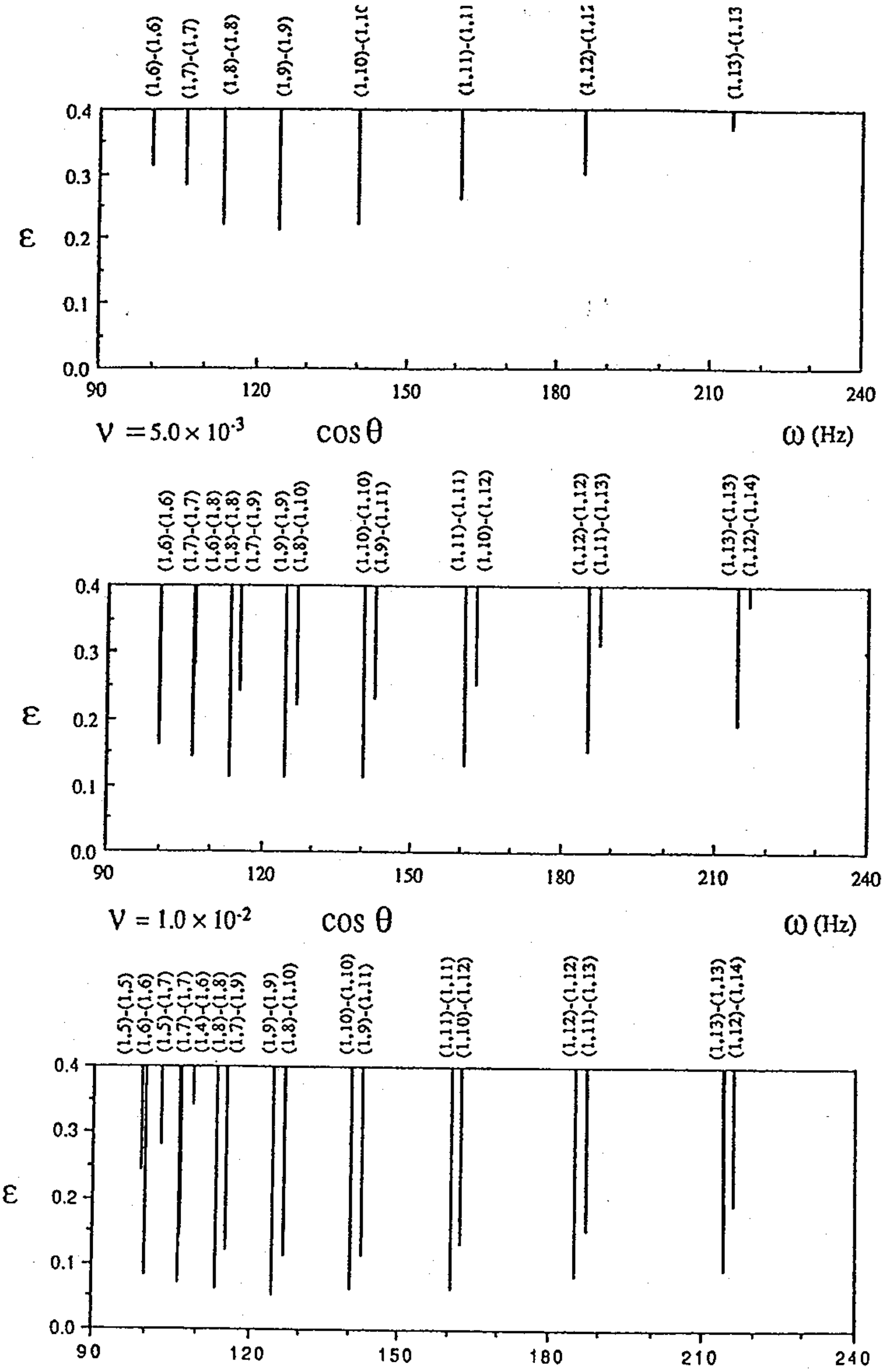
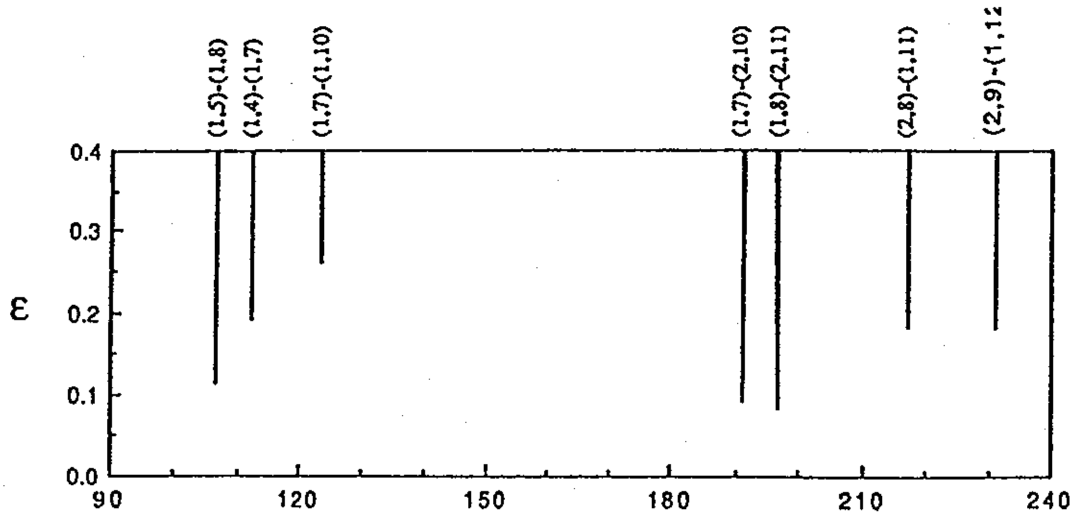
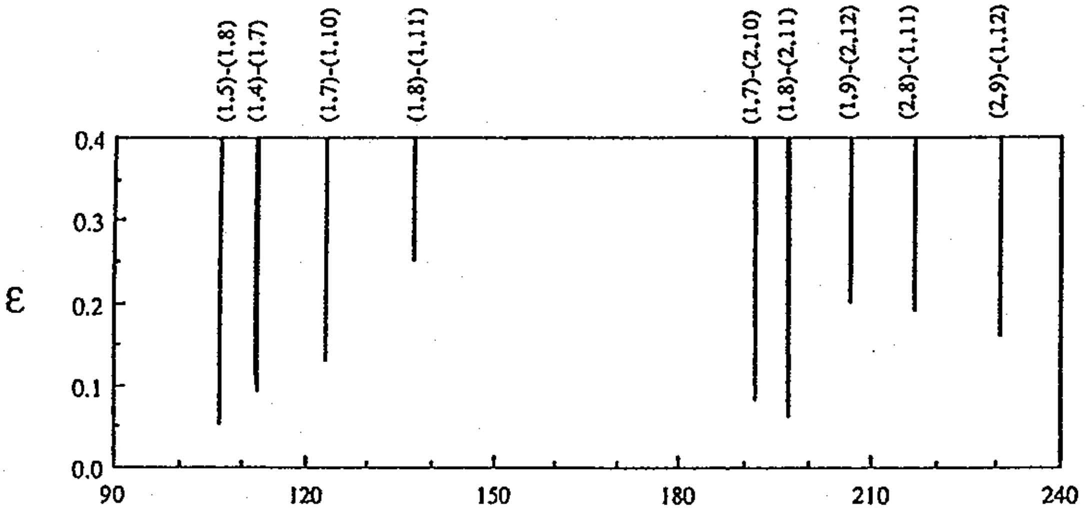


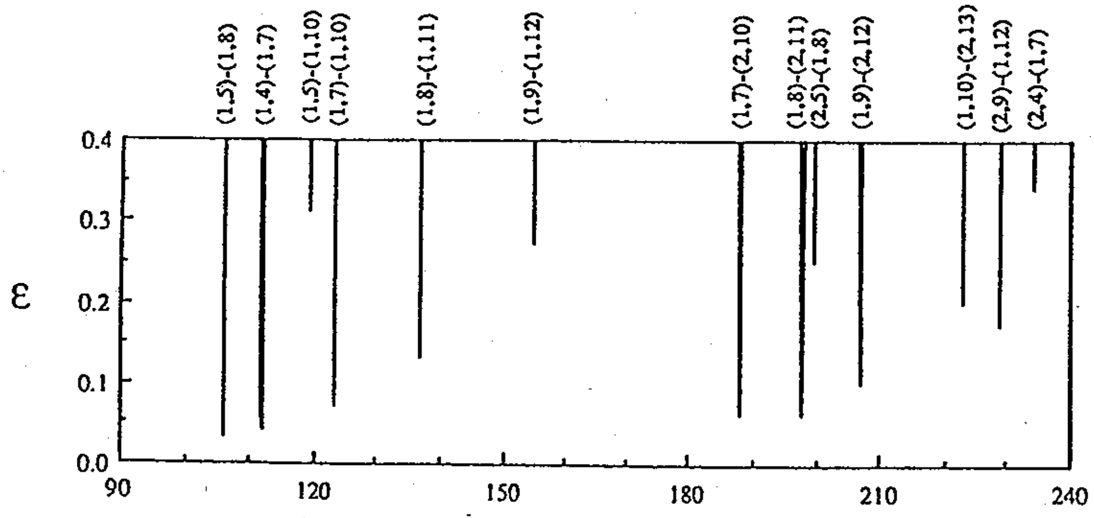
Fig. 7-7 High order instability regions (cosθ imperfections)



V = 5.0× 10^{-3} cos 2θ



V = 1.0× 10^{-2} cos 2θ



V = 2.0× 10^{-2} cos 2θ

Fig. 7-8 (cos2θ imperfection)

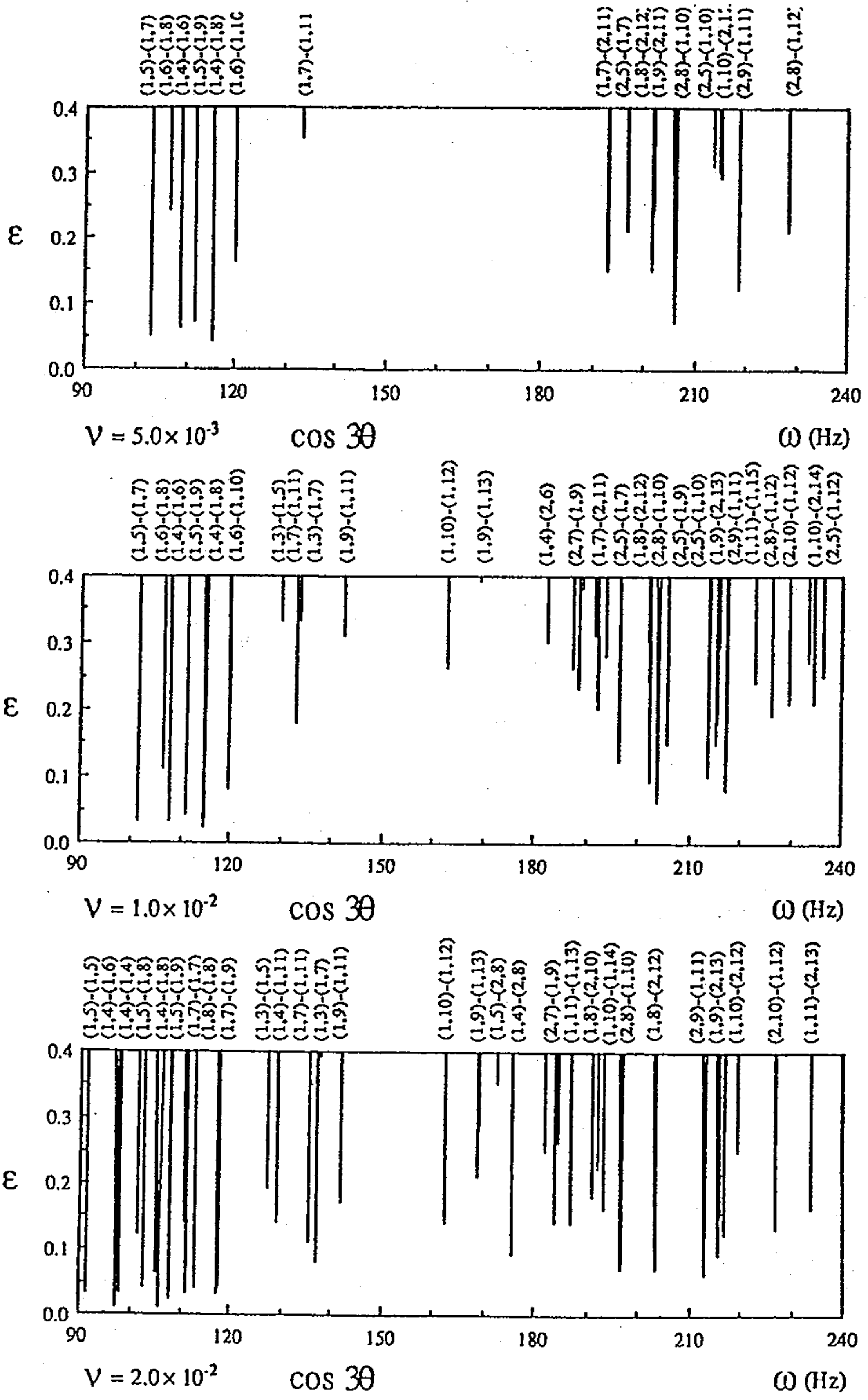


Fig. 7-9 (cos 3θ imperfection)

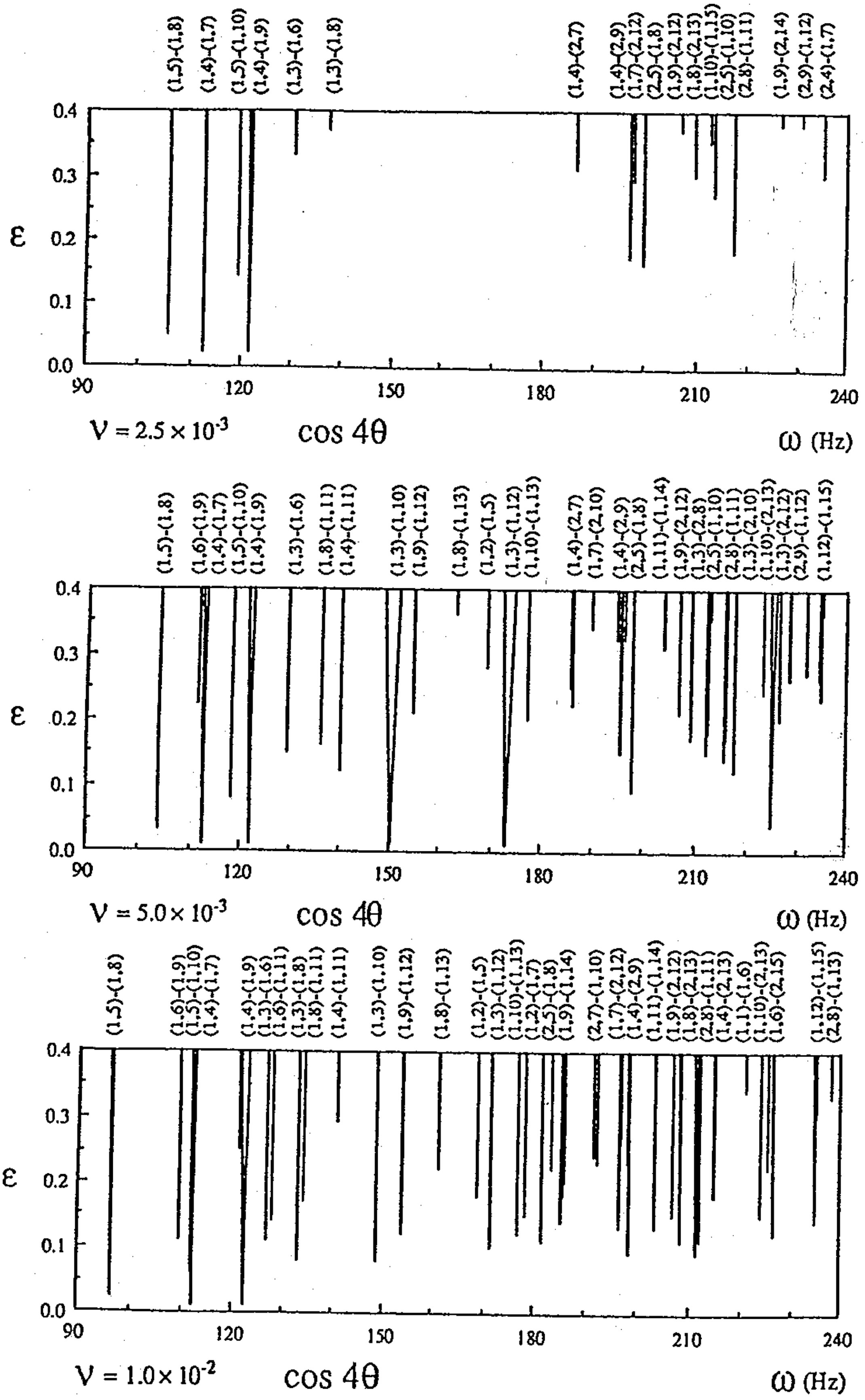
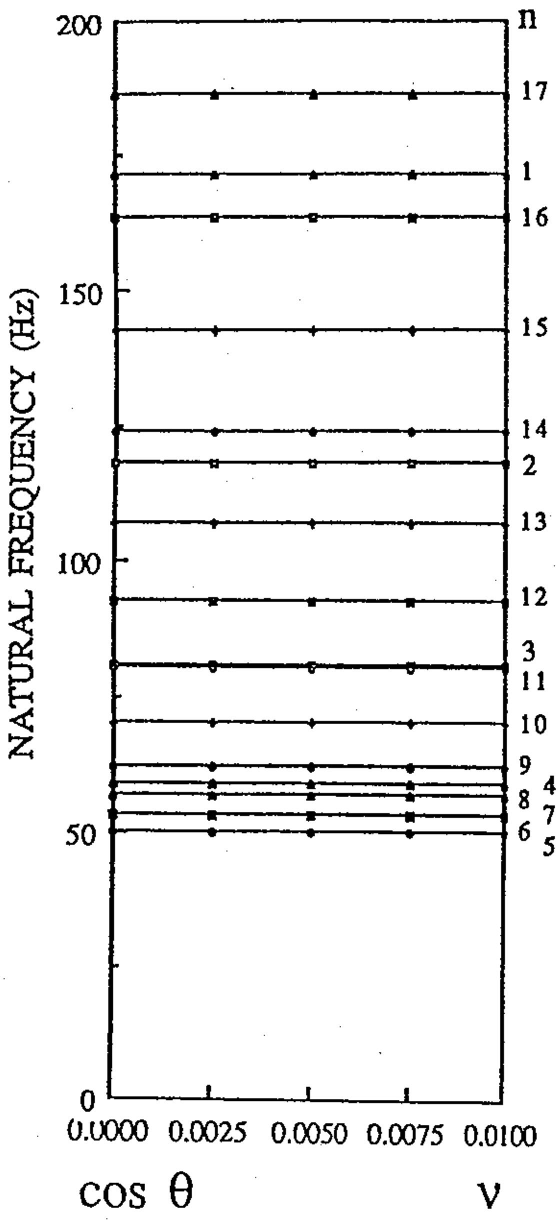
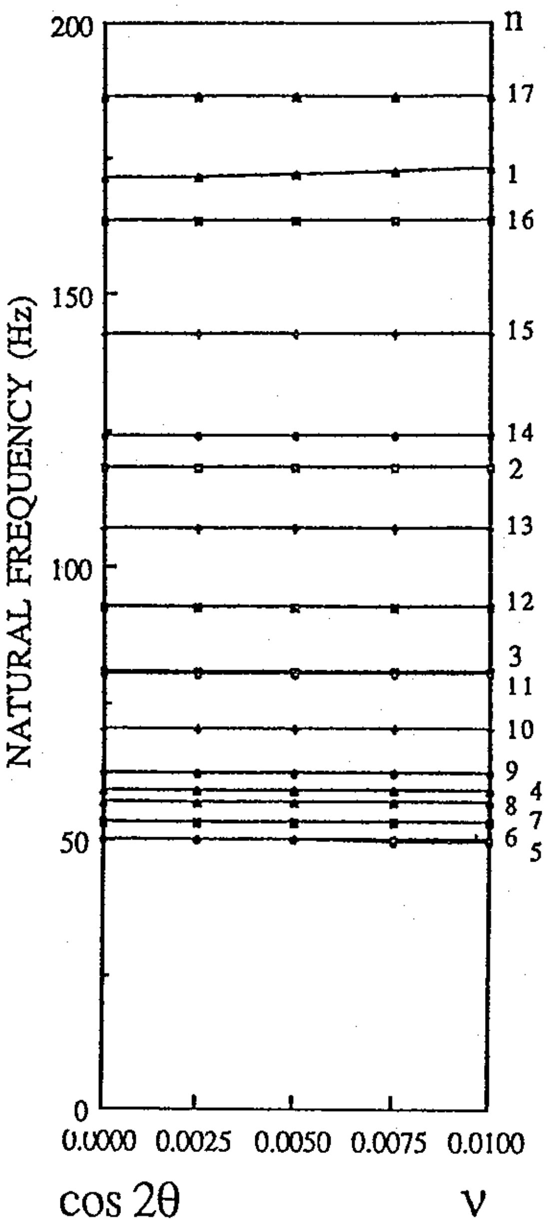


Fig. 7-10

(cos4θ imperfection)



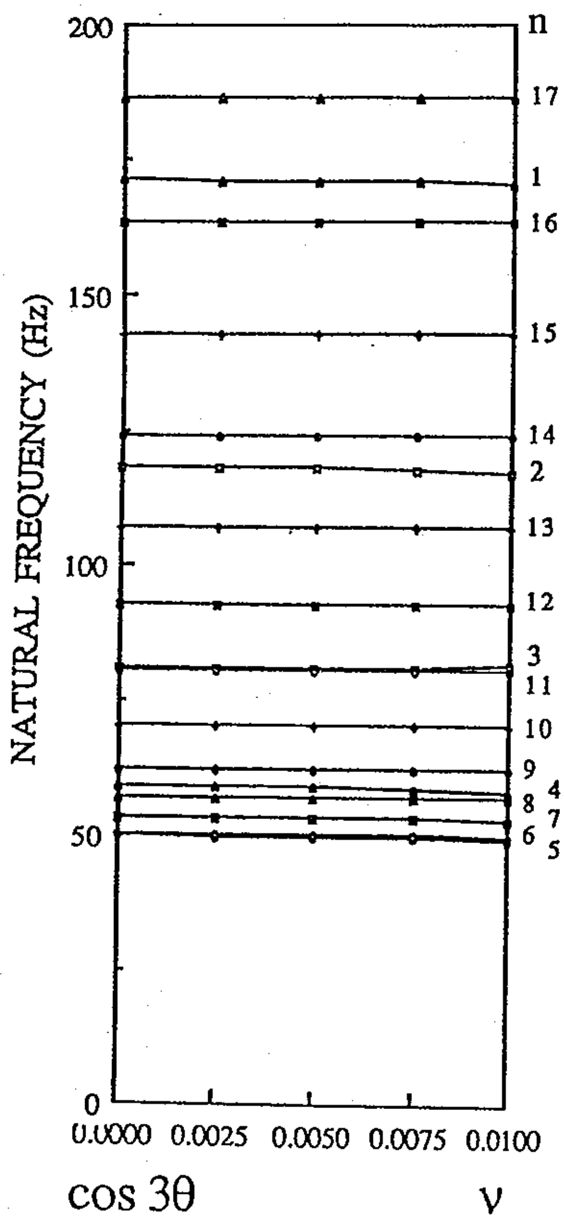


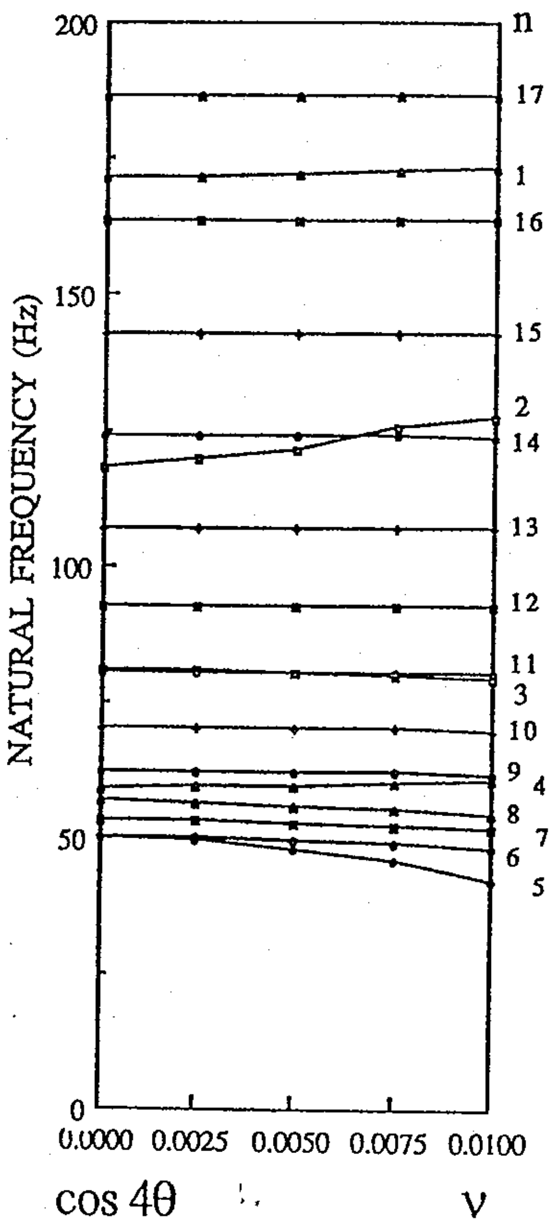
Imperfection amplitude

Fig. 7-11

Fig. 7-12

Changes of natural frequeneies (1/2)





Imperfection amplitude

Fig. 7-13

Fig. 7-14

Changes of natural frequancies

(2) 75% full tall shell [4]

perfect shell

Table 7-3 Comparison of experimental and theoretical buckling frequencies due to nth and (n+1) th circumferential coupling for a 75 full tall shell

n	n+1	Experimental[4] frequencies (Hz)	Theoretical[8] frequencies (Hz)	Theoretical (present) frequencies (Hz)
5	6	58	58	
6	7	66	67	
7	8	80	81	
8	9	96	97	
9	10	114	116	
10	11	136	138	

imperfect shell

Fig.7-16 through Fig.7-19 show the stability charts with excluding the main instability regions for the imperfection patterns of cosθ, cos2θ, cos3θ and cos4θ, respectively. The patterns of the combination of two different modes, (i,n)-(j,m), due to the imperfections are the same as the previous example.

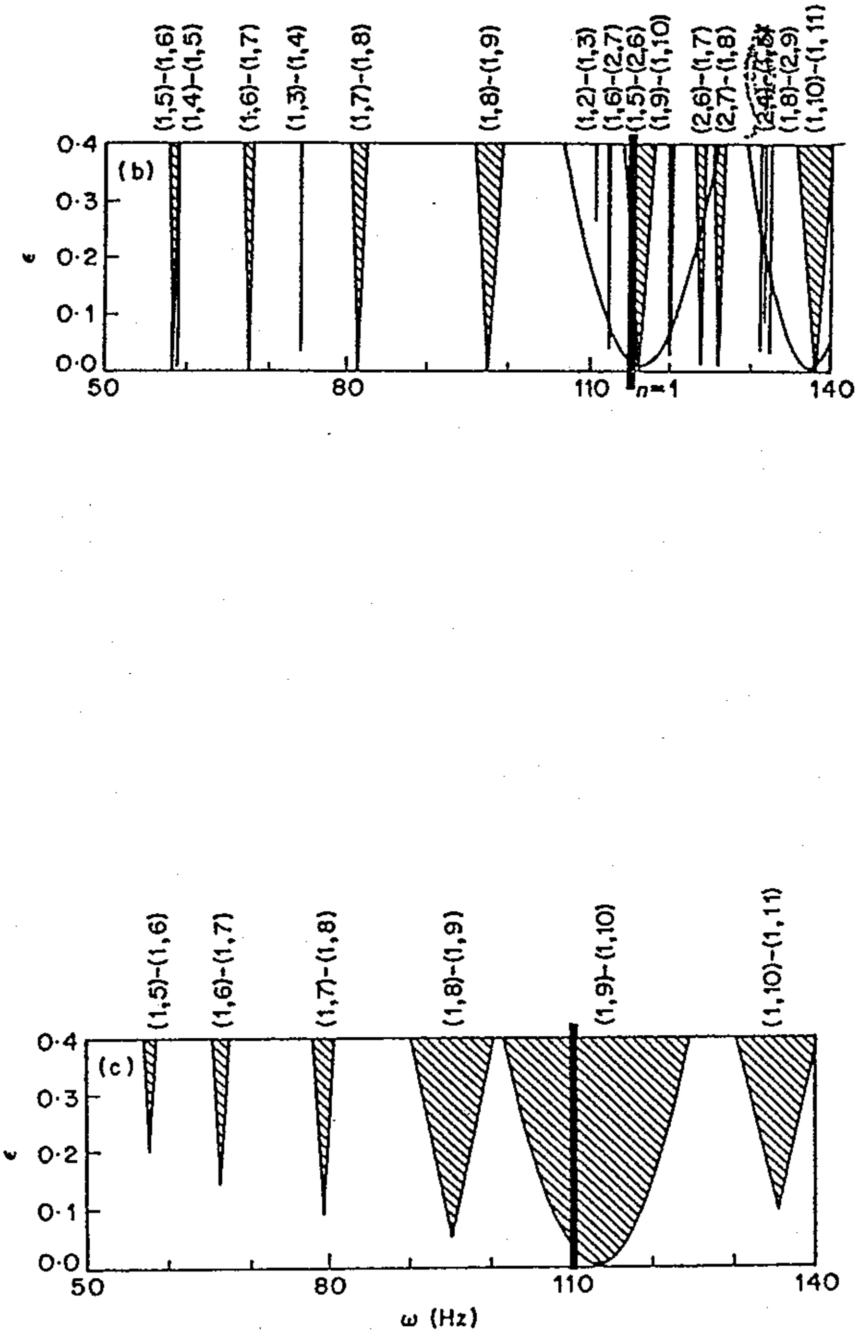


Fig. 7-15 stability Charts for 75 percent full tall tank (perfect shall)

HT75-C1

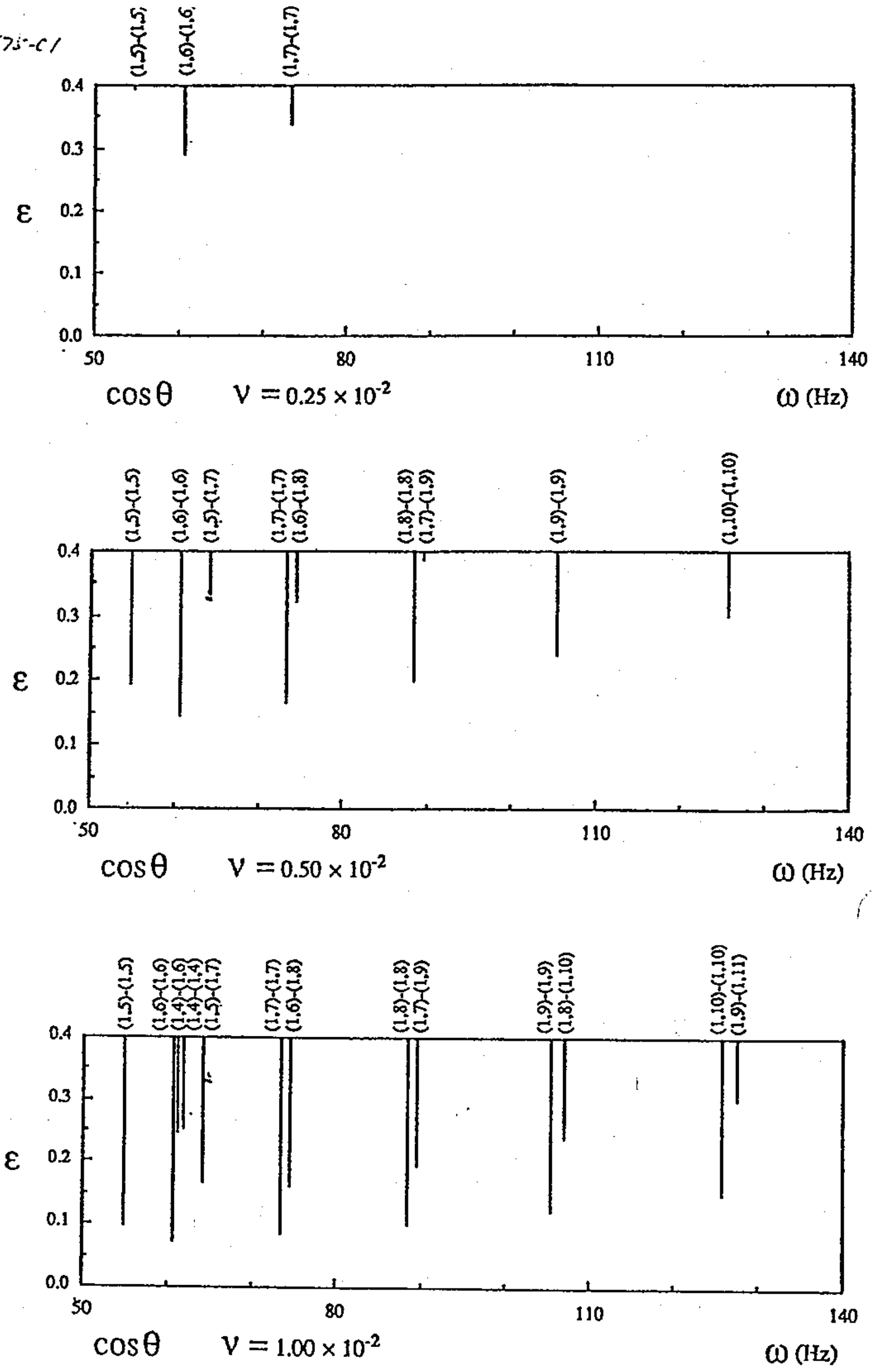


Fig. 7-16 High order instability regions (cosθ imperfection)

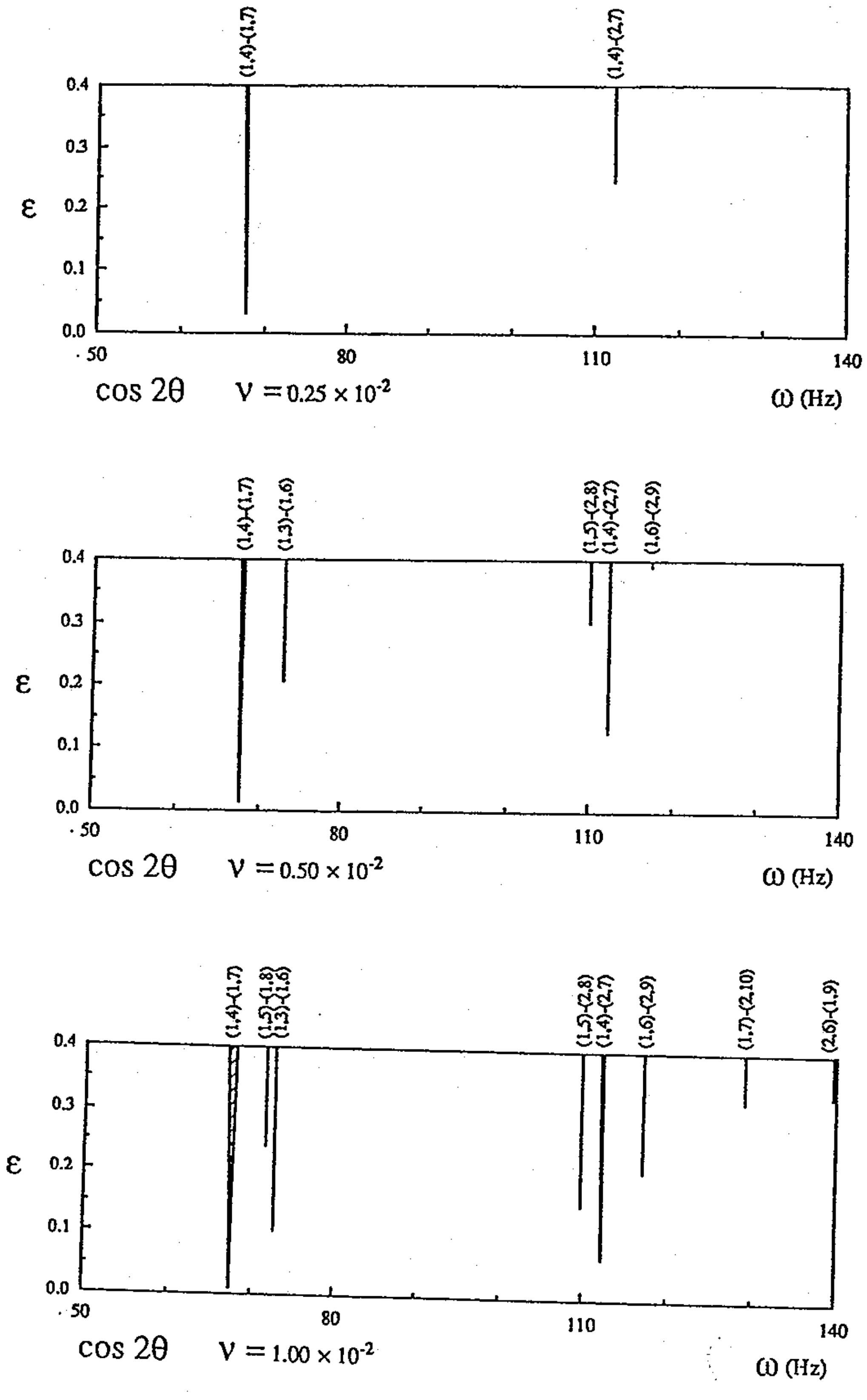
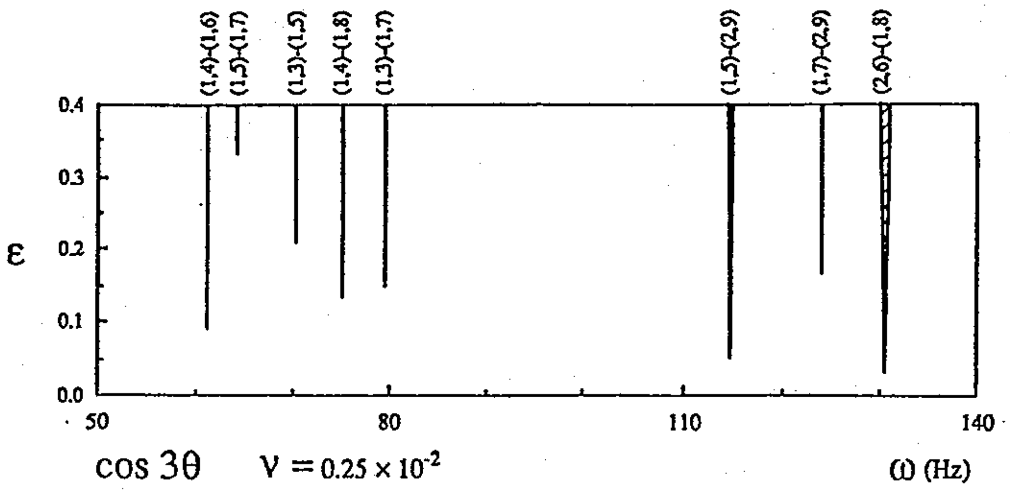
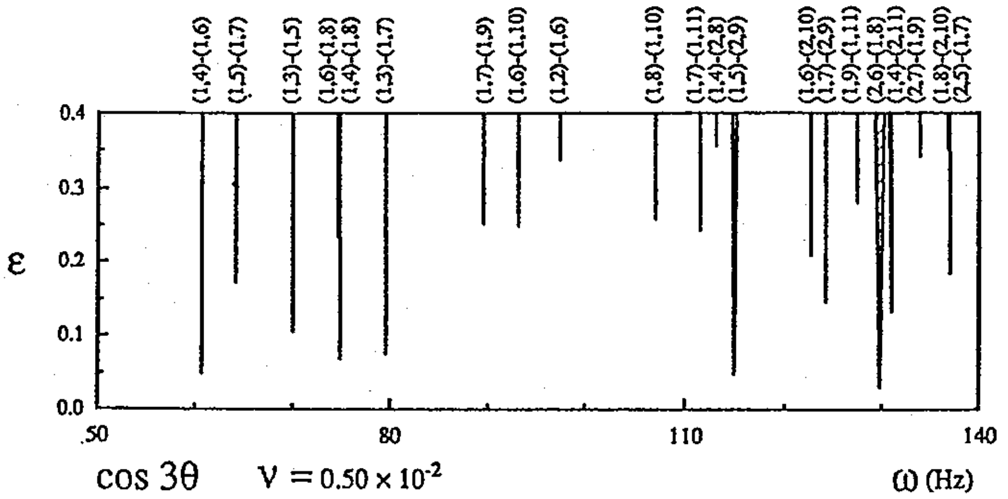


Fig. 7-17 High order instability regions (cos 2θ imperfection)





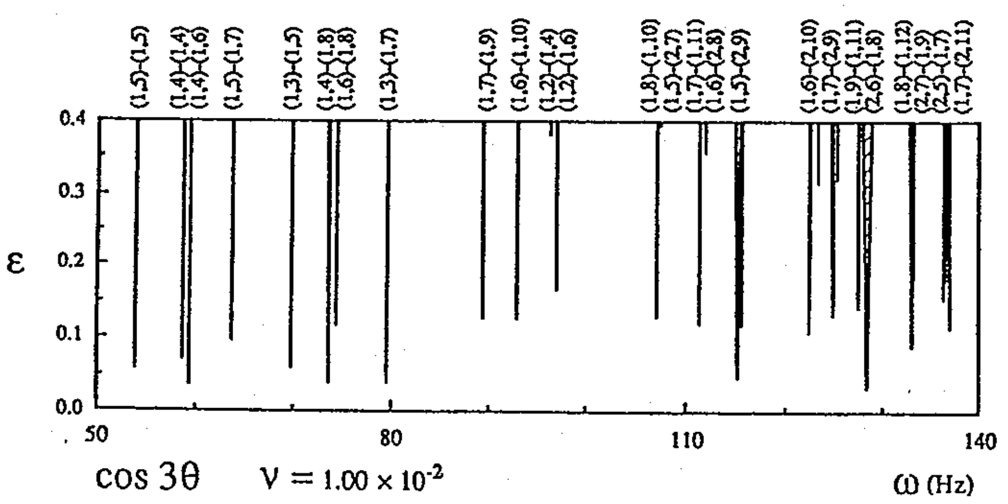


Fig. 7-18 High order instability regions (cos 3θ imperfection)

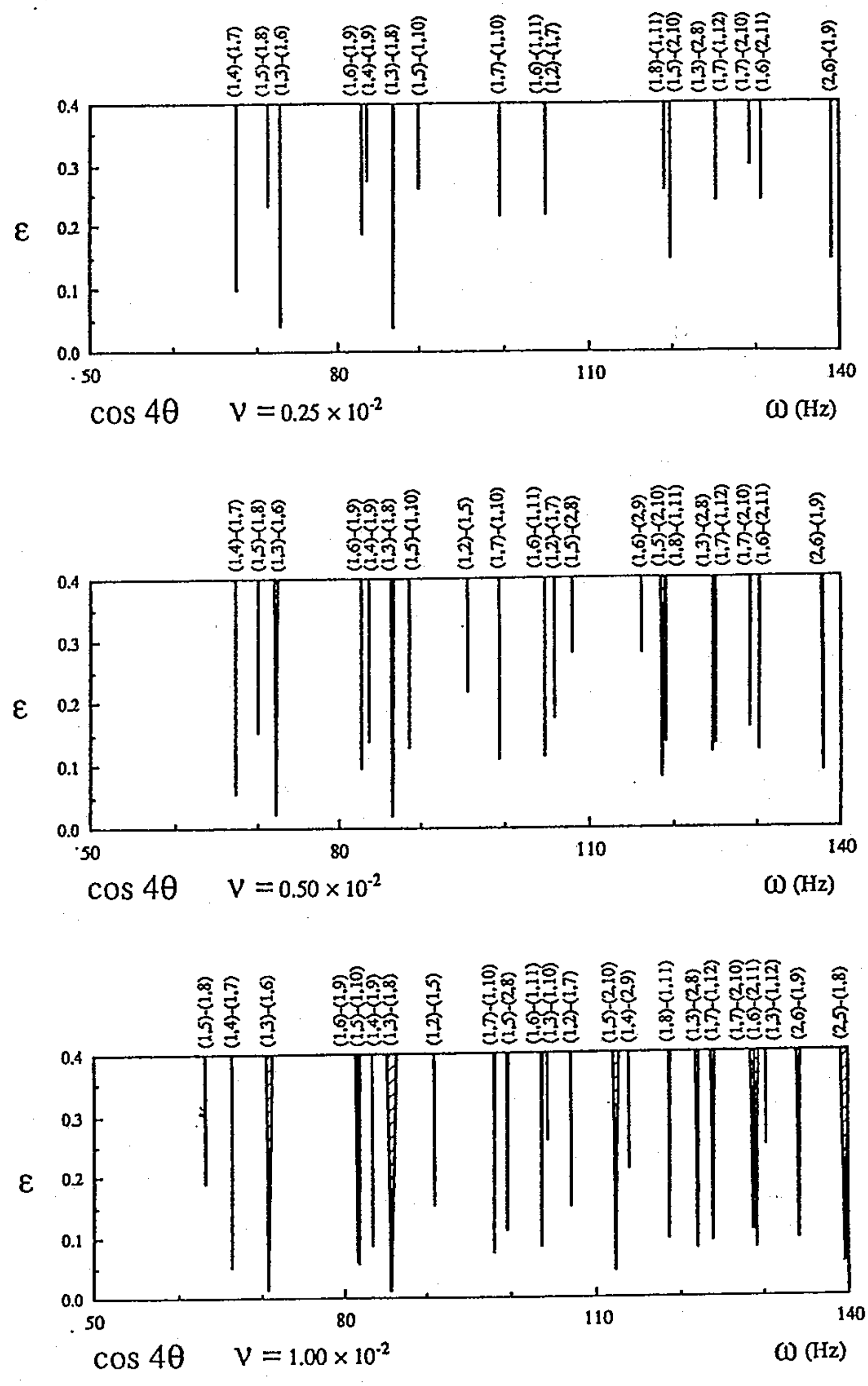
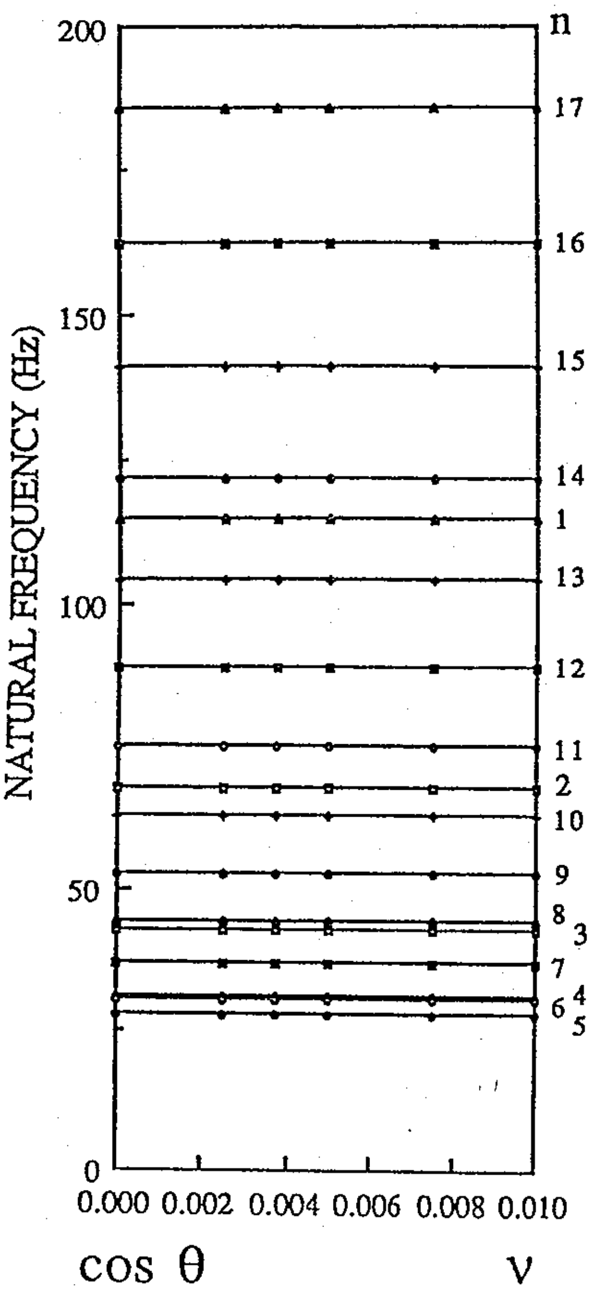
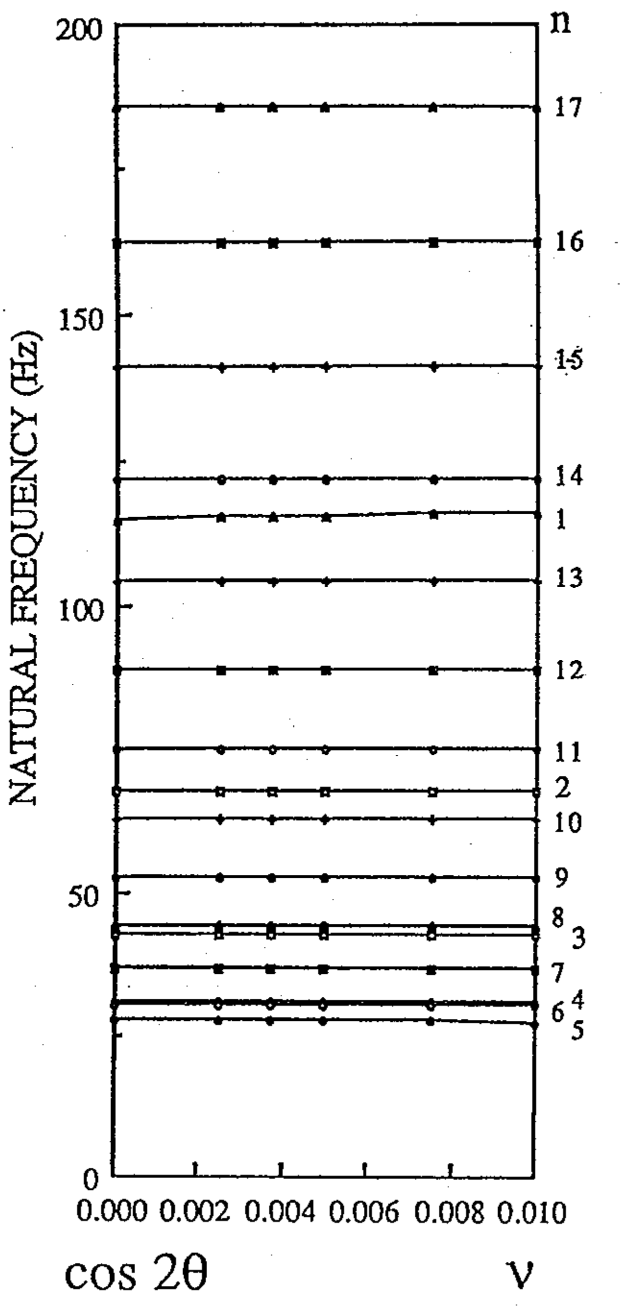


Fig. 7-19 High order instability regions (cos4θ imperfection)



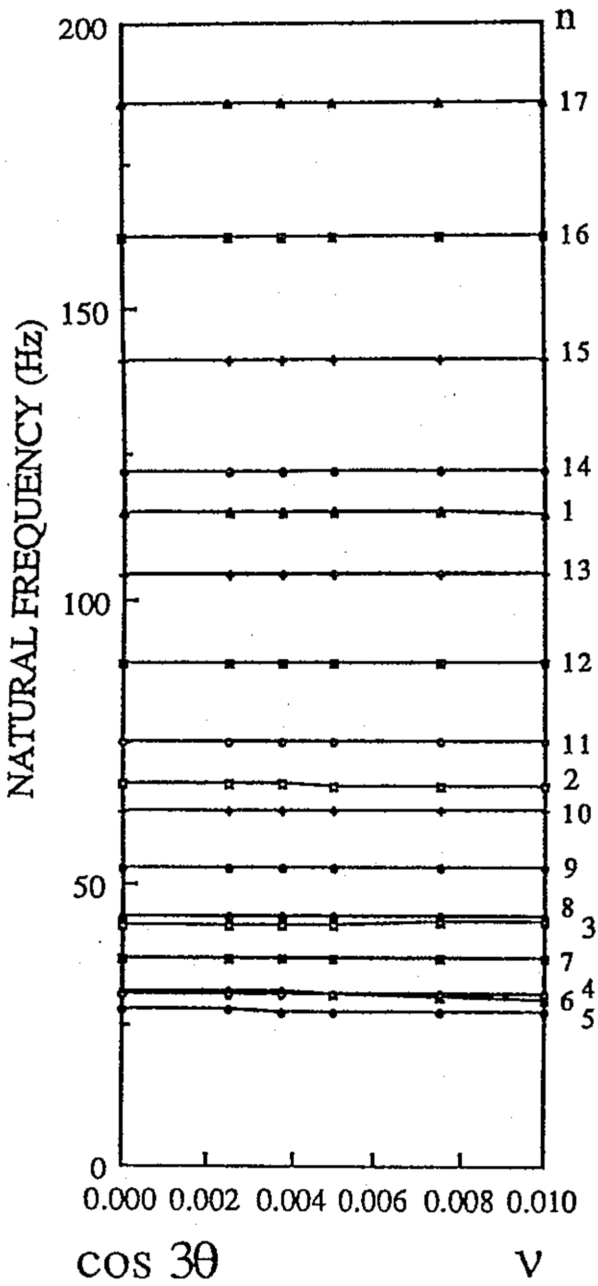


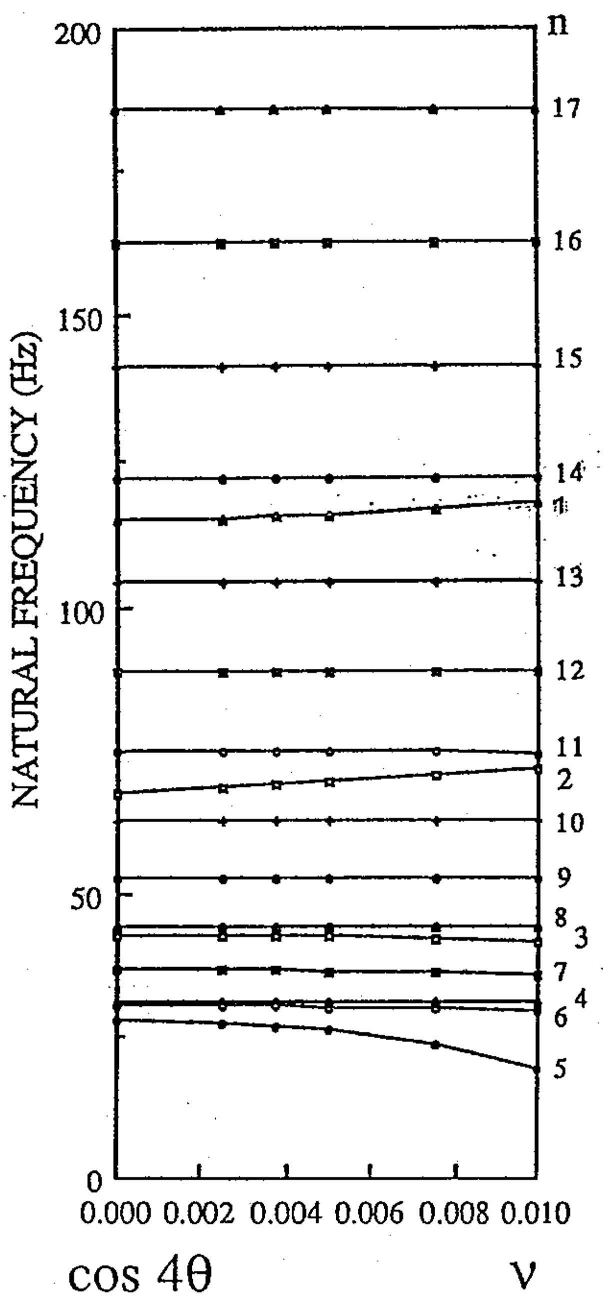
Imperfection amplitude

Fig. 7-20

Fig. 7-21

Changes of natural frequencies





Imperfection amplitude

Fig. 7-22

Fig. 7-23

Changes of natural frequencies

8. Discussion

In this chapter the main points obtained by the numerical analyses are summarized and some discussions are made

Identification of high order instability regions

The high order instability regions, which cannot be obtained by the analysis assuming a perfect shell, have been observed from the experiments [4].
The possibility of the occurance of these high order instalibtiy regions has been verified by the present anelyses which include the effects of circumferential imperfections of a shell.
In the numerical analyses only one circumferential wave of imperfection is assumed for each case.
However,the actual imperfections should be the combination of many waves of imperfections.
Therefore, the actual measurement data will be needed in order to compare the analysis with the experiment.

Shift of the natural frequencies

The natural frequencies are affected by the circumferential imperfections. And which natural frequencies are affected most depends on which circumferential imperfection is assumed.
Also the amount of shift depends on the imperfection pattern.
From the numerical examples it is observed that the influence becomes larger according to the increase of the coefficient of θ in the imperfection pattern function for the same imperfection amplitude .

Size of high order instability regions

Generally, the high order instability regions grow larger as the imperfection amplitude increases. However, there is the possibility that some tricky behaviors occur by the interference among instability regions as mentioned before.

Practical application

In this paper very simple single imperfections are considered as the examples. However, the imperfections of actual shells will be expressed as the combination of many single waves of imperfections. And it is obvious that the instability regions are affected by not only the amplitudes of imperfections but also the patterns of imperfections.

Therefore, the actual measurement data of the imperfection are necessary to compare the experimental results with the analytical ones as precisely as possible. However, it is not better choice to do analyses for each shell in the real situation. In order to establish a rational design method based on analyses, it is important to understand the tendency of the imperfections based on the actual dimensional data base, considering the manufacturing processes.

Based on this data base, the analysis models should be composed.

9.Conclusion

The detailed analytical procedure to solve the dynamic buckling problems of cylindrical shells, which considers not only dynamic fluid - structure interaction and modal coupling in both axial and circumferential directions but also circumferential imperfections, has been described in Chapt. 2 through Chapt. 6. And some simple example analyses have been carried out to demonstrate the effect of the imperfection in Chapt. 7. In Chapt. 8 some key points for the analysis of imperfect shells have been discussed.

Through the example analyses, some characteristics due to the imperfections have been clarified. Also the possibility of the high order instability regions, which were observed in the experimental results, has been verified.

Appendix A

$$C_{nk}(\omega) = \frac{j}{\pi\bar{R}} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} m \left\{ \widehat{a}_{mi}^{nk} \int_0^H \bar{h}_{0m}(\zeta,\omega) \cos(\mu i \zeta) d\zeta \right.$$

$$\left. + \widehat{b}_{mi}^{nk} \int_0^{\bar{R}} \bar{f}_m(r,\omega) J_m(\tilde{\varepsilon}_i r) rdr \right\} \int_0^{2\pi} \eta'(\theta) e^{j(m-n)\theta} d\theta$$

$$- \frac{\bar{R}}{\pi} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} \left\{ \hat{a}_{mi}^{nk} \int_0^H \bar{h}_{0m}(\zeta,\omega) \cos(\mu i \zeta) d\zeta \right.$$

$$\left. + \hat{b}_{mi}^{nk} \int_0^{\bar{R}} \bar{f}_m(r,\omega) J_m(\tilde{\varepsilon}_i r) rdr \right\} \int_0^{2\pi} \eta(\theta) e^{j(m-n)\theta} d\theta$$

$$+ \frac{1}{\pi \left\{ H - \frac{g}{\omega^2} \sin^2(\mu_k H) \right\} \tilde{\mu}_k I_n'(\tilde{\mu}_k \bar{R})} \int_0^{2\pi} \int_0^H \bar{h}_1(\theta,\xi) \cos(\mu_k \zeta) e^{-jn\theta} d\zeta\, d\theta \qquad \text{(A-1)}$$

where

$$\widehat{a}_{mi}^{nk} = \frac{I_m(\tilde{\mu}_k \bar{R})}{\left\{ H - \frac{g}{\omega^2} \sin^2(\mu_k H) \right\} \tilde{\mu}_k^2 I_m'(\tilde{\mu}_k \bar{R}) I_n'(\tilde{\mu}_k \bar{R})} \delta_{ik} \qquad \text{(A-2a)}$$

$$\widehat{b}_{mi}^{nk} = \frac{-2}{\bar{R}^2 \tilde{\mu}_k (\varepsilon_i^2 + \mu_k^2) \left\{ H - \frac{g}{\omega^2} \sin^2(\mu_k H) \right\} \left(1 - \frac{m^2}{(\tilde{\varepsilon}_i \bar{R})^2} \right) J_m(\tilde{\varepsilon}_i \bar{R}) I_n'(\tilde{\mu}_k \bar{R})} \qquad \text{(A-2b)}$$

$$\hat{a}_{mi}^{nk} = \frac{I_m''(\tilde{\mu}_k \bar{R})}{\left\{ H - \frac{g}{\omega^2} \sin^2(\mu_k H) \right\} I_m'(\tilde{\mu}_k \bar{R}) I_n'(\tilde{\mu}_k \bar{R})} \delta_{ik} \qquad \text{(A-2c)}$$

$$\hat{b}_{mi}^{nk} = \frac{-2\, \tilde{\varepsilon}_i^2 J_m''(\tilde{\varepsilon}_i \bar{R})}{\bar{R}^2 \tilde{\mu}_k (\varepsilon_i^2 + \mu_k^2) \left\{ H - \frac{g}{\omega^2} \sin^2(\mu_k H) \right\} \left(1 - \frac{m^2}{(\tilde{\varepsilon}_i \bar{R})^2} \right) J_m^2(\tilde{\varepsilon}_i \bar{R}) I_n'(\tilde{\mu}_k \bar{R})} \qquad \text{(A-2d)}$$

Appendix B

$$F_{in}\,(r,z,t) = \frac{\rho_F}{\left\{1 - \frac{g}{\omega^2 H}\sin^2(\mu_i H)\right\}(\mu_i H)\; I_n'\,(\tilde{\mu}_i \bar{R})}\; I_n\,(\tilde{\mu}_i r)\cos(\mu_i z)\, e^{-j\omega t} \qquad \text{(B-1)}$$

$$Q_{in}\,(r,z,t) = \frac{\rho_F}{(\varepsilon_i \bar{R})\left(1 - \frac{n^2}{(\tilde{\varepsilon}_i \bar{R})^2}\right) J_n^2\,(\tilde{\varepsilon}_i \bar{R})}$$

$$\times\left\{\sinh\,(\varepsilon_i z) - \frac{\omega^2 \sinh\,(\varepsilon_i H) - \varepsilon_i g\cosh\,(\varepsilon_i H)}{\omega^2 \cosh\,(\varepsilon_i H) - \varepsilon_i g\sinh\,(\varepsilon_i H)}\cosh\,(\varepsilon_i z)\right\} J_n\,(\tilde{\varepsilon}_i r)\, e^{-j\omega t} \qquad \text{(B-2)}$$

$$\hat{\hat{A}}{}^m_{in}\,(r,z,t) = \frac{\rho_F}{\pi \bar{R}}\, m\hat{\hat{a}}{}^{ni}_{mi}\, I_n\,(\tilde{\mu}_i r)\cos\,(\mu_i z)\, e^{-j\omega t}$$

$$= \frac{\rho_F\, m\, I_m\,(\tilde{\mu}_i \bar{R})\, I_n\,(\tilde{\mu}_i r)\cos\,(\mu_i z)\, e^{-j\omega t}}{\pi \bar{R}\left\{H - \frac{g}{\omega^2}\sin^2(\mu_i H)\right\}\tilde{\mu}_i^2\, I_m'\,(\tilde{\mu}_i \bar{R})\, I_n'\,(\tilde{\mu}_i \bar{R})} \qquad \text{(B-3a)}$$

$$\hat{A}{}^m_{in}\,(r,z,t) = \frac{\rho_F \bar{R}}{\pi}\,\hat{a}{}^{i}_{mi}\, I_n\,(\tilde{\mu}_i r)\cos\,(\mu_i z)\, e^{-j\omega t}$$

$$= \frac{\rho_F\, \bar{R}\, I_m''(\tilde{\mu}_i \bar{R})\cdot I_n\,(\tilde{\mu}_i r)\cos\,(\mu_i z)\, e^{-j\omega t}}{\pi\left\{H - \frac{g}{\omega^2}\sin^2(\mu_i H)\right\} I_m'\,(\tilde{\mu}_i \bar{R})\, I_n'\,(\tilde{\mu}_i \bar{R})} \qquad \text{(B-3b)}$$

$$\left(\hat{\hat{A}}{}^m_{in} = \frac{m\, I_m\,(\tilde{\mu}_i \bar{R})}{\bar{R}^2\,\tilde{\mu}_i^2\, I_m''(\tilde{\mu}_i \bar{R})}\,\hat{A}{}^m_{in}\right)$$

$$\hat{\hat{B}}{}^{im}_{kn}\,(r,z,t) = \frac{\rho_F}{\pi \bar{R}}\, m\,\hat{\hat{b}}{}^{nk}_{mi}\, I_n\,(\tilde{\mu}_k r)\cos\,(\mu_k z)\, e^{-j\omega t}$$

$$= -\frac{2\,\rho_F\, m\, I_n\,(\tilde{\mu}_k r)\cos\,(\mu_k z)\, e^{-j\omega t}}{\pi \bar{R}^3\,\tilde{\mu}_k\,(\varepsilon_i^2 + \mu_k^2)\left\{H - \frac{g}{\omega^2}\sin^2(\mu_k H)\right\}\left(1 - \frac{m^2}{(\tilde{\varepsilon}_i \bar{R})^2}\right) J_m\,(\tilde{\varepsilon}_i \bar{R})\, I_n'\,(\tilde{\mu}_k \bar{R})} \qquad \text{(B-4a)}$$

$$\hat{B}^{im}_{kn}(r,z,t) = -\frac{2\,\rho_F\,\tilde{\varepsilon}_i^2\,J_m''(\tilde{\varepsilon}_i\bar{R})\,I_n(\tilde{\mu}_k r)\cos(\mu_k z)\,e^{-j\omega t}}{\pi\bar{R}\,\tilde{\mu}_k\,(\varepsilon_i^2+\mu_k^2)\left\{H-\frac{g}{\omega^2}\sin^2(\mu_k H)\right\}\left(1-\frac{m^2}{(\tilde{\varepsilon}_i\bar{R})^2}\right)J_m^2(\tilde{\varepsilon}_i\bar{R})\,I_n'(\tilde{\mu}_k\bar{R})} \tag{B-4b}$$

$$\left(\hat{B}^{im}_{kn} = \frac{m\,J_m(\tilde{\varepsilon}_i\bar{R})}{\bar{R}^2\,\tilde{\varepsilon}_i^2\,J_m''(\tilde{\varepsilon}_i\bar{R})}\hat{B}^{im}_{kn}\right)$$

$$\tilde{C}_{in} = \left(\frac{I_1(\tilde{\mu}_i\bar{R})}{\bar{R}\,\tilde{\mu}_i\,I_1'(\tilde{\mu}_i\bar{R})}-1\right)\int_0^{2\pi}\eta'(\theta)\sin\theta\,e^{-jn\theta}\,d\theta + \frac{\bar{R}\,\tilde{\mu}_i\,I_1''(\tilde{\mu}_i\bar{R})}{I_1'(\tilde{\mu}_i\bar{R})}\int_0^{2\pi}\eta(\theta)\cos\theta\,e^{-jn\theta}\,d\theta \tag{B–5}$$

$$\tilde{D}_{in} = \int_0^{2\pi}\eta'(\theta)\sin\theta\,e^{-jn\theta}\,d\theta + \frac{\tilde{\varepsilon}_i^2\,\bar{R}^2\,J_1''(\tilde{\varepsilon}_i\bar{R})}{J_1(\tilde{\varepsilon}_i\bar{R})}\int_0^{2\pi}\eta(\theta)\cos\theta\,e^{-jn\theta}\,d\theta \tag{B–6}$$

Appendix C : The effect of a lumped mass (see Fig.C-1)

By integrating eqn.(4.1b) after substitution of eqn.(4.1c), an integration constant which is the function of θ is obtained. If we assume the shearing membrane force by

$$N_{z\psi}^{(M)} = A \sin\psi \approx A\,(\sin\theta - \nu\eta'\cos\theta) \tag{C-1}$$

the total shear force should be equivalent to the shearing inertia force by the lumped mass.

$$\oint N_{z\psi}^{(M)} \sin\psi\, R_s\, d\psi \approx A\,\bar{R} \int_0^{2\pi} (\sin\theta - \nu\eta'\cos\theta)^2\,(1 + \nu\eta)\, d\theta$$

$$= A\,\pi\bar{R}\,(1 + \frac{3}{2}\nu\, c_2^c)$$

$$\equiv M\,(G_h(t) + L_1\,\ddot{q}\,(t)) \tag{C-2}$$

Therefore,

$$A \approx \frac{M}{\pi\bar{R}}\,(G_h(t) + L_1\,\ddot{q}\,(t))\,(1 - \frac{3}{2}\nu\, c_2^c) \tag{C-3}$$

Consequently,

$$N_{z\psi}^{(M)} = \frac{M}{\pi\bar{R}}\,(G_h(t) + L_1\,\ddot{q}\,(t))\,(1 - \frac{3}{2}\nu\, c_2^c)\,(\sin\theta - \nu\eta'\cos\theta)$$

$$\approx \frac{M}{\pi\bar{R}}\,(G_h(t) + L_1\,\ddot{q}\,(t))\,\{\sin\theta - \nu\,(\eta'\cos\theta + \frac{3}{2}\nu\, c_2^c\,\sin\theta)\} \tag{C-4}$$

By integrating eqn.(4.1a) after substitution of eqn.(4.4a), a linear function of z can be obtaind. The axial membrane force induced by the horizontal ground motion and / or rocking motion has to be zero at the location of the lumped mass($z=L_1$), and the axial membrane force induced by the vertical ground motion and the gravity has to be a constant value which is equivalent to the value divided the axial inertia force by the developed length of the tank cross-section.

$$N_z^{(M)} = \frac{M}{\pi \bar{R}^2} (G_h(t) + L_1 \ddot{q}(t)) (L_1 - z)$$

$$\{\cos\theta - \nu ((\eta + \eta'')\cos\theta - \eta'\sin\theta + \frac{3}{2} c_2^c \cos\theta)\} + \frac{M}{2\pi \bar{R}} (G_v(t) - g) \quad \text{(C-5)}$$

It can be shown by using eqns.(C-4) and (C-5) that the equilibrium equations are satisfied.

<u>Total axial force</u>

$$\oint N_z^{(M)} R_s \, d\psi = M (G_v(t) - g) \quad \text{(C-6)}$$

<u>Bending moment</u>

$$\oint N_z^{(M)} \cdot R \cos\theta \cdot R_s \, d\psi = M (G_h(t) + L_1 \ddot{q}(t)) (L_1 - z) \quad \text{(C-7)}$$

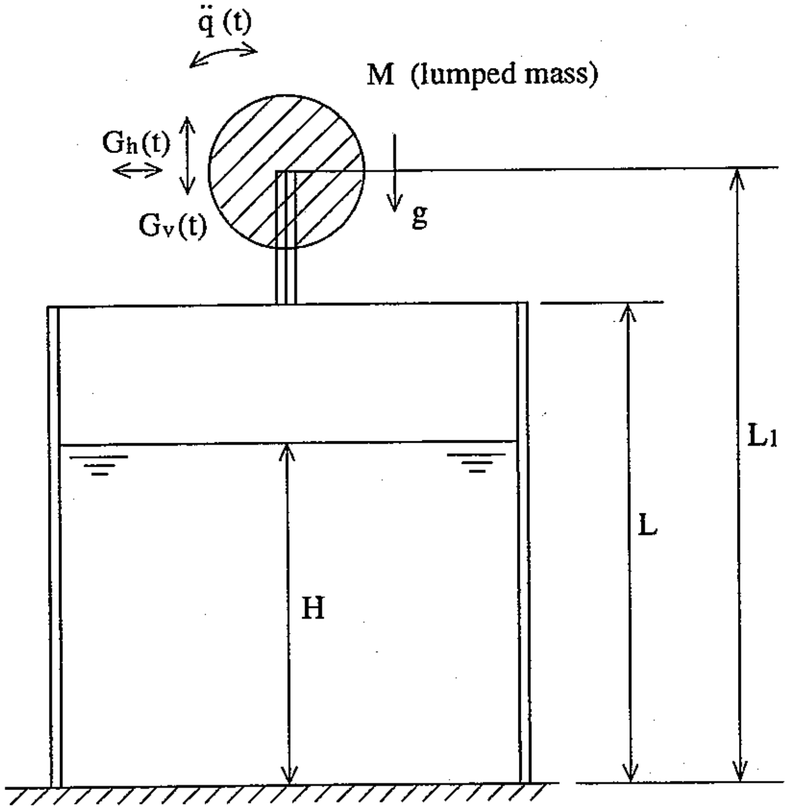


Fig.C-1 Effect of a lumped mass

Appendix D

$$\eta_{nm}^{ss} = \int_0^{2\pi} \eta(\theta)\ \sin n\theta\ \sin m\theta\ d\theta = \frac{\pi}{2}\left(c_{|n-m|}^{c} - c_{n+m}^{c}\right) \tag{D-1a}$$

$$\eta_{nm}^{sc} = \int_0^{2\pi} \eta(\theta)\ \sin n\theta\ \cos m\theta\ d\theta = \frac{\pi}{2}\left\{\mathrm{sign}\ (n-m)\ c_{|n-m|}^{s} + c_{n+m}^{s}\right\} \tag{D-1b}$$

$$\eta_{nm}^{cs} = \int_0^{2\pi} \eta(\theta)\ \cos n\theta\ \sin m\theta\ d\theta = \frac{\pi}{2}\left\{\mathrm{sign}\ (m-n)\ c_{|n-m|}^{s} + c_{n+m}^{s}\right\} \tag{D-1c}$$

$$\eta_{nm}^{cc} = \int_0^{2\pi} \eta(\theta)\ \cos n\theta\ \cos m\theta\ d\theta = \frac{\pi}{2}\left(c_{|n-m|}^{c} + c_{n+m}^{c}\right) \tag{D-1d}$$

$$\eta_{nm}^{'ss} = \int_0^{2\pi} \eta'(\theta)\ \sin n\theta\ \sin m\theta\ d\theta = \frac{\pi}{2}\left\{|n-m|\ c_{|n-m|}^{s} - (n+m)\ c_{n+m}^{s}\right\} \tag{D-2a}$$

$$\eta_{nm}^{'sc} = \int_0^{2\pi} \eta'(\theta)\ \sin n\theta\ \cos m\theta\ d\theta = \frac{\pi}{2}\left\{-\ (n-m)\ c_{|n-m|}^{c} - (n+m)\ c_{n+m}^{c}\right\} \tag{D-2b}$$

$$\eta_{nm}^{'cs} = \int_0^{2\pi} \eta'(\theta)\ \cos n\theta\ \sin m\theta\ d\theta = \frac{\pi}{2}\left\{(n-m)\ c_{|n-m|}^{c} - (n+m)\ c_{n+m}^{c}\right\} \tag{D-2c}$$

$$\eta_{nm}^{'cc} = \int_0^{2\pi} \eta'(\theta)\ \sin n\theta\ \sin m\theta\ d\theta = \frac{\pi}{2}\left\{|n-m|\ c_{|n-m|}^{s} + (n+m)\ c_{n+m}^{s}\right\} \tag{D-2d}$$

$$\eta_{nm}^{''ss} = \int_0^{2\pi} \eta''(\theta)\ \sin n\theta\ \sin m\theta\ d\theta = -\ \frac{\pi}{2}\left\{(n-m)^2\ c_{|n-m|}^{c} - (n+m)^2\ c_{n+m}^{c}\right\} \tag{D-3a}$$

$$\eta_{nm}^{''sc} = \int_0^{2\pi} \eta''(\theta)\ \sin n\theta\ \cos m\theta\ d\theta = -\ \frac{\pi}{2}\left\{\mathrm{sign}\ (n-m)\cdot(n-m)^2\ c_{|n-m|}^{s} + (n+m)^2\ c_{n+m}^{s}\right\} \tag{D-3b}$$

$$\eta_{nm}^{''cs} = \int_0^{2\pi} \eta''(\theta)\ \cos n\theta\ \sin m\theta\ d\theta = -\frac{\pi}{2}\left\{\mathrm{sign}\ (m-n)\cdot(n-m)^2\, c_{|n-m|}^{s} + (n+m)^2\, c_{n+m}^{s}\right\} \qquad \text{(D-3c)}$$

$$\eta_{nm}^{''cc} = \int_0^{2\pi} \eta''(\theta)\ \cos n\theta\ \cos m\theta\ d\theta = -\frac{\pi}{2}\left\{(n-m)^2\, c_{|n-m|}^{c} + (n+m)^2\, c_{n+m}^{c}\right\} \qquad \text{(D-3d)}$$

$$\eta_{nm}^{'''ss} = \int_0^{2\pi} \eta'''(\theta)\ \sin n\theta\ \sin m\theta\ d\theta = -\frac{\pi}{2}\left\{|n-m|^3\, c_{|n-m|}^{s} - (n+m)^3\, c_{n+m}^{s}\right\} \qquad \text{(D-4a)}$$

$$\eta_{nm}^{'''sc} = \int_0^{2\pi} \eta'''(\theta)\ \sin n\theta\ \cos m\theta\ d\theta = -\frac{\pi}{2}\left\{-\ (n-m)^3\, c_{|n-m|}^{c} - (n+m)^3\, c_{n+m}^{c}\right\} \qquad \text{(D-4b)}$$

$$\eta_{nm}^{'''cs} = \int_0^{2\pi} \eta'''(\theta)\ \cos n\theta\ \sin m\theta\ d\theta = -\frac{\pi}{2}\left\{(n-m)^3\, c_{|n-m|}^{c} - (n+m)^3\, c_{n+m}^{c}\right\} \qquad \text{(D-4c)}$$

$$\eta_{nm}^{'''cc} = \int_0^{2\pi} \eta'''(\theta)\ \cos n\theta\ \cos m\theta\ d\theta = -\frac{\pi}{2}\left\{|n-m|^3\, c_{|n-m|}^{s} + (n+m)^3\, c_{n+m}^{s}\right\} \qquad \text{(D-4d)}$$

$$\eta_{nm}^{''''ss} = \int_0^{2\pi} \eta''''(\theta)\ \sin n\theta\ \sin m\theta\ d\theta = \frac{\pi}{2}\left\{(n-m)^4\, c_{|n-m|}^{c} - (n+m)^4\, c_{n+m}^{c}\right\} \qquad \text{(D-5a)}$$

$$\eta_{nm}^{''''sc} = \int_0^{2\pi} \eta''''(\theta)\ \sin n\theta\ \cos m\theta\ d\theta = \frac{\pi}{2}\left\{\mathrm{sign}\ (n-m)\cdot(n-m)^4\, c_{|n-m|}^{s} + (n+m)^4\, c_{n+m}^{s}\right\} \qquad \text{(D-5b)}$$

$$\eta_{nm}^{''''cs} = \int_0^{2\pi} \eta''''(\theta)\ \cos n\theta\ \sin m\theta\ d\theta = \frac{\pi}{2}\left\{\mathrm{sign}\ (m-n)\cdot(n-m)^4\, c_{|n-m|}^{s} + (n+m)^4\, c_{n+m}^{s}\right\} \qquad \text{(D-5c)}$$

$$\eta_{nm}^{''''cc} = \int_0^{2\pi} \eta''''(\theta)\ \cos n\theta\ \cos m\theta\ d\theta = \frac{\pi}{2}\left\{(n-m)^4\, c_{|n-m|}^{c} + (n+m)^4\, c_{n+m}^{c}\right\} \qquad \text{(D-5d)}$$

$$\eta^{ss}_{snm} = \int_{0}^{2\pi} \eta(\theta)\ \sin\theta\ \sin n\theta\ \sin m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ -\ \mathrm{sign}\ (n-m-1)\ c^{s}_{|n-m-1|} + \mathrm{sign}\ (n-m+1)\ c^{s}_{|n-m+1|} + c^{s}_{n+m-1} - c^{s}_{n+m+1} \right\} \qquad \text{(D-6a)}$$

$$\eta^{sc}_{snm} = \int_{0}^{2\pi} \eta(\theta)\ \sin\theta\ \sin n\theta\ \cos m\theta\ d\theta$$

$$= \frac{\pi}{4} \left(c^{c}_{|n-m-1|} - c^{c}_{|n-m+1|} + c^{c}_{n+m-1} - c^{c}_{n+m+1} \right) \qquad \text{(D-6b)}$$

$$\eta^{cs}_{snm} = \int_{0}^{2\pi} \eta(\theta)\ \sin\theta\ \cos n\theta\ \sin m\theta\ d\theta$$

$$= \frac{\pi}{4} \left(-\ c^{c}_{|n-m-1|} + c^{c}_{|n-m+1|} + c^{c}_{n+m-1} - c^{c}_{n+m+1} \right) \qquad \text{(D-6c)}$$

$$\eta^{cc}_{snm} = \int_{0}^{2\pi} \eta(\theta)\ \sin\theta\ \cos n\theta\ \cos m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ -\ \mathrm{sign}\ (n-m-1)\ c^{s}_{|n-m-1|} + \mathrm{sign}\ (n-m+1)\ c^{s}_{|n-m+1|} - c^{s}_{n+m-1} + c^{s}_{n+m+1} \right\} \qquad \text{(D-6d)}$$

$$\eta^{\prime ss}_{snm} = \int_{0}^{2\pi} \eta'(\theta)\ \sin\theta\ \sin n\theta\ \sin m\theta\ d\theta$$

$$= -\frac{\pi}{4} \left\{ -\ (n-m-1)\ c^{c}_{|n-m-1|} + (n-m+1)\ c^{c}_{|n-m+1|} + (n+m-1)\ c^{c}_{n+m-1} - (n+m+1)\ c^{c}_{n+m+1} \right\} \qquad \text{(D-7a)}$$

$$\eta^{\prime sc}_{snm} = \int_{0}^{2\pi} \eta'(\theta)\ \sin\theta\ \sin n\theta\ \cos m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ |n-m-1|\ c^{s}_{|n-m-1|} - |n-m+1|\ c^{s}_{|n-m+1|} + (n+m-1)\ c^{s}_{n+m-1} - (n+m+1)\ c^{s}_{n+m+1} \right\} \qquad \text{(D-7b)}$$

$$\eta^{\prime cs}_{snm} = \int_{0}^{2\pi} \eta'(\theta)\ \sin\theta\ \cos n\theta\ \sin m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ -\ |n-m-1|\ c^{s}_{|n-m-1|} + |n-m+1|\ c^{s}_{|n-m+1|} + (n+m-1)\ c^{s}_{n+m-1} - (n+m+1)\ c^{s}_{n+m+1} \right\} \qquad \text{(D-7c)}$$

$$\eta^{\prime cc}_{snm} = \int_{0}^{2\pi} \eta'(\theta)\ \sin\theta\ \cos n\theta\ \cos m\theta\ d\theta$$

$$= -\frac{\pi}{4} \left\{ -\ (n-m-1)\ c^{c}_{|n-m-1|} + (n-m+1)\ c^{c}_{|n-m+1|} - (n+m-1)\ c^{c}_{n+m-1} + (n+m+1)\ c^{c}_{n+m+1} \right\} \qquad \text{(D-7d)}$$

$$\eta''^{ss}_{snm} = \int_0^{2\pi} \eta''(\theta) \sin\theta \sin n\theta \sin m\theta \, d\theta$$

$$= -\frac{\pi}{4} \{ -\mathrm{sign}\,(n-m-1)\cdot(n-m-1)^2 c^{s}_{|n-m-1|} + \mathrm{sign}\,(n-m+1)\cdot(n-m+1)^2 c^{s}_{|n-m+1|}$$

$$+ (n+m-1)^2 c^{s}_{n+m-1} - (n+m+1)^2 c^{s}_{n+m+1} \} \qquad \text{(D-8a)}$$

$$\eta''^{sc}_{snm} = \int_0^{2\pi} \eta''(\theta) \sin\theta \sin n\theta \cos m\theta \, d\theta$$

$$= -\frac{\pi}{4} \{ (n-m-1)^2 c^{c}_{|n-m-1|} - (n-m+1)^2 c^{c}_{|n-m+1|} + (n+m-1)^2 c^{c}_{n+m-1} - (n+m+1)^2 c^{c}_{n+m+1} \}$$

(D-8b)

$$\eta''^{cs}_{snm} = \int_0^{2\pi} \eta''(\theta) \sin\theta \cos n\theta \sin m\theta \, d\theta$$

$$= -\frac{\pi}{4} \{ -(n-m-1)^2 c^{c}_{|n-m-1|} + (n-m+1)^2 c^{c}_{n-m+1} + (n+m-1)^2 c^{c}_{n+m-1} - (n+m+1)^2 c^{c}_{n+m+1} \}$$

(D-8c)

$$\eta''^{cc}_{snm} = \int_0^{2\pi} \eta''(\theta) \sin\theta \cos n\theta \cos m\theta \, d\theta$$

$$= -\frac{\pi}{4} \{ -\mathrm{sign}\,(n-m-1)\cdot(n-m-1)^2 c^{s}_{|n-m-1|} + \mathrm{sign}\,(n-m+1)\cdot(n-m+1)^2 c^{s}_{|n-m+1|}$$

$$- (n+m-1)^2 c^{s}_{n+m-1} + (n+m+1)^2 c^{s}_{n+m+1} \} \qquad \text{(D-8d)}$$

$$\eta'''^{ss}_{snm} = \int_0^{2\pi} \eta'''(\theta) \sin\theta \sin n\theta \sin m\theta \, d\theta$$

$$= \frac{\pi}{4} \{ -(n-m-1)^3 c^{c}_{|n-m-1|} + (n-m+1)^3 c^{c}_{n-m+1} + (n+m-1)^3 c^{c}_{n+m-1} - (n+m+1)^3 c^{c}_{n+m+1} \}$$

(D-9a)

$$\eta'''^{sc}_{snm} = \int_0^{2\pi} \eta'''(\theta) \sin\theta \sin n\theta \cos m\theta \, d\theta$$

$$= -\frac{\pi}{4} \{ |n-m-1|^3 c^{s}_{|n-m-1|} - |n-m+1|^3 c^{s}_{|n-m+1|} + (n+m-1)^3 c^{s}_{n+m-1} - (n+m+1)^3 c^{s}_{n+m+1} \} \qquad \text{(D-9b)}$$

$$\eta'''^{cs}_{snm} = \int_0^{2\pi} \eta'''(\theta) \sin\theta \cos n\theta \sin m\theta \, d\theta$$

$$= -\frac{\pi}{4} \{ -|n-m-1|^3 c^{s}_{|n-m-1|} + |n-m+1|^3 c^{s}_{|n-m+1|} + (n+m-1)^3 c^{s}_{n+m-1} - (n+m+1)^3 c^{s}_{n+m+1} \}$$

(D-9c)

$$\eta_{snm}^{'''cc} = \int_{0}^{2\pi} \eta'''(\theta) \sin\theta \cos n\theta \cos m\theta \, d\theta$$

$$= \frac{\pi}{4} \{ - (n-m-1)^3 c^{c}_{|n-m-1|} + (n-m+1)^3 c^{c}_{|n-m+1|} - (n+m-1)^3 c^{c}_{n+m-1} + (n+m+1)^3 c^{c}_{n+m+1} \} \tag{D-9d}$$

$$\eta_{snm}^{''''ss} = \int_{0}^{2\pi} \eta''''(\theta) \sin\theta \sin n\theta \sin m\theta \, d\theta$$

$$= \frac{\pi}{4} \{ - \mathrm{sign}\,(n-m-1)\cdot(n-m-1)^4 c^{s}_{|n-m-1|} + \mathrm{sign}\,(n-m+1)\cdot(n-m+1)^4 c^{s}_{|n-m+1|}$$

$$+ (n+m-1)^4 c^{s}_{n+m-1} - (n+m+1)^4 c^{s}_{n+m+1} \} \tag{D-10a}$$

$$\eta_{snm}^{''''sc} = \int_{0}^{2\pi} \eta''''(\theta) \sin\theta \sin n\theta \cos m\theta \, d\theta$$

$$= \frac{\pi}{4} \{ (n-m-1)^4 c^{c}_{|n-m-1|} - (n-m+1)^4 c^{c}_{|n-m+1|} + (n+m-1)^4 c^{c}_{n+m-1} - (n+m+1)^4 c^{c}_{n+m+1} \} \tag{D-10b}$$

$$\eta_{snm}^{''''cs} = \int_{0}^{2\pi} \eta''''(\theta) \sin\theta \cos n\theta \sin m\theta \, d\theta$$

$$= \frac{\pi}{4} \{ - (n-m-1)^4 c^{c}_{|n-m-1|} + (n-m+1)^4 c^{c}_{|n-m+1|} + (n+m-1)^4 c^{c}_{n+m-1} - (n+m+1)^4 c^{c}_{n+m+1} \} \tag{D-10c}$$

$$\eta_{snm}^{''''cc} = \int_{0}^{2\pi} \eta''''(\theta) \sin\theta \cos n\theta \cos m\theta \, d\theta$$

$$= \frac{\pi}{4} \{ - \mathrm{sign}\,(n-m-1)\cdot(n-m-1)^4 c^{s}_{|n-m-1|} + \mathrm{sign}\,(n-m+1)\cdot(n-m+1)^4 c^{s}_{|n-m+1|}$$

$$- (n+m-1)^4 c^{s}_{n+m-1} + (n+m+1)^4 c^{s}_{n+m+1} \} \tag{D-10d}$$

$$\eta_{cnm}^{ss} = \int_{0}^{2\pi} \eta(\theta) \cos\theta \sin n\theta \sin m\theta \, d\theta$$

$$= \frac{\pi}{4} (c^{c}_{|n-m-1|} + c^{c}_{|n-m+1|} - c^{c}_{n+m-1} - c^{c}_{n+m+1}) \tag{D-11a}$$

$$\eta^{sc}_{cnm} = \int_0^{2\pi} \eta(\theta)\ \cos\theta\ \sin n\theta\ \cos m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ \mathrm{sign}\ (n-m-1)\ c^{s}_{|n-m-1|} + \mathrm{sign}\ (n-m+1)\ c^{s}_{|n-m+1|} + c^{s}_{n+m-1} + c^{s}_{n+m+1} \right\} \qquad \text{(D-11b)}$$

$$\eta^{cs}_{cnm} = \int_0^{2\pi} \eta(\theta)\ \cos\theta\ \cos n\theta\ \sin m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ -\ \mathrm{sign}\ (n-m-1)\ c^{s}_{|n-m-1|} - \mathrm{sign}\ (n-m+1)\ c^{s}_{|n-m+1|} + c^{s}_{n+m-1} + c^{s}_{n+m+1} \right\} \qquad \text{(D-11c)}$$

$$\eta^{cc}_{cnm} = \int_0^{2\pi} \eta(\theta)\ \cos\theta\ \cos n\theta\ \cos m\theta\ d\theta$$

$$= \frac{\pi}{4} \left(c^{c}_{|n-m-1|} + c^{c}_{|n-m+1|} + c^{c}_{n+m-1} + c^{c}_{n+m+1} \right) \qquad \text{(D-11d)}$$

$$\eta'^{ss}_{cnm} = \int_0^{2\pi} \eta'(\theta)\ \cos\theta\ \sin n\theta\ \sin m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ |n-m-1|\ c^{s}_{|n-m-1|} + |n-m+1|\ c^{s}_{|n-m+1|} - (n+m-1)\ c^{s}_{n+m-1} - (n+m+1)\ c^{s}_{n+m+1} \right\} \qquad \text{(D-12a)}$$

$$\eta'^{sc}_{cnm} = \int_0^{2\pi} \eta'(\theta)\ \cos\theta\ \sin n\theta\ \cos m\theta\ d\theta$$

$$= -\frac{\pi}{4} \left\{ (n-m-1)\ c^{c}_{|n-m-1|} + (n-m+1)\ c^{c}_{|n-m+1|} + (n+m-1)\ c^{c}_{n+m-1} + (n+m+1)\ c^{c}_{n+m+1} \right\}$$

$$\text{(D-12b)}$$

$$\eta'^{cs}_{cnm} = \int_0^{2\pi} \eta'(\theta)\ \cos\theta\ \cos n\theta\ \sin m\theta\ d\theta$$

$$= -\frac{\pi}{4} \left\{ -\ (n-m-1)\ c^{c}_{|n-m-1|} - (n-m+1)\ c^{c}_{|n-m+1|} + (n+m-1)\ c^{c}_{n+m-1} + (n+m+1)\ c^{c}_{n+m+1} \right\}$$

$$\text{(D-12c)}$$

$$\eta'^{cc}_{cnm} = \int_0^{2\pi} \eta'(\theta)\ \cos\theta\ \cos n\theta\ \cos m\theta\ d\theta$$

$$= \frac{\pi}{4} \left\{ |n-m-1|\ c^{s}_{|n-m-1|} + |n-m+1|\ c^{s}_{|n-m+1|} + (n+m-1)\ c^{s}_{n+m-1} + (n+m+1)\ c^{s}_{n+m+1} \right\} \qquad \text{(D-12d)}$$

$$\eta^{''ss}_{cnm} = \int_{0}^{2\pi} \eta''(\theta)\ \cos\theta\ \sin n\theta\ \sin m\theta\ d\theta$$

$$= -\frac{\pi}{4}\left\{ (n-m-1)^2\, c^{c}_{|n-m-1|} + (n-m+1)^2\, c^{c}_{|n-m+1|} - (n+m-1)^2\, c^{c}_{n+m-1} - (n+m+1)^2\, c^{c}_{n+m+1} \right\} \tag{D-13a}$$

$$\eta^{''sc}_{cnm} = \int_{0}^{2\pi} \eta''(\theta)\ \cos\theta\ \sin n\theta\ \cos m\theta\ d\theta$$

$$= -\frac{\pi}{4}\left\{ \mathrm{sign}(n-m-1)\cdot(n-m-1)^2\, c^{s}_{|n-m-1|} + \mathrm{sign}(n-m+1)\cdot(n-m+1)^2\, c^{s}_{|n-m+1|} \right.$$

$$\left. + (n+m-1)^2\, c^{s}_{n+m-1} + (n+m+1)^2\, c^{s}_{n+m+1} \right\} \tag{D-13b}$$

$$\eta^{''cs}_{cnm} = \int_{0}^{2\pi} \eta''(\theta)\ \cos\theta\ \cos n\theta\ \sin m\theta\ d\theta$$

$$= -\frac{\pi}{4}\left\{ -\,\mathrm{sign}(n-m-1)\cdot(n-m-1)^2\, c^{s}_{|n-m-1|} - \mathrm{sign}(n-m+1)\cdot(n-m+1)^2\, c^{s}_{|n-m+1|} \right.$$

$$\left. + (n+m-1)^2\, c^{s}_{n+m-1} + (n+m+1)^2\, c^{s}_{n+m+1} \right\} \tag{D-13c}$$

$$\eta^{''cc}_{cnm} = \int_{0}^{2\pi} \eta''(\theta)\ \cos\theta\ \cos n\theta\ \cos m\theta\ d\theta$$

$$= -\frac{\pi}{4}\left\{ (n-m-1)^2\, c^{c}_{|n-m-1|} + (n-m+1)^2\, c^{c}_{|n-m+1|} + (n+m-1)^2\, c^{c}_{n+m-1} + (n+m+1)^2\, c^{c}_{n+m+1} \right\} \tag{D-13d}$$

$$\eta^{'''ss}_{cnm} = \int_{0}^{2\pi} \eta'''(\theta)\ \cos\theta\ \sin n\theta\ \sin m\theta\ d\theta$$

$$= -\frac{\pi}{4}\left\{ |n-m-1|^3\, c^{s}_{|n-m-1|} + |n-m+1|^3\, c^{s}_{|n-m+1|} - (n+m-1)^3\, c^{s}_{n+m-1} - (n+m+1)^3\, c^{s}_{n+m+1} \right\} \tag{D-14a}$$

$$\eta^{'''sc}_{cnm} = \int_{0}^{2\pi} \eta'''(\theta)\ \cos\theta\ \sin n\theta\ \cos m\theta\ d\theta$$

$$= \frac{\pi}{4}\left\{ (n-m-1)^3\, c^{c}_{|n-m-1|} + (n-m+1)^3\, c^{c}_{|n-m+1|} + (n+m-1)^3\, c^{c}_{n+m-1} + (n+m+1)^3\, c^{c}_{n+m+1} \right\} \tag{D-14b}$$

$$\eta'''^{cs}_{cnm} = \int_{0}^{2\pi} \eta'''(\theta)\cos\theta\cos n\theta\sin m\theta\, d\theta$$

$$= \frac{\pi}{4}\left\{ -(n-m-1)^3 c^{c}_{|n-m-1|} - (n-m+1)^3 c^{c}_{|n-m+1|} + (n+m-1)^3 c^{c}_{n+m-1} + (n+m+1)^3 c^{c}_{n+m+1} \right\} \quad \text{(D-14c)}$$

$$\eta'''^{cc}_{cnm} = \int_{0}^{2\pi} \eta'''(\theta)\cos\theta\cos n\theta\cos m\theta\, d\theta$$

$$= -\frac{\pi}{4}\left\{ |n-m-1|^3 c^{s}_{|n-m-1|} + |n-m+1|^3 c^{s}_{|n-m+1|} + (n+m-1)^3 c^{s}_{n+m-1} + (n+m+1)^3 c^{s}_{n+m+1} \right\} \quad \text{(D-14d)}$$

$$\eta''''^{ss}_{cnm} = \int_{0}^{2\pi} \eta''''(\theta)\cos\theta\sin n\theta\sin m\theta\, d\theta$$

$$= \frac{\pi}{4}\left\{ (n-m-1)^4 c^{c}_{|n-m-1|} + (n-m+1)^4 c^{c}_{|n-m+1|} - (n+m-1)^4 c^{c}_{n+m-1} - (n+m+1)^4 c^{c}_{n+m+1} \right\} \quad \text{(D-15a)}$$

$$\eta''''^{sc}_{cnm} = \int_{0}^{2\pi} \eta''''(\theta)\cos\theta\sin n\theta\cos m\theta\, d\theta$$

$$= \frac{\pi}{4}\left\{ \mathrm{sign}(n-m-1)\cdot(n-m-1)^4 c^{s}_{|n-m-1|} + \mathrm{sign}(n-m+1)\cdot(n-m+1)^4 c^{s}_{|n-m+1|} \right.$$

$$\left. + (n+m-1)^4 c^{s}_{n+m-1} + (n+m+1)^4 c^{s}_{n+m+1} \right\} \quad \text{(D-15b)}$$

$$\eta''''^{cs}_{cnm} = \int_{0}^{2\pi} \eta''''(\theta)\cos\theta\cos n\theta\sin m\theta\, d\theta$$

$$= \frac{\pi}{4}\left\{ -\mathrm{sign}(n-m-1)\cdot(n-m-1)^4 c^{s}_{|n-m-1|} - \mathrm{sign}(n-m+1)\cdot(n-m+1)^4 c^{s}_{|n-m+1|} \right.$$

$$\left. + (n+m-1)^4 c^{s}_{n+m-1} + (n+m+1)^4 c^{s}_{n+m+1} \right\} \quad \text{(D-15c)}$$

$$\eta''''^{cc}_{cnm} = \int_{0}^{2\pi} \eta''''(\theta)\cos\theta\cos n\theta\cos m\theta\, d\theta$$

$$= \frac{\pi}{4}\left\{ (n-m-1)^4 c^{c}_{|n-m-1|} + (n-m+1)^4 c^{c}_{|n-m+1|} + (n+m-1)^4 c^{c}_{n+m-1} + (n+m+1)^4 c^{c}_{n+m+1} \right\} \quad \text{(D-15d)}$$

Appendix E : Cases of m=0 or n=0

(i) structural mass matix

(n=m=0)

$$\mathbf{M}^{s0}_{ij00} = 2\pi\rho h\,\bar{\mathrm{R}}\,\mathbf{M}^{*s0}_{(0)ij} \tag{E-1}$$

where

$$\mathbf{M}^{*s0}_{(0)ij} = \begin{bmatrix} \int_0^L \mathbf{H}^{0}_{(0)ij}\,dz & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \int_0^L \mathbf{H}^{1}_{(0)ij}\,dz & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \int_0^L \mathbf{H}^{1}_{(0)ij}\,dz \end{bmatrix} \tag{E-2}$$

and

$$\mathbf{H}^{0}_{(0)ij} = \begin{bmatrix} 0 & 0 \\ 0 & H^{I0}_{i0}(z)H^{I0}_{j0} \end{bmatrix} \tag{E-3a}$$

$$\mathbf{H}^{1}_{(0)ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & H^{I1}_{i0}(z)H^{I1}_{j0}(z) & H^{I1}_{i0}(z)H^{I1}_{j1}(z) \\ 0 & 0 & H^{I1}_{i1}(z)H^{I1}_{j0}(z) & H^{I1}_{i1}(z)H^{I1}_{j1}(z) \end{bmatrix} \tag{E-3b}$$

(ii) added mass matrix

$$\mathbf{M}^{ad0}_{ij00} = 2\pi \sum_{q=1}^{\infty} \frac{2\rho_F\bar{\mathrm{R}}\;I_0(\mu_q\bar{\mathrm{R}})}{\mu_q H\;I_0'(\mu_q\bar{\mathrm{R}})}\,\gamma^T_{iq}\,\mathbf{A}\,\gamma_{jq} \tag{E-4a}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{E-4b}$$

(iii) material stiffness matrices

$$\mathbf{K}_{ij00}^{D01} = 2\pi \int_0^L \mathbf{B}_{zi}^{10T}\, \mathbf{C}_{(0)}^{10}\, \mathbf{B}_{zj}^{10}\, dz \tag{E-5}$$

wfere

$$\mathbf{C}_{(0)}^{10} = \frac{Eh\bar{R}}{1-\bar{\nu}^2} \begin{bmatrix} \mathbf{C}_{A(0)}^{10} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{B(0)}^{10} \end{bmatrix} \tag{E-6a}$$

and

$$\mathbf{C}_{A(0)}^{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \bar{\nu} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\nu} & 0 & 0 & 0 & 1 \end{bmatrix} \tag{E-6b}$$

$$\mathbf{C}_{B(0)}^{10} = \frac{1-\bar{\nu}}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E-6c}$$

$$\mathbf{K}_{ij00}^{D02} = 2\pi \int_0^L \mathbf{B}_{zi}^{20T}\, \mathbf{C}_{(0)}^{20}\, \mathbf{B}_{zj}^{20}\, dz \tag{E-7}$$

where

$$\mathbf{C}_{(0)}^{20} = \frac{Eh^3\bar{R}}{12(1-\bar{\nu}^2)} \begin{bmatrix} \mathbf{C}_{A(0)}^{20} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{B(0)}^{20} \end{bmatrix} \tag{E-7a}$$

and

$$\mathbf{C}_{A(0)}^{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{E-7b}$$

$$\mathbf{C}^{20}_{B(0)} = \frac{1-\bar{\nu}}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{C}^{10}_{B(0)} \tag{E-7c}$$

$$\mathbf{K}^{D3}_{ij00} = -2\pi \int_0^L \mathbf{B}^{10T}_{zi}\, \mathbf{C}^{3}_{(0)}\, \mathbf{B}^{20}_{zj}\, dz \tag{E-8}$$

where

$$\mathbf{C}^{3}_{(0)} = \frac{Eh^3}{12(1-\bar{\nu}^2)}\begin{bmatrix} \mathbf{C}^{3}_{A(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{3}_{B(0)} \end{bmatrix} \tag{E-9a}$$

and

$$\mathbf{C}^{3}_{A(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\nu} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{E-9b}$$

$$\mathbf{C}^{3}_{B(0)} = \frac{1-\bar{\nu}}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E-9c}$$

(iv) geometric stiffness matrices

(m=n=0)

$$\mathbf{A}^{01}_{100} = \bar{R}\int_0^{2\pi} \mathbf{B}^{G0}_{100}(\theta)d\theta = 2\pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{E-10a}$$

$$\mathbf{A}^{01}_{200} = \bar{R}\int_0^{2\pi} \mathbf{B}^{G0}_{200}(\theta)d\theta = 2\pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{E-10b}$$

$$\mathbf{A}^{01}_{300} = \bar{R}\int_{0}^{2\pi} \mathbf{B}^{G0}_{300}(\theta)d\theta = 2\pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{E-10c}$$

(n=1,m=0)

$$\mathbf{A}^{0s}_{110} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}^{G0}_{110}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{E-11a}$$

$$\mathbf{A}^{0s}_{210} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}^{G0}_{210}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{E-11b}$$

$$\mathbf{A}^{0s}_{310} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}^{G0}_{310}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{E-11c}$$

(n=0,m=1)

$$\mathbf{A}^{0s}_{101} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}^{G0}_{101}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{E-11d}$$

$$\mathbf{A}^{0s}_{201} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}^{G0}_{201}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \tag{E-11e}$$

$$\mathbf{A}_{301}^{0s} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}_{301}^{G0}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \qquad \text{(E-11f)}$$

(n=1,m=0)

$$\mathbf{A}_{110}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta\cdot\mathbf{B}_{110}^{G0}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(E-12a)}$$

$$\mathbf{A}_{210}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta\cdot\mathbf{B}_{210}^{G0}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(E-12b)}$$

$$\mathbf{A}_{310}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta\cdot\mathbf{B}_{310}^{G0}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \qquad \text{(E-12c)}$$

(n=0,m=1)

$$\mathbf{A}_{101}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta\cdot\mathbf{B}_{101}^{G0}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(E-12d)}$$

$$\mathbf{A}_{201}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta\cdot\mathbf{B}_{201}^{G0}(\theta)d\theta = \pi\bar{R}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(E-12e)}$$

$$\mathbf{A}^{0c}_{301} = \bar{R} \int_{0}^{2\pi} \cos\theta \cdot \mathbf{B}^{G0}_{301}(\theta) d\theta = \pi \bar{R} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \qquad \text{(E-12f)}$$

(v) load correction matrices

(m=n=0)

$$\mathbf{C}^{01}_{00} = 2\pi \left[\begin{array}{ccccc|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \frac{2}{\bar{R}}\mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{0}_{10\times4} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \\ -\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \hline \multicolumn{5}{c|}{\mathbf{0}_{4\times10}} & \mathbf{0}_{4\times4} \end{array} \right] \qquad \text{(E-13a)}$$

where $\mathbf{J} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (E-13b)

(m=0,n=1)

$$\mathbf{C}^{0c}_{10} = \pi \left[\begin{array}{ccccc|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \frac{2}{\bar{R}}\mathbf{J} & \mathbf{J} & \mathbf{0} & \mathbf{0}_{14\times4} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \\ -\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \mathbf{J}^{*T} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & -\mathbf{J}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{array} \right] \qquad \text{(E-14a)}$$

where $\mathbf{J}^{*} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (E-14b)

(m=1,n=0)

$$\mathbf{C}^{0c}_{01} = \mathbf{C}^{0c\,T}_{10} \qquad \text{(E-14c)}$$

(m=0,n=1)

$$\mathbf{C}_{10}^{0s} = \pi \left[\begin{array}{ccccc|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J}^{*T} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{J}^{*T} & \mathbf{J}^{*T} & \mathbf{0} & \mathbf{0}_{14\times4} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}^{*T} & \mathbf{0} & \mathbf{0} & \\ -\mathbf{J}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & -\mathbf{J} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{array} \right] \qquad \text{(E-15a)}$$

and

(m=1,n=0)

$$\mathbf{C}_{01}^{0s} = \mathbf{C}_{10}^{0s^{T}} \qquad \text{(E-15b)}$$

Appendix F

$$\mathbf{C}^{10} = \frac{Eh\bar{R}}{1-\bar{v}^2}\begin{bmatrix} \mathbf{C}_A^{10} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_B^{10} \end{bmatrix} \tag{F-1a}$$

where

$$\underset{6\times 6}{\mathbf{C}_A^{10}} = \begin{bmatrix} 1 & 0 & 0 & -\bar{v}m & \bar{v} & 0 \\ 0 & 1 & \bar{v}m & 0 & 0 & \bar{v} \\ 0 & \bar{v}n & nm & 0 & 0 & n \\ -\bar{v}n & 0 & 0 & nm & -n & 0 \\ \bar{v} & 0 & 0 & -m & 1 & 0 \\ 0 & \bar{v} & m & 0 & 0 & 1 \end{bmatrix} \tag{F-1b}$$

$$\underset{4\times 4}{\mathbf{C}_B^{10}} = \begin{bmatrix} \frac{1-\bar{v}}{2}nm & 0 & 0 & \frac{1-\bar{v}}{2}n \\ 0 & \frac{1-\bar{v}}{2}nm & -\frac{1-\bar{v}}{2}n & 0 \\ 0 & -\frac{1-\bar{v}}{2}m & \frac{1-\bar{v}}{2} & 0 \\ \frac{1-\bar{v}}{2}m & 0 & 0 & \frac{1-\bar{v}}{2} \end{bmatrix} \tag{F-1c}$$

$$\underset{10\times 10}{\mathbf{C}_{nm}^{110}} = \frac{Eh\bar{R}}{1-\bar{v}^2}\begin{bmatrix} \mathbf{C}_{Anm}^{110} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{Bnm}^{110} \end{bmatrix} \tag{F-2a}$$

where

$$\underset{6\times 6}{\mathbf{C}_{Anm}^{110}} = \begin{bmatrix} (\eta_1)_{nm} & \bar{v}m(\eta_3)_{nm} & \bar{v}(\eta_1)_{nm} \\ \bar{v}n(\eta_3)_{mn}^T & nm(\eta_2)_{nm} & n(\eta_3)_{mn}^T \\ \bar{v}(\eta_1)_{nm} & m(\eta_3)_{nm} & (\eta_1)_{nm} \end{bmatrix} \tag{F-2b}$$

$$\mathbf{C}_{Bnm}^{110} = \frac{1-\bar{v}}{2}\begin{bmatrix} nm(\eta_2)_{nm} & n(\eta_3)_{mn}^T \\ m(\eta_3)_{nm} & (\eta_1)_{nm} \end{bmatrix} \tag{F-2c}$$

* $(\eta_1)_{nm}$, $(\eta_2)_{nm}$, $(\eta_3)_{nm}$, : see Appendix G.

$$\underset{10\times6}{\mathbf{C}^{111}_{nm}} = \frac{Eh\bar{R}}{1-\bar{\nu}^2}\begin{bmatrix} \mathbf{0} & \mathbf{C}^{111}_{Cnm} \\ \mathbf{C}^{111}_{Dnm} & \mathbf{0} \end{bmatrix} \tag{F-3a}$$

where

$$\underset{6\times4}{\mathbf{C}^{111}_{Cnm}} = \begin{bmatrix} \bar{\nu}m(\eta_3)_{nm} & \bar{\nu}\{(\eta_1)_{nm}+(\eta''_1)_{nm}\} \\ nm(\eta_2)_{nm} & n\{(\eta_3)^T_{mn}+(\eta''_3)^T_{mn}\} \\ m(\eta_3)_{nm} & \{(\eta_1)_{nm}+(\eta''_1)_{nm}\} \end{bmatrix} \tag{F-3b}$$

$$\underset{4\times2}{\mathbf{C}^{111}_{Dnm}} = \begin{bmatrix} \frac{1-\bar{\nu}}{2}nm(\eta_2)_{nm} \\ \\ \frac{1-\bar{\nu}}{2}m\ (\eta_3)_{nm} \end{bmatrix} \tag{F-3c}$$

* $(\eta_1)_{nm}$, $(\eta_2)_{nm}$, $(\eta_3)_{nm}$, $(\eta''_1)_{nm}$, $(\eta''_3)_{nm}$: see Appendix G.

$$\underset{10\times10}{\mathbf{C}^{20}} = \frac{Eh^3\bar{R}}{12(1-\bar{\nu}^2)}\begin{bmatrix} \mathbf{C}^{20}_{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{20}_{B} \end{bmatrix} \tag{F-4a}$$

where

$$\underset{6\times6}{\mathbf{C}^{20}_{A}} = \begin{bmatrix} 1 & 0 & -\bar{\nu}m^2 & 0 & 0 & -\bar{\nu}m \\ 0 & 1 & 0 & -\bar{\nu}m^2 & \bar{\nu}m & 0 \\ -\bar{\nu}n^2 & 0 & n^2m^2 & 0 & 0 & n^2m \\ 0 & -\bar{\nu}n^2 & 0 & n^2m^2 & -n^2m & 0 \\ 0 & \bar{\nu}n & 0 & -nm^2 & nm & 0 \\ -\bar{\nu}n & 0 & nm^2 & 0 & 0 & nm \end{bmatrix} \tag{F-4b}$$

$$\underset{4\times 4}{\mathbf{C}_{B}^{20}} = \begin{bmatrix} \frac{1-\bar{v}}{2}nm & 0 & 0 & \frac{1-\bar{v}}{2}n \\ 0 & \frac{1-\bar{v}}{2}nm & -\frac{1-\bar{v}}{2}n & 0 \\ 0 & -\frac{1-\bar{v}}{2}m & \frac{1-\bar{v}}{2} & 0 \\ \frac{1-\bar{v}}{2}m & 0 & 0 & \frac{1-\bar{v}}{2} \end{bmatrix} \tag{F-4c}$$

$$\underset{10\times 10}{\mathbf{C}_{nm}^{210}} = \frac{Eh^3\bar{R}}{12(1-\bar{v}^2)} \begin{bmatrix} \mathbf{C}_{Anm}^{210} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{Bnm}^{210} \end{bmatrix} \tag{F-5a}$$

where

$$\underset{6\times 6}{\mathbf{C}_{Anm}^{210}} = \begin{bmatrix} (\eta_1)_{nm} & -\bar{v}m^2(\eta_1)_{nm} & \bar{v}m(\eta_3)_{nm} \\ -\bar{v}n^2(\eta_1)_{nm} & n^2m^2(\eta_1)_{nm} & -n^2m(\eta_3)_{nm} \\ \bar{v}n(\eta_3)_{mn}^{T} & -nm^2(\eta_3)_{mn}^{T} & nm(\eta_2)_{nm} \end{bmatrix} \tag{F-5b}$$

$$\underset{4\times 4}{\mathbf{C}_{Bnm}^{210}} = \frac{1-\bar{v}}{2} \begin{bmatrix} nm(\eta_2)_{nm} & n(\eta_3)_{mn}^{T} \\ m(\eta_3)_{nm} & (\eta_1)_{nm} \end{bmatrix} \tag{F-5c}$$

$*$ $(\eta_\alpha)_{nm}$ $(\alpha=1,3)$: see Appendix G.

$$\underset{10\times 8}{\mathbf{C}_{nm}^{211}} = \frac{Eh^3\bar{R}}{12(1-\bar{v}^2)} \begin{bmatrix} \mathbf{C}_{Anm}^{211} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{Bnm}^{211} \end{bmatrix} \tag{F-6a}$$

where

$$\underset{6\times 4}{\mathbf{C}_{Anm}^{211}} = \begin{bmatrix} \bar{v}m\{2(\eta_3)_{nm}+(\eta_3'')_{nm}\} & -\bar{v}m\{2m(\eta_1)_{nm}-(\eta_3')_{nm}\} \\ -n^2m\{2(\eta_3)_{nm}+(\eta_3'')_{nm}\} & n^2m\{2m(\eta_1)_{nm}-(\eta_3')_{nm}\} \\ nm\{2(\eta_2)_{nm}+(\eta_2'')_{nm}\} & -nm\{2m(\eta_3)_{mn}^{T}-(\eta_2')_{nm}\} \end{bmatrix} \tag{F-6b}$$

$$\underset{4\times 4}{\mathbf{C}_{Bnm}^{211}} = \frac{1-\bar{v}}{2} \begin{bmatrix} n\{(\eta_3)_{mn}^{T}+(\eta_3'')_{mn}^{T}\} & nm(\eta_2)_{nm} \\ (\eta_1)_{nm}+(\eta_1'')_{nm} & m(\eta_3)_{nm} \end{bmatrix} \tag{F-6c}$$

$*$ $(\eta_\alpha)_{nm}$, $(\eta_\alpha')_{nm}$, $(\eta_\alpha'')_{nm}$, $(\alpha=1,3)$: see Appendix G.

$$\underset{10\times 10}{\mathbf{C}^3} = \frac{Eh^3}{12(1-\bar{\nu}^2)} \begin{bmatrix} \mathbf{C}_A^3 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_B^3 \end{bmatrix} \tag{F-7a}$$

where

$$\mathbf{C}_A^3 = \begin{bmatrix} 1 & 0 & -\bar{\nu}m^2 & 0 & 0 & -\bar{\nu}m \\ 0 & 1 & 0 & -\bar{\nu}m^2 & \bar{\nu}m & 0 \\ 0 & \bar{\nu}n & 0 & -nm^2 & nm & 0 \\ -\bar{\nu}n & 0 & nm^2 & 0 & 0 & nm \\ \bar{\nu} & 0 & -m^2 & 0 & 0 & -m \\ 0 & \bar{\nu} & 0 & -m^2 & m & 0 \end{bmatrix} \tag{F-7b}$$

$$\mathbf{C}_B^3 = \frac{1-\bar{\nu}}{2} \begin{bmatrix} nm & 0 & 0 & n \\ 0 & nm & -n & 0 \\ 0 & -m & 1 & 0 \\ m & 0 & 0 & 1 \end{bmatrix} \tag{F-7c}$$

Appendix G

$$(\eta_1)_{nm}=\begin{bmatrix}\eta^{ss}_{nm} & \eta^{sc}_{nm}\\ \eta^{cs}_{nm} & \eta^{cc}_{nm}\end{bmatrix}=\frac{\pi}{2}\begin{bmatrix}c^{c}_{|n-m|}-c^{c}_{n+m} & :\ \mathrm{sign}(n-m)c^{s}_{|n-m|}+c^{s}_{n+m}\\ \mathrm{sign}(m-n)c^{s}_{|n-m|}+c^{s}_{n+m}\ : & c^{c}_{|n-m|}+c^{c}_{n+m}\end{bmatrix} \tag{G-1a}$$

$$(\eta_2)_{nm}=\begin{bmatrix}\eta^{cc}_{nm} & -\eta^{cs}_{nm}\\ -\eta^{sc}_{nm} & \eta^{ss}_{nm}\end{bmatrix}=\frac{\pi}{2}\begin{bmatrix}c^{c}_{|n-m|}+c^{c}_{n+m} & :\ \mathrm{sign}(n-m)c^{s}_{|n-m|}-c^{s}_{n+m}\\ \mathrm{sign}(m-n)c^{s}_{|n-m|}-c^{s}_{n+m}\ : & c^{c}_{|n-m|}-c^{c}_{n+m}\end{bmatrix} \tag{G-1b}$$

$$(\eta_3)_{nm}=\begin{bmatrix}\eta^{sc}_{nm} & -\eta^{ss}_{nm}\\ \eta^{cc}_{nm} & -\eta^{cs}_{nm}\end{bmatrix}=\frac{\pi}{2}\begin{bmatrix}\mathrm{sign}(n-m)c^{s}_{|n-m|}+c^{s}_{n+m}\ : & -c^{c}_{|n-m|}+c^{c}_{n+m}\\ c^{c}_{|n-m|}+c^{c}_{n+m} & :\ \mathrm{sign}(n-m)c^{s}_{|n-m|}-c^{s}_{n+m}\end{bmatrix} \tag{G-1c}$$

$$(\ (\eta_1)^{T}_{mn}=(\eta_1)_{nm},\ (\eta_2)^{T}_{mn}=(\eta_2)_{nm} \qquad (\text{"T"},\ m\leftrightarrow n)\ )$$

$$(\eta_1')_{nm}=\begin{bmatrix}\eta'^{ss}_{nm} & \eta'^{sc}_{nm}\\ \eta'^{cs}_{nm} & \eta'^{cc}_{nm}\end{bmatrix}=\frac{\pi}{2}\begin{bmatrix}|n-m|c^{s}_{|n-m|}-(n+m)c^{s}_{n+m}\ : & (m-n)c^{c}_{|n-m|}-(n+m)c^{c}_{n+m}\\ (n-m)c^{c}_{|n-m|}-(n+m)c^{c}_{n+m} & :\ |n-m|c^{s}_{|n-m|}+(n+m)c^{s}_{n+m}\end{bmatrix} \tag{G-2a}$$

$$(\eta_2')_{nm}=\begin{bmatrix}\eta'^{cc}_{nm} & -\eta'^{cs}_{nm}\\ -\eta'^{sc}_{nm} & \eta'^{ss}_{nm}\end{bmatrix}=\frac{\pi}{2}\begin{bmatrix}|n-m|c^{s}_{|n-m|}+(n+m)c^{s}_{n+m}\ : & (m-n)c^{c}_{|n-m|}+(n+m)c^{c}_{n+m}\\ (n-m)c^{c}_{|n-m|}+(n+m)c^{c}_{n+m} & :\ |n-m|c^{s}_{|n-m|}-(n+m)c^{s}_{n+m}\end{bmatrix} \tag{G-2b}$$

$$(\eta_3')_{nm}=\begin{bmatrix}\eta'^{sc}_{nm} & -\eta'^{ss}_{nm}\\ \eta'^{cc}_{nm} & -\eta'^{cs}_{nm}\end{bmatrix}=\frac{\pi}{2}\begin{bmatrix}(m-n)c^{c}_{|n-m|}-(n+m)c^{c}_{n+m}\ : & -|n-m|c^{s}_{|n-m|}+(n+m)c^{s}_{n+m}\\ |n-m|c^{s}_{|n-m|}+(n+m)c^{s}_{n+m} & :\ (m-n)c^{c}_{|n-m|}+(n+m)c^{c}_{n+m}\end{bmatrix} \tag{G-2c}$$

$$(\ (\eta_1')^{T}_{mn}=(\eta_1')_{nm},\ (\eta_2')^{T}_{mn}=(\eta_2')_{nm}\ )$$

$$(\eta_1'')_{nm} = \begin{bmatrix} \eta_{nm}^{\prime\prime ss} & \eta_{nm}^{\prime\prime sc} \\ \eta_{nm}^{\prime\prime cs} & \eta_{nm}^{\prime\prime cc} \end{bmatrix}$$

$$= -\frac{\pi}{2}\begin{bmatrix} (n-m)^2 c_{|n-m|}^{c} - (n+m)^2 c_{n+m}^{c} & : \mathrm{sign}(n-m)\cdot(n-m)^2 c_{|n-m|}^{s} + (n+m)^2 c_{n+m}^{s} \\ \mathrm{sign}(m-n)\cdot(n-m)^2 c_{|n-m|}^{s} + (n+m)^2 c_{n+m}^{s} : & (n-m)^2 c_{|n-m|}^{c} + (n+m)^2 c_{n+m}^{c} \end{bmatrix} \quad \text{(G-3a)}$$

$$(\eta_2'')_{nm} = \begin{bmatrix} \eta_{nm}^{\prime\prime cc} & -\eta_{nm}^{\prime\prime cs} \\ -\eta_{nm}^{\prime\prime sc} & \eta_{nm}^{\prime\prime ss} \end{bmatrix}$$

$$= -\frac{\pi}{2}\begin{bmatrix} (n-m)^2 c_{|n-m|}^{c} + (n+m)^2 c_{n+m}^{c} & : \mathrm{sign}(n-m)\cdot(n-m)^2 c_{|n-m|}^{s} - (n+m)^2 c_{n+m}^{s} \\ \mathrm{sign}(m-n)\cdot(n-m)^2 c_{|n-m|}^{s} - (n+m)^2 c_{n+m}^{s} : & (n-m)^2 c_{|n-m|}^{c} - (n+m)^2 c_{n+m}^{c} \end{bmatrix} \quad \text{(G-3b)}$$

$$(\eta_3'')_{nm} = \begin{bmatrix} \eta_{nm}^{\prime\prime sc} & -\eta_{nm}^{\prime\prime ss} \\ \eta_{nm}^{\prime\prime cc} & -\eta_{nm}^{\prime\prime cs} \end{bmatrix}$$

$$= -\frac{\pi}{2}\begin{bmatrix} \mathrm{sign}(n-m)\cdot(n-m)^2 c_{|n-m|}^{s} + (n+m)^2 c_{n+m}^{s} : & -(n-m)^2 c_{|n-m|}^{c} + (n+m)^2 c_{n+m}^{c} \\ (n-m)^2 c_{|n-m|}^{c} + (n+m)^2 c_{n+m}^{c} & : \mathrm{sign}(n-m)\cdot(n-m)^2 c_{|n-m|}^{s} - (n+m)^2 c_{n+m}^{s} \end{bmatrix} \quad \text{(G-3c)}$$

$$\left((\eta_1'')_{mn}^{T} = (\eta_1'')_{nm} \, , (\eta_2'')_{mn}^{T} = (\eta_2'')_{nm} \right)$$

$$(\eta_1''')_{nm} = \begin{bmatrix} \eta_{nm}^{\prime\prime\prime ss} & \eta_{nm}^{\prime\prime\prime sc} \\ \eta_{nm}^{\prime\prime\prime cs} & \eta_{nm}^{\prime\prime\prime cc} \end{bmatrix} = -\frac{\pi}{2}\begin{bmatrix} |n-m|^3 c_{|n-m|}^{s} - (n+m)^3 c_{n+m}^{s} & : (m-n)^3 c_{|n-m|}^{c} - (n+m)^3 c_{n+m}^{c} \\ (n-m)^3 c_{|n-m|}^{c} - (n+m)^3 c_{n+m}^{c} & : |n-m|^3 c_{|n-m|}^{s} + (n+m)^3 c_{n+m}^{s} \end{bmatrix} \quad \text{(G-4a)}$$

$$(\eta_2''')_{nm} = \begin{bmatrix} \eta_{nm}^{\prime\prime\prime cc} & -\eta_{nm}^{\prime\prime\prime cs} \\ -\eta_{nm}^{\prime\prime\prime sc} & \eta_{nm}^{\prime\prime\prime ss} \end{bmatrix} = -\frac{\pi}{2}\begin{bmatrix} |n-m|^3 c_{|n-m|}^{s} + (n+m)^3 c_{n+m}^{s} & : (m-n)^3 c_{|n-m|}^{c} + (n+m)^3 c_{n+m}^{c} \\ (n-m)^3 c_{|n-m|}^{c} + (n+m)^3 c_{n+m}^{c} & : |n-m|^3 c_{|n-m|}^{s} - (n+m)^3 c_{n+m}^{s} \end{bmatrix} \quad \text{(G-4b)}$$

$$(\eta_3''')_{nm} = \begin{bmatrix} \eta'''^{sc}_{nm} & -\eta'''^{ss}_{nm} \\ \eta'''^{cc}_{nm} & -\eta'''^{cs}_{nm} \end{bmatrix} = -\frac{\pi}{2}\begin{bmatrix} (m-n)^3 c^{c}_{|n-m|} - (n+m)^3 c^{c}_{n+m} & : -|n-m|^3 c^{s}_{|n-m|} + (n+m)^3 c^{s}_{n+m} \\ |n-m|^3 c^{s}_{|n-m|} + (n+m)^3 c^{s}_{n+m} & : (m-n)^3 c^{c}_{|n-m|} + (n+m)^3 c^{c}_{n+m} \end{bmatrix}$$

(G-4c)

$$(\,(\eta_1''')^T_{mn} = (\eta_1''')_{nm}\,,\,(\eta_2''')^T_{mn} = (\eta_2''')_{nm}\,)$$

$$(\eta_1'''')_{nm} = \begin{bmatrix} \eta''''^{ss}_{nm} & \eta''''^{sc}_{nm} \\ \eta''''^{cs}_{nm} & \eta''''^{cc}_{nm} \end{bmatrix}$$

$$= \frac{\pi}{2}\begin{bmatrix} (n-m)^4 c^{c}_{|n-m|} - (n+m)^4 c^{c}_{n+m} & : \mathrm{sign}(n-m)\cdot(n-m)^4 c^{s}_{|n-m|} + (n+m)^4 c^{s}_{n+m} \\ \mathrm{sign}(m-n)\cdot(n-m)^4 c^{s}_{|n-m|} + (n+m)^4 c^{s}_{n+m} & : (n-m)^4 c^{c}_{|n-m|} + (n+m)^4 c^{c}_{n+m} \end{bmatrix}$$

(G-5a)

$$(\eta_2'''')_{nm} = \begin{bmatrix} \eta''''^{cc}_{nm} & -\eta''''^{cs}_{nm} \\ -\eta''''^{sc}_{nm} & \eta''''^{ss}_{nm} \end{bmatrix}$$

$$= \frac{\pi}{2}\begin{bmatrix} (n-m)^4 c^{c}_{|n-m|} + (n+m)^4 c^{c}_{n+m} & : \mathrm{sign}(n-m)\cdot(n-m)^4 c^{s}_{|n-m|} - (n+m)^4 c^{s}_{n+m} \\ \mathrm{sign}(m-n)\cdot(n-m)^4 c^{s}_{|n-m|} - (n+m)^4 c^{s}_{n+m} & : (n-m)^4 c^{c}_{|n-m|} - (n+m)^4 c^{c}_{n+m} \end{bmatrix}$$

(G-5b)

$$(\eta_3'''')_{nm} = \begin{bmatrix} \eta''''^{sc}_{nm} & -\eta''''^{ss}_{nm} \\ \eta''''^{cc}_{nm} & -\eta''''^{cs}_{nm} \end{bmatrix}$$

$$= \frac{\pi}{2}\begin{bmatrix} \mathrm{sign}(n-m)\cdot(n-m)^4 c^{s}_{|n-m|} + (n+m)^4 c^{s}_{n+m} & : -(n-m)^4 c^{c}_{|n-m|} + (n+m)^4 c^{c}_{n+m} \\ (n-m)^4 c^{c}_{|n-m|} + (n+m)^4 c^{c}_{n+m} & : \mathrm{sign}(n-m)\cdot(n-m)^4 c^{s}_{|n-m|} - (n+m)^4 c^{s}_{n+m} \end{bmatrix}$$

(G-5c)

$$(\,(\eta_1'''')^T_{mn} = (\eta_1'''')_{nm}\,,\,(\eta_2'''')^T_{mn} = (\eta_2'''')_{nm}\,)$$

The details of the components of these matrices are given in Appendix D (eqns. (D-1)~(D-5)).

$$(\eta_{1s}^{(\alpha)})_{nm}=\begin{bmatrix}\eta_{snm}^{(\alpha)ss} & \eta_{snm}^{(\alpha)sc}\\ \eta_{snm}^{(\alpha)cs} & \eta_{snm}^{(\alpha)cc}\end{bmatrix},\ (\eta_{2s}^{(\alpha)})_{nm}=\begin{bmatrix}\eta_{snm}^{(\alpha)cc} & -\eta_{snm}^{(\alpha)cs}\\ -\eta_{snm}^{(\alpha)sc} & \eta_{snm}^{(\alpha)ss}\end{bmatrix},\ (\eta_{3s}^{(\alpha)})_{nm}=\begin{bmatrix}\eta_{snm}^{(\alpha)sc} & -\eta_{snm}^{(\alpha)ss}\\ \eta_{snm}^{(\alpha)cc} & -\eta_{snm}^{(\alpha)cs}\end{bmatrix}$$

$$(\,(\eta_{1s}^{(\alpha)})_{mn}^{T}=(\eta_{1s}^{(\alpha)})_{nm},\ (\eta_{2s}^{(\alpha)})_{mn}^{T}=(\eta_{2s}^{(\alpha)})_{nm}\,) \qquad \text{(G-6a,b,c)}$$

$$(\eta_{1c}^{(\alpha)})_{nm}=\begin{bmatrix}\eta_{cnm}^{(\alpha)ss} & \eta_{cnm}^{(\alpha)sc}\\ \eta_{cnm}^{(\alpha)cs} & \eta_{cnm}^{(\alpha)cc}\end{bmatrix},\ (\eta_{2c}^{(\alpha)})_{nm}=\begin{bmatrix}\eta_{cnm}^{(\alpha)cc} & -\eta_{cnm}^{(\alpha)cs}\\ -\eta_{cnm}^{(\alpha)sc} & \eta_{cnm}^{(\alpha)ss}\end{bmatrix},\ (\eta_{3c}^{(\alpha)})_{nm}=\begin{bmatrix}\eta_{cnm}^{(\alpha)sc} & -\eta_{cnm}^{(\alpha)ss}\\ \eta_{cnm}^{(\alpha)cc} & -\eta_{cnm}^{(\alpha)cs}\end{bmatrix}$$

$$(\,(\eta_{1c}^{(\alpha)})_{mn}^{T}=(\eta_{1c}^{(\alpha)})_{nm},\ (\eta_{2c}^{(\alpha)})_{mn}^{T}=(\eta_{2c}^{(\alpha)})_{nm}\,) \qquad \text{(G-7a,b,c)}$$

where “α” represents the number of times of defferentiation, i, e, “ ”, “,”, “,,”, “,,,” and “,,,,”.

The details of the components of eqns. (G-6) and (G-7) are given in Appendix D (eqns. (D-6) ~ (D-15)).

Appendix H

$$\mathbf{B}_{1nm}^{G0}(\theta)=\begin{bmatrix} \sin n\theta\,\sin m\theta & \sin n\theta\,\cos m\theta & 0 & 0 & 0 & 0 \\ \cos n\theta\,\sin m\theta & \cos n\theta\,\cos m\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin n\theta\,\sin m\theta & \sin n\theta\,\cos m\theta & 0 & 0 \\ 0 & 0 & \cos n\theta\,\sin m\theta & \cos n\theta\,\cos m\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin n\theta\,\sin m\theta & \sin n\theta\,\cos m\theta \\ 0 & 0 & 0 & 0 & \cos n\theta\,\sin m\theta & \cos n\theta\,\cos m\theta \end{bmatrix}$$

(H-1a)

$$\mathbf{B}_{2nm}^{G0}(\theta)=\left[\begin{array}{cccc} nm\cos n\theta\cos m\theta & -nm\cos n\theta\sin m\theta & 0 & 0 \\ -nm\sin n\theta\cos m\theta & nm\sin n\theta\sin m\theta & 0 & 0 \\ 0 & 0 & nm\cos n\theta\cos m\theta+\sin n\theta\sin m\theta & -nm\cos n\theta\sin m\theta+\sin n\theta\cos m\theta \\ 0 & 0 & -nm\sin n\theta\cos m\theta+\cos n\theta\sin m\theta & nm\sin n\theta\sin m\theta+\cos n\theta\cos m\theta \\ 0 & 0 & m\sin n\theta\cos m\theta-n\cos n\theta\sin m\theta & -m\sin n\theta\sin m\theta-n\cos n\theta\cos m\theta \\ 0 & 0 & m\cos n\theta\cos m\theta+n\sin n\theta\sin m\theta & -m\cos n\theta\sin m\theta+n\sin n\theta\cos m\theta \end{array}\right.$$

$$\left.\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ n\cos n\theta\sin m\theta-m\sin n\theta\cos m\theta & n\cos n\theta\cos m\theta+m\sin n\theta\sin m\theta \\ -n\sin n\theta\sin m\theta-m\cos n\theta\cos m\theta & -n\sin n\theta\cos m\theta+m\cos n\theta\sin m\theta \\ \sin n\theta\sin m\theta+nm\cos n\theta\cos m\theta & \sin n\theta\cos m\theta-nm\cos n\theta\sin m\theta \\ \cos n\theta\sin m\theta-nm\sin n\theta\cos m\theta & \cos n\theta\cos m\theta+nm\sin n\theta\sin m\theta \end{array}\right]$$

(H-1b)

$$\mathbf{B}_{3nm}^{G0}(\theta)=\begin{bmatrix} m\sin n\theta\cos m\theta & -m\sin n\theta\sin m\theta & 0 & 0 & 0 & 0 \\ m\cos n\theta\cos m\theta & -m\cos n\theta\sin m\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & m\sin n\theta\cos m\theta & -m\sin n\theta\sin m\theta & \sin n\theta\sin m\theta & \sin n\theta\cos m\theta \\ 0 & 0 & m\cos n\theta\cos m\theta & -m\cos n\theta\sin m\theta & \cos n\theta\sin m\theta & \cos n\theta\cos m\theta \\ 0 & 0 & -\sin n\theta\sin m\theta & -\sin n\theta\cos m\theta & m\sin n\theta\cos m\theta & -m\sin n\theta\sin m\theta \\ 0 & 0 & -\cos n\theta\sin m\theta & -\cos n\theta\cos m\theta & m\cos n\theta\cos m\theta & -m\cos n\theta\sin m\theta \end{bmatrix}$$

(H-1c)

$$\mathbf{B}_{1nm}^{G1}(\theta)=\begin{bmatrix} \eta m\sin n\theta\cos m\theta & -\eta m\sin n\theta\sin m\theta \\ \eta m\cos n\theta\cos m\theta & -\eta m\cos n\theta\sin m\theta \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(H-2a)}$$

$$\mathbf{B}_{2nm}^{G0}(\theta)=\left[\begin{matrix} 0 & 0 \\ 0 & 0 \\ \eta nm\cos n\theta\cos m\theta+(\eta+\eta'')\sin n\theta\sin m\theta & -\eta nm\cos n\theta\sin m\theta+(\eta+\eta'')\sin n\theta\cos m\theta \\ -\eta nm\sin n\theta\cos m\theta+(\eta+\eta'')\cos n\theta\sin m\theta & \eta nm\sin n\theta\sin m\theta+(\eta+\eta'')\cos n\theta\cos m\theta \\ \eta m\sin n\theta\cos m\theta-(\eta+\eta'')n\cos n\theta\sin m\theta & -\eta m\sin n\theta\sin m\theta-(\eta+\eta'')n\cos n\theta\cos m\theta \\ \eta m\cos n\theta\cos m\theta+(\eta+\eta'')n\sin n\theta\sin m\theta & -\eta m\cos n\theta\sin m\theta+(\eta+\eta'')n\sin n\theta\cos m\theta \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 \\ 0 & 0 \\ (\eta+\eta'')n\cos n\theta\sin m\theta-\eta m\sin n\theta\cos m\theta & (\eta+\eta'')n\cos n\theta\cos m\theta+\eta m\sin n\theta\sin m\theta \\ -(\eta+\eta'')n\sin n\theta\sin m\theta-\eta m\cos n\theta\cos m\theta & -(\eta+\eta'')n\sin n\theta\cos m\theta+\eta m\cos n\theta\sin m\theta \\ (\eta+\eta'')\sin n\theta\sin m\theta+\eta nm\cos n\theta\cos m\theta & (\eta+\eta'')\sin n\theta\cos m\theta-\eta nm\cos n\theta\sin m\theta \\ (\eta+\eta'')\cos n\theta\sin m\theta-\eta nm\sin n\theta\cos m\theta & (\eta+\eta'')\cos n\theta\cos m\theta+\eta nm\sin n\theta\sin m\theta \end{matrix}\right]$$

$$\text{(H-2b)}$$

$$\mathbf{B}_{3nm}^{G1}(\theta)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta m\sin n\theta\cos m\theta & -\eta m\sin n\theta\sin m\theta & (\eta+\eta'')\sin n\theta\sin m\theta & (\eta+\eta'')\sin n\theta\cos m\theta \\ \eta m\cos n\theta\cos m\theta & -\eta m\cos n\theta\sin m\theta & (\eta+\eta'')\cos n\theta\sin m\theta & (\eta+\eta'')\cos n\theta\cos m\theta \\ -(\eta+\eta'')\sin n\theta\sin m\theta & -(\eta+\eta'')\sin n\theta\cos m\theta & \eta m\sin n\theta\cos m\theta & -\eta m\sin n\theta\sin m\theta \\ -(\eta+\eta'')\cos n\theta\sin m\theta & -(\eta+\eta'')\cos n\theta\cos m\theta & \eta m\cos n\theta\cos m\theta & -\eta m\cos n\theta\sin m\theta \end{bmatrix}$$

$$\text{(H-2c)}$$

$$\mathbf{B}_{4nm}^{G1}(\theta)=\begin{bmatrix} \eta nm\cos n\theta\cos m\theta & -\eta nm\cos n\theta\sin m\theta \\ -\eta nm\sin n\theta\cos m\theta & \eta nm\sin n\theta\sin m\theta \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(H-2d)}$$

$$\underset{14\times 14}{\mathbf{B}^{*}_{0nm}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -(\mathbf{SC}_1)_{nm} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -m(\mathbf{SC}_3)_{nm} \\ 0 & 0 & \frac{2}{R}(\mathbf{SC}_1)_{nm} & (\mathbf{SC}_1)_{nm} & 0 & m(\mathbf{SC}_3)_{nm} & 0 \\ 0 & 0 & (\mathbf{SC}_1)_{nm} & 0 & 0 & 0 & 0 \\ -(\mathbf{SC}_1)_{nm} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n(\mathbf{SC}_3)^{T}_{mn} & 0 & 0 & 0 & 0 \\ 0 & -n(\mathbf{SC}_3)^{T}_{mn} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{H-3a}$$

$$\underset{14\times 14}{\mathbf{B}^{*}_{1nm}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\eta(\mathbf{SC}_1)_{nm} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{R}\eta''(\mathbf{SC}_1)_{nm} & \eta(\mathbf{SC}_1)_{nm} & 0 & 0 & 0 \\ 0 & 0 & \eta(\mathbf{SC}_1)_{nm} & 0 & 0 & 0 & 0 \\ -\eta(\mathbf{SC}_1)_{nm} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{H-3b}$$

where

$$(\mathbf{SC}_1)_{nm} = \begin{bmatrix} \sin n\theta \sin m\theta & \sin n\theta \cos m\theta \\ \cos n\theta \sin m\theta & \cos n\theta \cos m\theta \end{bmatrix} \tag{H-4a}$$

$$(\mathbf{SC}_3)_{nm} = \begin{bmatrix} \sin n\theta \cos m\theta & -\sin n\theta \sin m\theta \\ \cos n\theta \cos m\theta & -\cos n\theta \sin m\theta \end{bmatrix} \tag{H-4b}$$

Appendix I

$$\mathbf{A}_{1nm}^{01} = \overline{R} \int_0^{2\pi} \mathbf{B}_{1nm}^{G0}(\theta)\, d\theta = \delta_{nm}\, \pi\overline{R}\, \underset{6\times6}{\mathbf{I}} \tag{I-1a}$$

$$\mathbf{A}_{2nm}^{01} = \overline{R} \int_0^{2\pi} \mathbf{B}_{2nm}^{G0}(\theta)\, d\theta = \delta_{nm}\, \pi\overline{R} \begin{bmatrix} nm & 0 & 0 & 0 & 0 & 0 \\ 0 & nm & 0 & 0 & 0 & 0 \\ 0 & 0 & nm+1 & 0 & 0 & n+m \\ 0 & 0 & 0 & nm+1 & -n-m & 0 \\ 0 & 0 & 0 & -m-n & nm+1 & 0 \\ 0 & 0 & m+n & 0 & 0 & nm+1 \end{bmatrix} \tag{I-1b}$$

$$\mathbf{A}_{3nm}^{01} = \overline{R} \int_0^{2\pi} \mathbf{B}_{3nm}^{G0}(\theta)\, d\theta = \delta_{nm}\, \pi\overline{R} \begin{bmatrix} 0 & -m & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & 1 & 0 \\ 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -m \\ 0 & 0 & 0 & -1 & m & 0 \end{bmatrix} \tag{I-1c}$$

∗ in the case of n=m=0, see Appendix E (eqns.(E-10)).

$$\mathbf{A}_{1nm}^{0s} = \overline{R} \int_0^{2\pi} \sin\theta\, \mathbf{B}_{1nm}^{G0}(\theta)\, d\theta = \frac{1}{2}\, \mathrm{sign}(n-m)\delta_{1,|n-m|}\cdot\pi\overline{R} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{I-2a}$$

$$\mathbf{A}_{2nm}^{0s} = \overline{R} \int_0^{2\pi} \sin\theta\, \mathbf{B}_{2nm}^{G0}(\theta)\, d\theta = \frac{1}{2}\, \mathrm{sign}(n-m)\delta_{1,|n-m|}\cdot\pi\overline{R} \begin{bmatrix} 0 & nm & 0 & 0 & 0 & 0 \\ -nm & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & nm+1 & -n-m & 0 \\ 0 & 0 & -nm-1 & 0 & 0 & -n-m \\ 0 & 0 & m+n & 0 & 0 & nm+1 \\ 0 & 0 & 0 & m+n & -nm-1 & 0 \end{bmatrix} \tag{I-2b}$$

$$\mathbf{A}_{3nm}^{0s} = \overline{R} \int_0^{2\pi} \sin\theta\, \mathbf{B}_{3nm}^{G0}(\theta)\, d\theta = \frac{1}{2}\mathrm{sign}(n-m)\delta_{1,|n-m|}\cdot\pi\overline{R} \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & 0 & m & -1 & 0 \\ 0 & 0 & 0 & -1 & m & 0 \\ 0 & 0 & 1 & 0 & 0 & m \end{bmatrix} \tag{I-2c}$$

∗ in the case of (m=0, n=1) or (n=0, m=1), see Appendix E (eqns. (E-11)).

$$\mathbf{A}_{1nm}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta \cdot \mathbf{B}_{1nm}^{G0}(\theta)\, d\theta = \frac{1}{2}\delta_{1,|n-m|} \cdot \pi \bar{R} \underset{6\times6}{\mathbf{I}} \qquad \text{(I-3a)}$$

$$\mathbf{A}_{2nm}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta \cdot \mathbf{B}_{2nm}^{G0}(\theta)\, d\theta = \frac{1}{2}\delta_{1,|n-m|} \cdot \pi \bar{R} \begin{bmatrix} nm & 0 & 0 & 0 & 0 & 0 \\ 0 & nm & 0 & 0 & 0 & 0 \\ 0 & 0 & nm+1 & 0 & 0 & n+m \\ 0 & 0 & 0 & nm+1 & -n-m & 0 \\ 0 & 0 & 0 & -m-n & nm+1 & 0 \\ 0 & 0 & m+n & 0 & 0 & nm+1 \end{bmatrix} \qquad \text{(I-3b)}$$

$$\mathbf{A}_{3nm}^{0c} = \bar{R}\int_{0}^{2\pi} \cos\theta \cdot \mathbf{B}_{3nm}^{G0}(\theta)\, d\theta = \frac{1}{2}\delta_{1,|n-m|} \cdot \pi \bar{R} \begin{bmatrix} 0 & -m & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & 1 & 0 \\ 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -m \\ 0 & 0 & 0 & -1 & m & 0 \end{bmatrix} \qquad \text{(I-3c)}$$

* in the case of (m=0, n=1) or (n=0, m=1), see Appendix E (eqns.(E-12)).

$$\mathbf{A}_{1nm}^{0\eta\alpha} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\, \mathbf{B}_{1nm}^{G0}(\theta)\, d\theta = \bar{R} \begin{bmatrix} (\eta_1^{\alpha})_{nm} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\eta_1^{\alpha})_{nm} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\eta_1^{\alpha})_{nm} \end{bmatrix} \qquad \text{(I-4a)}$$

$$\mathbf{A}_{2nm}^{0\eta\alpha} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\, \mathbf{B}_{2nm}^{G0}(\theta)\, d\theta = \bar{R} \begin{bmatrix} nm(\eta_2^{\alpha})_{nm} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & nm(\eta_2^{\alpha})_{nm}+(\eta_1^{\alpha})_{nm} & n(\eta_3^{\alpha})_{mn}^{T}-m(\eta_3^{\alpha})_{nm} \\ \mathbf{0} & m(\eta_3^{\alpha})_{nm}-n(\eta_3^{\alpha})_{mn}^{T} & (\eta_1^{\alpha})_{nm}+nm(\eta_2^{\alpha})_{nm} \end{bmatrix}$$

$$\text{(I-4b)}$$

$$\mathbf{A}_{3nm}^{0\eta\alpha} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\, \mathbf{B}_{3nm}^{G0}(\theta)\, d\theta = \bar{R} \begin{bmatrix} m(\eta_3^{\alpha})_{nm} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & m(\eta_3^{\alpha})_{nm} & (\eta_1^{\alpha})_{nm} \\ \mathbf{0} & -(\eta_1^{\alpha})_{nm} & m(\eta_3^{\alpha})_{nm} \end{bmatrix} \qquad \text{(I-4c)}$$

where α represents the times of differentiation, i. e. “0” to “4” or “ ”, “,”, “,,”, “,,,”, and “,,,,”.

The components of $(\eta_1^{\alpha})_{nm}$, $(\eta_2^{\alpha})_{nm}$ and $(\eta_3^{\alpha})_{nm}$ are given in Appendix G (eqns.(G-1) through (G-5)).

$$\mathbf{A}^{0\eta\alpha s}_{1nm} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\ \sin\theta\ \mathbf{B}^{G0}_{1nm}(\theta)\, d\theta = \bar{R}\begin{bmatrix} (\eta^{\alpha}_{1s})_{nm} & 0 & 0 \\ 0 & (\eta^{\alpha}_{1s})_{nm} & 0 \\ 0 & 0 & (\eta^{\alpha}_{1s})_{nm} \end{bmatrix} \tag{I-5a}$$

$$\mathbf{A}^{0\eta\alpha s}_{2nm} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\cdot\sin\theta\cdot\mathbf{B}^{G0}_{2nm}(\theta)d\theta = \bar{R}\begin{bmatrix} nm(\eta^{\alpha}_{2s})_{nm} & 0 & 0 \\ 0 & nm(\eta^{\alpha}_{2s})_{nm}+(\eta^{\alpha}_{1s})_{nm} & n(\eta^{\alpha}_{3s})^{T}_{mn}-m(\eta^{\alpha}_{3s})_{nm} \\ 0 & m(\eta^{\alpha}_{3s})_{nm}-n(\eta^{\alpha}_{3s})^{T}_{mn} & nm(\eta^{\alpha}_{2s})_{nm}+(\eta^{\alpha}_{1s})_{nm} \end{bmatrix} \tag{I-5b}$$

$$\mathbf{A}^{0\eta\alpha s}_{3nm} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\cdot\sin\theta\cdot\mathbf{B}^{G0}_{3nm}(\theta)\, d\theta = \bar{R}\begin{bmatrix} m(\eta^{\alpha}_{3s})_{nm} & 0 & 0 \\ 0 & m(\eta^{\alpha}_{3s})_{nm} & (\eta^{\alpha}_{1s})_{nm} \\ 0 & -(\eta^{\alpha}_{1s})_{nm} & m(\eta^{\alpha}_{3s})_{nm} \end{bmatrix} \tag{I-5c}$$

where α represents the times of differentiation, i. e. 0 to 4 or “ ”, “,”, “,,”, “,,,” and “,,,,”.

The components of $(\eta^{\alpha}_{1s})_{nm}$, $(\eta^{\alpha}_{2s})_{nm}$ and $(\eta^{\alpha}_{3s})_{nm}$ are given in Appendix G (eqns.(G-6)).

$$\mathbf{A}^{0\eta\alpha c}_{1nm} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\cdot\cos\theta\cdot\mathbf{B}^{G0}_{1nm}(\theta)\, d\theta = \bar{R}\begin{bmatrix} (\eta^{\alpha}_{1c})_{nm} & 0 & 0 \\ 0 & (\eta^{\alpha}_{1c})_{nm} & 0 \\ 0 & 0 & (\eta^{\alpha}_{1c})_{nm} \end{bmatrix} \tag{I-6a}$$

$$\mathbf{A}^{0\eta\alpha c}_{2nm} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\cdot\cos\theta\cdot\mathbf{B}^{G0}_{2nm}(\theta)d\theta = \bar{R}\begin{bmatrix} nm(\eta^{\alpha}_{2c})_{nm} & 0 & 0 \\ 0 & nm(\eta^{\alpha}_{2c})_{nm}+(\eta^{\alpha}_{1c})_{nm} & n(\eta^{\alpha}_{3c})^{T}_{mn}-m(\eta^{\alpha}_{3c})_{nm} \\ 0 & m(\eta^{\alpha}_{3c})_{nm}-n(\eta^{\alpha}_{3c})^{T}_{mn} & nm(\eta^{\alpha}_{2c})_{nm}+(\eta^{\alpha}_{1c})_{nm} \end{bmatrix} \tag{I-6b}$$

$$\mathbf{A}^{0\eta\alpha c}_{3nm} = \bar{R}\int_{0}^{2\pi} \eta^{(\alpha)}(\theta)\cdot\cos\theta\cdot\mathbf{B}^{G0}_{3nm}(\theta)\, d\theta = \bar{R}\begin{bmatrix} m(\eta^{\alpha}_{3c})_{nm} & 0 & 0 \\ 0 & m(\eta^{\alpha}_{3c})_{nm} & (\eta^{\alpha}_{1c})_{nm} \\ 0 & -(\eta^{\alpha}_{1c})_{nm} & m(\eta^{\alpha}_{3c})_{nm} \end{bmatrix} \tag{I-6c}$$

where α represents the times of differentiation, i. e. 0 to 4 or “ ”, “,”, “,,”, “,,,” and “,,,,”.

The components of $(\eta^{\alpha}_{1c})_{nm}$, $(\eta^{\alpha}_{2c})_{nm}$, and $(\eta^{\alpha}_{3c})_{nm}$ are given in Appendix G (eqns.(G-7)).

$$\mathbf{A}_{1nmp}^{0s} = \overline{R} \int_{0}^{2\pi} \sin p\theta \cdot \mathbf{B}_{1nm}^{G0}(\theta)\, d\theta$$

$$= \frac{\pi \overline{R}}{2} \left\{ \mathrm{sign}(n-m)\, \delta_{p,|n-m|} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} + \delta_{p,n+m} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right\} \tag{I-7a}$$

$$\mathbf{A}_{2nmp}^{0s} = \overline{R} \int_{0}^{2\pi} \sin p\theta \cdot \mathbf{B}_{2nm}^{G0}(\theta)\, d\theta$$

$$= \frac{\pi \overline{R}}{2} \left\{ \mathrm{sign}(n-m)\, \delta_{p,|n-m|} \begin{bmatrix} 0 & nm & 0 & 0 & 0 & 0 \\ -nm & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & nm+1 & -(n+m) & 0 \\ 0 & 0 & -(nm+1) & 0 & 0 & -(n+m) \\ 0 & 0 & n+m & 0 & 0 & nm+1 \\ 0 & 0 & 0 & n+m & -(nm+1) & 0 \end{bmatrix} \right.$$

$$\left. - \delta_{p,n+m} \begin{bmatrix} 0 & nm & 0 & 0 & 0 & 0 \\ nm & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & nm-1 & m-n & 0 \\ 0 & 0 & nm-1 & 0 & 0 & n-m \\ 0 & 0 & n-m & 0 & 0 & nm-1 \\ 0 & 0 & 0 & m-n & nm-1 & 0 \end{bmatrix} \right\} \tag{I-7b}$$

$$\mathbf{A}_{3nmp}^{0s} = \overline{R} \int_{0}^{2\pi} \sin p\theta \cdot \mathbf{B}_{3nm}^{G0}(\theta)\, d\theta$$

$$= \frac{\pi \overline{R}}{2} \left\{ \mathrm{sign}(n-m)\, \delta_{p,|n-m|} \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & 0 & m & -1 & 0 \\ 0 & 0 & 0 & -1 & m & 0 \\ 0 & 0 & 1 & 0 & 0 & m \end{bmatrix} + \delta_{p,n+m} \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & 0 & -m & 1 & 0 \\ 0 & 0 & 0 & -1 & m & 0 \\ 0 & 0 & -1 & 0 & 0 & -m \end{bmatrix} \right\} \tag{I-7c}$$

$$\mathbf{A}_{1nmp}^{0c} = \overline{R} \int_{0}^{2\pi} \cos p\theta \cdot \mathbf{B}_{1nm}^{G0}(\theta)\, d\theta$$

$$= \frac{\pi \overline{R}}{2} \left\{ \delta_{p,|n-m|} \underset{6\times6}{\mathbf{I}} + \delta_{p,n+m} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix} \right\} \qquad \text{(I-8a)}$$

$$\mathbf{A}_{2nmp}^{0c} = \overline{R} \int_{0}^{2\pi} \cos p\theta \cdot \mathbf{B}_{2nm}^{G0}(\theta)\, d\theta$$

$$= \frac{\pi \overline{R}}{2} \left\{ \delta_{p,|n-m|} \begin{bmatrix} nm & 0 & 0 & 0 & 0 & 0 \\ 0 & nm & 0 & 0 & 0 & 0 \\ 0 & 0 & nm+1 & 0 & 0 & m+n \\ 0 & 0 & 0 & nm+1 & -(m+n) & 0 \\ 0 & 0 & 0 & -(m+n) & nm+1 & 0 \\ 0 & 0 & m+n & 0 & 0 & nm+1 \end{bmatrix} \right.$$

$$\left. + \delta_{p,n+m} \begin{bmatrix} nm & 0 & 0 & 0 & 0 & 0 \\ 0 & -nm & 0 & 0 & 0 & 0 \\ 0 & 0 & nm-1 & 0 & 0 & n-m \\ 0 & 0 & 0 & -(nm-1) & n-m & 0 \\ 0 & 0 & 0 & m-n & nm-1 & 0 \\ 0 & 0 & m-n & 0 & 0 & -(nm-1) \end{bmatrix} \right\} \qquad \text{(I-8b)}$$

$$\mathbf{A}_{3nmp}^{0c} = \overline{R} \int_{0}^{2\pi} \cos p\theta \cdot \mathbf{B}_{3nm}^{G0}(\theta)\, d\theta$$

$$= \frac{\pi \overline{R}}{2} \left\{ \delta_{p,|n-m|} \begin{bmatrix} 0 & -m & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & 1 & 0 \\ 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -m \\ 0 & 0 & 0 & -1 & m & 0 \end{bmatrix} + \delta_{p,n+m} \begin{bmatrix} 0 & m & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -1 & 0 \\ 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & m \\ 0 & 0 & 0 & -1 & m & 0 \end{bmatrix} \right\} \qquad \text{(I-8c)}$$

$$\underset{6\times2}{\mathbf{A}}{}_{1nm}^{11} = \overline{R} \int_{0}^{2\pi} \mathbf{B}_{1nm}^{G1}(\theta)\, d\theta = \overline{R} m \begin{bmatrix} (\eta_3)_{nm} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \text{(I-9a)}$$

$$\underset{6\times4}{\mathbf{A}}{}_{2nm}^{11} = \bar{R}\int_{0}^{2\pi} \mathbf{B}_{2nm}^{G1}(\theta)\,d\theta$$

$$= \bar{R}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ nm(\eta_2)_{nm}+\{(\eta_1)_{nm}+(\eta_1'')_{nm}\} & n\{(\eta_3)_{mn}^{T}+(\eta_3'')_{mn}^{T}\}-m(\eta_3)_{nm} \\ -n\{(\eta_3)_{mn}^{T}+(\eta_3'')_{mn}^{T}\}+m(\eta_3)_{nm} & nm(\eta_2)_{nm}+\{(\eta_1)_{nm}+(\eta_1'')_{nm}\} \end{bmatrix} \tag{I-9b}$$

$$\underset{6\times4}{\mathbf{A}}{}_{3nm}^{11} = \bar{R}\int_{0}^{2\pi} \mathbf{B}_{3nm}^{G1}(\theta)\,d\theta$$

$$= \bar{R}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ m(\eta_3)_{nm} & (\eta_1)_{nm}+(\eta_1'')_{nm} \\ -\{(\eta_1)_{nm}+(\eta_1'')_{nm}\} & m(\eta_3)_{nm} \end{bmatrix} \tag{I-9c}$$

$$\underset{6\times2}{\mathbf{A}}{}_{4nm}^{11} = \bar{R}\int_{0}^{2\pi} \mathbf{B}_{4nm}^{G1}(\theta)\,d\theta = \bar{R}nm\begin{bmatrix} (\eta_2)_{nm} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{I-9d}$$

The components of $(\eta_\alpha)_{nm}$ and $(\eta_\alpha'')_{nm}$ (α=1,3) are given in Appendix G (eqns.(G-1) and (G-3)).

$$\mathbf{A}_{1nm}^{1s} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}_{1nm}^{G1}(\theta)\,d\theta = \bar{R}nm\begin{bmatrix} (\eta_{3s})_{nm} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{I-10a}$$

$$\mathbf{A}_{2nm}^{1s} = \bar{R}\int_{0}^{2\pi} \sin\theta\cdot\mathbf{B}_{2nm}^{G1}(\theta)\,d\theta$$

$$= \bar{R}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \{(\eta_{1s})_{nm}+(\eta_{1s}'')_{nm}\}+nm(\eta_{2s})_{nm} & n\{(\eta_{3s})_{mn}^{T}+(\eta_{3s}'')_{mn}^{T}\}-m(\eta_{3s})_{nm} \\ -n\{(\eta_{3s})_{mn}^{T}+(\eta_{3s}'')_{mn}^{T}\}+m(\eta_{3s})_{nm} & \{(\eta_{1s})_{nm}+(\eta_{1s}'')_{nm}\}+nm(\eta_{2s})_{nm} \end{bmatrix} \tag{I-10b}$$

$$\mathbf{A}_{3nm}^{1s} = \bar{R}\int_0^{2\pi} \sin\theta\cdot\mathbf{B}_{3nm}^{G1}(\theta)\,d\theta$$

$$= \bar{R}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ m(\eta_{3s})_{nm} & (\eta_{1s})_{nm}+(\eta''_{1s})_{nm} \\ -\{(\eta_{1s})_{nm}+(\eta''_{1s})_{nm}\} & m(\eta_{3s})_{nm} \end{bmatrix} \qquad \text{(I-10c)}$$

$$\mathbf{A}_{4nm}^{1s} = \bar{R}\int_0^{2\pi} \sin\theta\cdot\mathbf{B}_{4nm}^{G1}(\theta)\,d\theta = \bar{R}nm\begin{bmatrix} (\eta_{2s})_{nm} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \text{(I-10d)}$$

The components of $(\eta_{\alpha s})_{nm}$ and $(\eta''_{\alpha s})_{nm}$ $(\alpha=1,3)$ are given in Appendix G (eqns. (G-6)).

$$\mathbf{A}_{1nm}^{1c} = \bar{R}\int_0^{2\pi} \cos\theta\cdot\mathbf{B}_{1nm}^{G1}(\theta)\,d\theta = \bar{R}nm\begin{bmatrix} (\eta_{3c})_{nm} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \text{(I-11a)}$$

$$\mathbf{A}_{2nm}^{1c} = \bar{R}\int_0^{2\pi} \cos\theta\cdot\mathbf{B}_{2nm}^{G1}(\theta)\,d\theta$$

$$= \bar{R}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \{(\eta_{1c})_{nm}+(\eta''_{1c})_{nm}\}+nm(\eta_{2c})_{nm} & n\{(\eta_{3c})_{mn}^{T}+(\eta''_{3c})_{mn}^{T}\}-m(\eta_{3c})_{nm} \\ -n\{(\eta_{3c})_{mn}^{T}+(\eta''_{3c})_{mn}^{T}\}+m(\eta_{3c})_{nm} & \{(\eta_{1c})_{nm}+(\eta''_{1c})_{nm}\}+nm(\eta_{2c})_{nm} \end{bmatrix} \qquad \text{(I-11b)}$$

$$\mathbf{A}_{3nm}^{1c} = \bar{R}\int_0^{2\pi} \cos\theta\cdot\mathbf{B}_{3nm}^{G1}(\theta)\,d\theta$$

$$= \bar{R}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ m(\eta_{3c})_{nm} & (\eta_{1c})_{nm}+(\eta''_{1c})_{nm} \\ -\{(\eta_{1c})_{nm}+(\eta''_{1c})_{nm}\} & m(\eta_{3c})_{nm} \end{bmatrix} \qquad \text{(I-11c)}$$

$$\mathbf{A}_{4nm}^{1c} = \bar{R}\int_0^{2\pi} \cos\theta\cdot\mathbf{B}_{4nm}^{G1}(\theta)\,d\theta = \bar{R}nm\begin{bmatrix} (\eta_{2c})_{nm} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \text{(I-11d)}$$

The components of $(\eta_{\alpha c})_{nm}$ and $(\eta''_{\alpha c})_{nm}$ $(\alpha=1,3)$ are given in Appendix G (eqns. (G-7)).

In all these equations, the matrices integrated, $\mathbf{B}_{\alpha nm}^{G0}$ (α=1,3), and $\mathbf{B}_{\alpha nm}^{G1}$ (α=1,4), are given in Appendix H.

$$\mathbf{A}_{1nmp}^{0c} = \frac{\pi\bar{R}}{2}\left(\delta_{p,|n-m|}\,\mathbf{I} + \delta_{p,n+m}\,\mathbf{L}_{12}^{c}\right) \qquad \text{(I-8a)}$$

$$\mathbf{A}_{2nmp}^{0c} = \frac{\pi\bar{R}}{2}\left(\delta_{p,|n-m|}\,\mathbf{L}_{21}^{c} + \delta_{p,n+m}\,\mathbf{L}_{22}^{c}\right) \qquad \text{(I-8b)}$$

$$\mathbf{A}_{3nmp}^{0c} = \frac{\pi\bar{R}}{2}\left(\delta_{p,|n-m|}\,\mathbf{L}_{31}^{c} + \delta_{p,n+m}\,\mathbf{L}_{32}^{c}\right) \qquad \text{(I-8c)}$$

$$\mathbf{A}_{1nmp}^{0s} = \frac{\pi\bar{R}}{2}\left(\text{sign }(n-m)\;\delta_{p,|n-m|}\,\mathbf{L}_{11}^{s} + \delta_{p,n+m}\,\mathbf{L}_{12}^{s}\right) \qquad \text{(I-7a)}$$

$$\mathbf{A}_{2nmp}^{0s} = \frac{\pi\bar{R}}{2}\left(\text{sign }(n-m)\;\delta_{p,|n-m|}\,\mathbf{L}_{21}^{s} + \delta_{p,n+m}\,\mathbf{L}_{22}^{s}\right) \qquad \text{(I-7b)}$$

$$\mathbf{A}_{3nmp}^{0s} = \frac{\pi\bar{R}}{2}\left(\text{sign }(n-m)\;\delta_{p,|n-m|}\,\mathbf{L}_{31}^{s} + \delta_{p,n+m}\,\mathbf{L}_{32}^{s}\right) \qquad \text{(I-7c)}$$

$$\mathbf{L}_{12}^{c} = \begin{bmatrix} -1 & & & & & \\ & +1 & & & & \\ & & -1 & & & \mathbf{0} \\ \mathbf{0} & & & +1 & & \\ & & & & -1 & \\ & & & & & +1 \end{bmatrix} \qquad \text{(I-12a)}$$

$$\mathbf{L}_{21}^{c} = \begin{bmatrix} nm & 0 & 0 & 0 & 0 & 0 \\ 0 & nm & 0 & 0 & 0 & 0 \\ 0 & 0 & nm+1 & 0 & 0 & n+m \\ 0 & 0 & 0 & nm+1 & -(n+m) & 0 \\ 0 & 0 & 0 & -(n+m) & nm+1 & 0 \\ 0 & 0 & n+m & 0 & 0 & nm+1 \end{bmatrix} \qquad \text{(I-12b)}$$

$$\mathbf{L}_{22}^{c} = \begin{bmatrix} nm & 0 & 0 & 0 & 0 & 0 \\ 0 & -nm & 0 & 0 & 0 & 0 \\ 0 & 0 & nm-1 & 0 & 0 & n-m \\ 0 & 0 & 0 & -(nm-1) & n-m & 0 \\ 0 & 0 & 0 & -(n-m) & nm-1 & 0 \\ 0 & 0 & -(n-m) & 0 & 0 & -(nm-1) \end{bmatrix} \qquad \text{(I-12c)}$$

$$\mathbf{L}^{c}_{31} = \left[\begin{array}{cc:cccc} 0 & -m & \multicolumn{4}{c}{\mathbf{0}} \\ m & 0 & & & & \\ \hdashline & & 0 & -m & 1 & 0 \\ \multicolumn{2}{c:}{\mathbf{0}} & m & 0 & 0 & 1 \\ & & -1 & 0 & 0 & -m \\ & & 0 & -1 & m & 0 \end{array}\right] \tag{I-12d}$$

$$\mathbf{L}^{c}_{32} = \left[\begin{array}{cc:cccc} 0 & m & \multicolumn{4}{c}{\mathbf{0}} \\ m & 0 & & & & \\ \hdashline & & 0 & m & -1 & 0 \\ \multicolumn{2}{c:}{\mathbf{0}} & m & 0 & 0 & 1 \\ & & 1 & 0 & 0 & m \\ & & 0 & -1 & m & 0 \end{array}\right] \tag{I-12e}$$

$$\mathbf{L}^{s}_{11} = \left[\begin{array}{cc:cc:cc} 0 & 1 & \multicolumn{2}{c:}{\mathbf{0}} & \multicolumn{2}{c}{\mathbf{0}} \\ -1 & 0 & & & & \\ \hdashline \multicolumn{2}{c:}{\mathbf{0}} & 0 & 1 & \multicolumn{2}{c}{\mathbf{0}} \\ & & -1 & 0 & & \\ \hdashline \multicolumn{2}{c:}{\mathbf{0}} & \multicolumn{2}{c:}{\mathbf{0}} & 0 & 1 \\ & & & & -1 & 0 \end{array}\right], \quad \mathbf{L}^{s}_{12} = \left[\begin{array}{cc:cc:cc} 0 & 1 & \multicolumn{2}{c:}{\mathbf{0}} & \multicolumn{2}{c}{\mathbf{0}} \\ 1 & 0 & & & & \\ \hdashline \multicolumn{2}{c:}{\mathbf{0}} & 0 & 1 & \multicolumn{2}{c}{\mathbf{0}} \\ & & 1 & 0 & & \\ \hdashline \multicolumn{2}{c:}{\mathbf{0}} & \multicolumn{2}{c:}{\mathbf{0}} & 0 & 1 \\ & & & & 1 & 0 \end{array}\right] \tag{I-13a,b}$$

$$\mathbf{L}^{s}_{21} = \left[\begin{array}{cc:cccc} 0 & nm & \multicolumn{4}{c}{\mathbf{0}} \\ -nm & 0 & & & & \\ \hdashline & & 0 & nm+1 & -(n+m) & 0 \\ \multicolumn{2}{c:}{\mathbf{0}} & -(nm+1) & 0 & 0 & -(n+m) \\ & & n+m & 0 & 0 & nm+1 \\ & & 0 & n+m & -(nm+1) & 0 \end{array}\right], \quad \mathbf{L}^{s}_{22} = \left[\begin{array}{cc:cccc} 0 & -nm & \multicolumn{4}{c}{\mathbf{0}} \\ -nm & 0 & & & & \\ \hdashline & & 0 & -nm+1 & n-m & 0 \\ \multicolumn{2}{c:}{\mathbf{0}} & -(nm+1) & 0 & 0 & -(n-m) \\ & & -(n-m) & 0 & 0 & -nm+1 \\ & & 0 & n-m & -nm+1 & 0 \end{array}\right] \tag{I-13c,d}$$

$$\mathbf{L}^{s}_{31} = \left[\begin{array}{cc:cccc} m & 0 & \multicolumn{4}{c}{\mathbf{0}} \\ 0 & m & & & & \\ \hdashline & & m & 0 & 0 & 1 \\ \multicolumn{2}{c:}{\mathbf{0}} & 0 & m & -1 & 0 \\ & & 0 & -1 & m & 0 \\ & & 1 & 0 & 0 & m \end{array}\right], \quad \mathbf{L}^{s}_{31} = \left[\begin{array}{cc:cccc} m & 0 & \multicolumn{4}{c}{\mathbf{0}} \\ 0 & m & & & & \\ \hdashline & & m & 0 & 0 & 1 \\ \multicolumn{2}{c:}{\mathbf{0}} & 0 & m & -1 & 0 \\ & & 0 & -1 & m & 0 \\ & & 1 & 0 & 0 & m \end{array}\right] \tag{I-13e,f}$$

Appendix J

$$\mathbf{C}^{01}_{nm} = \pi\delta_{nm} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{I}^* \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{I} & \mathbf{I} & \mathbf{0} & m\mathbf{I}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & n\mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n\mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{J-1}$$

where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{I}^* = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$ (J-2a,b)

(in the casc of m=n=0, see Appendix E. (eqn. (E-13)))

$$\mathbf{C}^{0c}_{nm} = \frac{\pi}{2}\,\delta_{1,|m-n|} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{I}^* \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{I} & \mathbf{I} & \mathbf{0} & m\mathbf{I}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & n\mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n\mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{J-3}$$

(in the casc of (m=0, n=1, and m=1, n=0), see Appendix E. (eqn. (E-14)))

$$\mathbf{C}^{0c}_{nm} = \frac{\pi}{2}\,\mathrm{sign}(n-m)\,\delta_{1,|n-m|} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{I}^{*T} & \mathbf{I}^{*T} & \mathbf{0} & m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n\mathbf{I}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & +n\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{J-4}$$

(in the casc of (m=0, n=1) or (m=1, n=0), see Appendix E. (eqn. (E-15)))

$$\mathbf{C}^{0\eta c}_{nm} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\eta_{1c})_{nm} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m(\eta_{1c})_{nm} \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}(\eta_{1c})_{nm} & (\eta_{1c})_{nm} & \mathbf{0} & m(\eta_{3c})_{nm} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\eta_{1c})_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\eta_{1c})_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & n(\eta_{3c})^{T}_{mn} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n(\eta_{3c})^{T}_{mn} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{J-5}$$

where $(\eta_{1c})_{nm}$, $(\eta_{3c})_{nm}$ are given in Appendix G. (eqns. (G-7))

$$\mathbf{C}_{nm}^{11} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\eta_1)_{nm} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{2}{R}(\eta_1'')_{nm} & (\eta_1)_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\eta_1)_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\eta_1)_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{J-6}$$

$$\mathbf{C}_{nm}^{1c} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\eta_{1c})_{nm} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{2}{R}(\eta_{1c}'')_{nm} & (\eta_{1c})_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\eta_{1c})_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\eta_{1c})_{nm} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{J-7}$$

where $(\eta_1'')_{nm}$, $(\eta_{1c})_{nm}$ and $(\eta_{1c}'')_{nm}$ are given in Appendix G. (eqns. (G-3) and (G-7))

$$\mathbf{C}_{nmp}^{0s} = \frac{\pi}{2} \left\{ \mathrm{sign}(n-m)\,\delta_{p,|n-m|} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\frac{2}{R}\mathbf{I}^* & -\mathbf{I}^* & \mathbf{0} & m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right.$$

$$\left. + \delta_{p,n+m} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & m\mathbf{J}^* \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{J} & \mathbf{J} & \mathbf{0} & -m\mathbf{J}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n\mathbf{J}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & n\mathbf{J}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \tag{J-8}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{J-9a,b}$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{J}^* = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{J-10a,b}$$

$$\mathbf{I}^T = \mathbf{I}\,, \ \mathbf{J}^T = \mathbf{J}\,, \ \mathbf{I}^{*T} = -\mathbf{I}^*\,, \ \mathbf{J}^{*T} = \mathbf{J}^* \tag{J-11}$$

$$\mathbf{C}_{nmp}^{0c} = \frac{\pi}{2}\left\{ \delta_{p,|n-m|} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{I}^* \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{I} & \mathbf{I} & \mathbf{0} & m\mathbf{I}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n\mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n\mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right.$$

$$\left. +\delta_{p,n+m} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{J} \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{J}^* & \mathbf{J}^* & \mathbf{0} & m\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{J}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & n\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \qquad \text{(J-12)}$$

$$\mathbf{C}_{nmp}^{0c} = \frac{\pi}{2}(\delta_{p,|n-m|}\,\mathbf{L}_{p1}^{c} + \delta_{p,n+m}\,\mathbf{L}_{p2}^{c}) \qquad \text{(J-12)}$$

$$\mathbf{C}_{nmp}^{0s} = \frac{\pi}{2}\{\text{sign}\,(n-m)\,\delta_{p,|n-m|}\,\mathbf{L}_{p1}^{s} + \delta_{p,n+m}\,\mathbf{L}_{p2}^{s}\} \qquad \text{(J-8)}$$

$$\mathbf{L}_{p1}^{c} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{I}^* \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{I} & \mathbf{I} & \mathbf{0} & m\mathbf{I}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n\mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & n\mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \text{(J-13a)}$$

$$\mathbf{L}_{p2}^{c} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{J} \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{J}^* & \mathbf{J}^* & \mathbf{0} & m\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{J}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & n\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \text{(J-13b)}$$

$$\mathbf{L}_{p1}^{S} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -m\mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\frac{2}{R}\mathbf{I}^* & -\mathbf{I}^* & \mathbf{0} & m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & n\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \text{(J-14a)}$$

$$\mathbf{L}_{p2}^{c} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & m\mathbf{J}^{*} \\ \mathbf{0} & \mathbf{0} & \frac{2}{R}\mathbf{J} & \mathbf{J} & \mathbf{0} & -m\mathbf{J}^{*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n\mathbf{J}^{*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & n\mathbf{J}^{*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{J-14b}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \ \mathbf{I}^{*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{J-9a,b}$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \ \mathbf{J}^{*} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{J-10a,b}$$

$$\mathbf{I}^{T} = \mathbf{I} , \ \mathbf{J}^{T} = \mathbf{J} , \ \mathbf{I}^{*T} = -\mathbf{I}^{*} , \ \mathbf{J}^{*T} = \mathbf{J}^{*} \tag{J-11}$$

Appendix K

eqn. (6.11) ⇒

(1) case of a open cantilever tank

$$\mathbf{d}^T = [\,\underbrace{\overbrace{\mathbf{d}_{20}}^{10}\ \ \mathbf{d}_{30} \cdots \mathbf{d}_{I0}}_{10(I-1)} : \underbrace{\mathbf{d}_{21} \cdots \mathbf{d}_{I1}}_{10(I-1)} : \cdots : \underbrace{\mathbf{d}_{2N} \cdots \mathbf{d}_{IN}}_{10(I-1)}\,] \tag{K-1a}$$

$$10(I\text{-}1)(N+1)$$

$$\Phi = \begin{bmatrix} \underset{10(I-1)\times a}{\phi_{10}} & & \mathbf{0} \\ & \phi_{20} & \\ \mathbf{0} & & \ddots\ \phi_{aN} \end{bmatrix} \qquad \{10(I\text{-}1)\times(N+1)\,\}\times a(N+1) \tag{K-1b}$$

$$\mathbf{d}^{(r)T} = [\,\underbrace{\overset{r}{\mathbf{d}}_{10}\ \ \overset{r}{\mathbf{d}}_{20} \cdots \overset{r}{\mathbf{d}}_{a0}}_{a} : \underbrace{\overset{r}{\mathbf{d}}_{11} \cdots \overset{r}{\mathbf{d}}_{a1}}_{a} : \cdots : \underbrace{\overset{r}{\mathbf{d}}_{1N} \cdots \overset{r}{\mathbf{d}}_{aN}}_{a}\,] \tag{K-1c}$$

(2) case of a closed cantilever tank with rigid roof

$$\mathbf{d}^T = [\,\underbrace{\overbrace{\mathbf{d}_{20}}^{10} \cdots \mathbf{d}_{I-1,0}}_{10(I-2)} : \underbrace{\mathbf{d}_{21} \cdots \mathbf{d}_{I1}}_{10(I-1)} : \underbrace{\mathbf{d}_{22} \cdots \mathbf{d}_{I-1,2}}_{10(I-2)} : \cdots : \underbrace{\mathbf{d}_{2N} \cdots \mathbf{d}_{I-1,N}}_{10(I-1)}\,] \tag{K-2a}$$

$$\{10(I\text{-}2)\times(N+1)\text{-}10\}$$

$$\Phi = \begin{bmatrix} \phi_{10} & \overset{10(I-1)\times a}{\downarrow} & & \mathbf{0} \\ \uparrow & \phi_{20} & & \\ 10(I-2)\times a \rightarrow & & \phi_{30} & \overset{10(I-2)\times a}{\downarrow} \\ \mathbf{0} & & & \ddots\ \phi_{aN} \end{bmatrix} \quad \{10(I\text{-}2)\times(N+1)+10\}\times a(N+1) \tag{K-2b}$$

$\mathbf{d}^{(r)}$: same as eqn. (K-1c) eqns. (6.13) ⇒

$$\bar{\Lambda} = \begin{bmatrix} \underset{a\times a}{\bar{\Lambda}_{00}} & & & \mathbf{0} \\ & \bar{\Lambda}_{11} & & \\ & & \ddots & \\ \mathbf{0} & & & \bar{\Lambda}_{NN} \end{bmatrix}_{a(N+1)\times a(N+1)} \tag{K-3a}$$

and

$$\bar{\Lambda}_{nm} = \begin{bmatrix} \hat{\omega}_{1n}^2 & & & \mathbf{0} \\ & \hat{\omega}_{2n}^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \hat{\omega}_{an}^2 \end{bmatrix}_{a\times a} \tag{K-3b}$$

where $\hat{\omega}_{in}$ (i=1,a) are natural frequencies without imperfections (ν=0).

$$\bar{\mathbf{G}}(t) = \begin{bmatrix} \overset{\overset{a\times a}{\swarrow}}{\bar{\mathbf{G}}_{00}} & \bar{\mathbf{G}}_{01} & & & & & \mathbf{0} \\ \bar{\mathbf{G}}_{10} & \bar{\mathbf{G}}_{11} & \bar{\mathbf{G}}_{12} & & & & \\ & \bar{\mathbf{G}}_{21} & \bar{\mathbf{G}}_{22} & \bar{\mathbf{G}}_{23} & & & \\ & & & \ddots & & & \\ & & & \bar{\mathbf{G}}_{n,n-1} & \bar{\mathbf{G}}_{nn} & \bar{\mathbf{G}}_{n,n+1} & \\ & & & & & \ddots & \\ & \mathbf{0} & & & & \bar{\mathbf{G}}_{N,N-1} & \bar{\mathbf{G}}_{NN} \end{bmatrix}_{a(N+1)\times a(N+1)} \tag{K-3c}$$

Appendix L

$$K^{c<1>}_{knm} = (I^{<1>}_{1k} - 1)\ \{(n-m-1)\ c^{c}_{|n-m-1|} - (n-m+1)\ c^{c}_{|n-m+1|}\} + I^{<2>}_{1k}\ (c^{c}_{|n-m-1|} + c^{c}_{|n-m+1|}) \tag{L-1a}$$

$$K^{c<2>}_{knm} = (I^{<1>}_{1k} - 1)\ \{(n+m-1)\ c^{c}_{n+m-1} - (n+m+1)\ c^{c}_{n+m+1}\} + I^{<2>}_{1k}\ (c^{c}_{n+m-1} + c^{c}_{n+m+1}) \tag{L-1b}$$

$$K^{s<1>}_{knm} = (I^{<1>}_{1k} - 1)\ (|n-m-1|\ c^{s}_{|n-m-1|} - |n-m+1|\ c^{s}_{|n-m+1|}) + I^{<2>}_{1k}\ \mathrm{sign}(n-m)\ (c^{s}_{|n-m-1|} + c^{s}_{|n-m+1|}) \tag{L-1c}$$

$$K^{s<2>}_{knm} = (I^{<1>}_{1k} - 1)\ \{(n+m-1)\ c^{s}_{n+m-1} - (n+m+1)\ c^{s}_{n+m+1}\} + I^{<2>}_{1k}\ (c^{s}_{n+m-1} + c^{s}_{n+m+1}) \tag{L-1d}$$

$$K^{c<3>}_{knm} = (I^{<1>}_{1k} - 1)\ (|n-m-1|\ c^{c}_{|n-m-1|} - |n-m+1|\ c^{c}_{|n-m+1|}) + I^{<2>}_{1k}\ \mathrm{sign}(n-m)\ (c^{c}_{|n-m-1|} + c^{c}_{|n-m+1|}) \tag{L-1e}$$

$$K^{s<3>}_{knm} = (I^{<1>}_{1k} - 1)\ \{(n-m-1)\ c^{s}_{|n-m-1|} - (n-m+1)\ c^{s}_{|n-m+1|}\} + I^{<2>}_{1k}\ (c^{s}_{|n-m-1|} + c^{s}_{|n-m+1|}) \tag{L-1f}$$

$$\bar{K}^{c<1>}_{knm} = \frac{\mu_k H(-1)^{k+1} - 1}{\mu_k^2} K^{c<1>}_{knm} + \sum_{i=1}^{\infty} \frac{J^{<2>}_{i}}{\varepsilon_i^2 + \mu_k^2}$$

$$* \{(n-m-1)\ c^{c}_{|n-m-1|} - (n-m+1)\ c^{c}_{|n-m+1|} + J^{<1>}_{i}\ (c^{c}_{|n-m-1|} + (c^{c}_{|n-m+1|})\} \tag{L-2a}$$

$$\bar{K}^{c<2>}_{knm} = \frac{\mu_k H(-1)^{k+1} - 1}{\mu_k^2} K^{c<2>}_{knm} + \sum_{i=1}^{\infty} \frac{J^{<2>}_{i}}{\varepsilon_i^2 + \mu_k^2}$$

$$* \{(n+m-1)\ c^{c}_{n+m-1} - (n+m+1)\ c^{c}_{n+m+1} + J^{<1>}_{i}\ (c^{c}_{n+m-1} + (c^{c}_{n+m+1})\} \tag{L-2b}$$

$$\bar{K}^{s<1>}_{knm} = \frac{\mu_k H(-1)^{k+1} - 1}{\mu_k^2} K^{s<1>}_{knm} + \sum_{i=1}^{\infty} \frac{J^{<2>}_{i}}{\varepsilon_i^2 + \mu_k^2}$$

$$* \{|n-m-1|\ c^{s}_{|n-m-1|} - |n-m+1|\ c^{s}_{|n-m+1|} + J^{<1>}_{i}\ \mathrm{sign}(n-m)\ (c^{s}_{|n-m-1|} + (c^{s}_{|n-m+1|})\} \tag{L-2c}$$

$$\bar{K}^{s<2>}_{knm} = \frac{\mu_k H(-1)^{k+1} - 1}{\mu_k^2} K^{s<2>}_{knm} + \sum_{i=1}^{\infty} \frac{J^{<2>}_{i}}{\varepsilon_i^2 + \mu_k^2}$$

$$* \{(n+m-1)\ c^{s}_{n+m-1} - (n+m+1)\ c^{s}_{n+m+1} + J^{<1>}_{i}\ (c^{s}_{n+m-1} + (c^{s}_{n+m+1})\} \tag{L-2d}$$

$$\bar{K}_{knm}^{c<3>} = \frac{\mu_k H(-1)^{k+1}-1}{\mu_k^2} K_{knm}^{c<3>} + \sum_{i=1}^{\infty} \frac{J_i^{<2>}}{\varepsilon_i^2+\mu_k^2}$$

$$* \{ |n-m-1| \, c_{|n-m-1|}^{c} - |n-m+1| \, c_{|n-m+1|}^{c} + J_i^{<1>} \operatorname{sign}(n-m) \, (c_{|n-m-1|}^{c} + c_{|n-m+1|}^{c}) \} \qquad \text{(L-2e)}$$

$$\bar{K}_{knm}^{s<3>} = \frac{\mu_k H(-1)^{k+1}-1}{\mu_k^2} K_{knm}^{s<3>} + \sum_{i=1}^{\infty} \frac{J_i^{<2>}}{\varepsilon_i^2+\mu_k^2}$$

$$* \{ (n-m-1) \, c_{|n-m-1|}^{s} - (n-m+1) \, c_{|n-m+1|}^{s} + J_i^{<1>} \, (c_{|n-m-1|}^{s} + (c_{|n-m+1|}^{s}) \} \qquad \text{(L-2f)}$$

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